Localization

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The procedure below generalizes the passage from an integral domain A to its field of fractions.

We work in $\boxed{\text{CRing.}}$

Def. A multiplicatively closed subset of a ring A is a subset $S \subset A$ s.t.:

(i) $1 \in S$.

(ii) If $s, t \in S$ then $s \cdot t \in S$.

Example. If A is an integral domain, then $S := A \setminus \{0\}$ is a m.c. subset.

Lemma Let $S \subset A$ be a m.c. subset, and let M be an A-module. Define a relation $\sim$ on the set $M \times S$ by:

$$\boxed{(m,s) \sim (n,t) \text{ if } \exists\, r \in S \text{ s.t. } r \cdot (sn - tm) = 0.}$$

Then $\sim$ is an equivalence relation.

pf. Exercise.

Def. In the setup of the lemma, we write

$$M_S := \frac{M \times S}{\sim} \quad \text{(the set of equiv. classes)},$$

and

$$\frac{m}{s} := \Big(\text{class of } (m,s)\Big).$$

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Theorem. Let A be a ring, $S \subset A$ a m.c. subset, and M ~~an~~ an A-module.

(1) The set M_S is an abelian group, with zero element $\frac{0}{1}$, and addition satisfying

$$(+) \qquad \frac{m}{s} + \frac{n}{t} = \frac{tm+sn}{st}.$$

(2) The abelian group A_S is a comm. ring, with unit $\frac{1}{1}$, and multiplication satisfying

$$(\cdot) \qquad \frac{a}{s} \cdot \frac{b}{t} = \frac{ab}{st}.$$

(3) The ab. grp. M_S is an A_S-module, with multiplication satis.

$$\frac{a}{s} \cdot \frac{m}{t} = \frac{am}{st}.$$

(4) The function $\lambda_{A,S} : A \to A_S$, $a \mapsto \frac{a}{1}$, is a ring hom.

(5) The function $\lambda_{M,S} : M \to M_S$, $m \mapsto \frac{m}{1}$, is an A-mod. hom.

(6) For $s \in S$, we have $\frac{1}{s} \cdot \frac{s}{1} = 1$. Thus $\lambda_A(s) = \frac{s}{1}$ is an invertible element of A_S.

(7) There is an A_S-module isomorphism $A_S \otimes_A M \xrightarrow{\simeq} M_S$ which sends

$$\frac{a}{s} \otimes m \mapsto \frac{am}{s}.$$

Proof. (1) We first prove that formula (†) gives a well-defined function (ie. indep. of representatives)

$$\alpha = + : M_S \times M_S \to M_S.$$

Suppose $\frac{m}{s} = \frac{m'}{s'}$ and $\frac{n}{t} = \frac{n'}{t'}$; we must show that

$$\frac{tm + sn}{st} \overset{?}{=} \frac{t'm' + s'n'}{s't'} \qquad (\heartsuit)$$

This can be done in two stages: first we assume $(t', n') = (t, n)$; and next we assume $(s', m') = (s, m)$. Due to the symmetry, it is enough to check the first stage only.

We know that $\exists\, r \in S$ s.t. $r(s'm - sm') = 0$. Then

$$r \cdot \Big(s't\,(tm + sn) - st\,(tm' + s'n) \Big) = 0 .$$

So indeed $(\heartsuit)$ is equality

Now that there is a well-defined operation of addition, it is trivial to check that M_S is an abelian group st, with zero $\frac{0}{1}$. (Same as for rational numbers!)

(2) Again, we must show that formula (††) determines a well-defined function

$\Downarrow$

(2) $\quad \cdot = \mu : A_S \times A_S \longrightarrow A_S$.

We change (a,s) to (a',s'); the other change, from (b,t) to (b',t'), is similar. So $\exists\, r \in S$ s.t. $r\cdot(s'a - sa') = 0$. We have to show that

$$\frac{ab}{st} \stackrel{?}{=} \frac{a'b}{s't}.$$

But $\quad r\cdot(s'tab - sta'b) = tbr\cdot(s'a - sa') = 0.$

After this, the verification that μ is the mult. of a comm. ring on the ab. grp. A_S is routine.

(3) Just like (2).

(4) & (5): Trivial.

(6) This is easy! $\frac{1}{s}\cdot\frac{s}{1} = \frac{s}{s}$.

But since $1\cdot(s\cdot 1 - 1\cdot s) = 0$, we get $\frac{s}{s} = \frac{1}{1}$.

(7) Consider the function

$$\beta : A_S \times M \longrightarrow M_S, \quad \beta\left(\frac{a}{s}, m\right) := \frac{a}{s}\cdot\frac{m}{1} = \frac{am}{s}.$$

This is A-bilinear, so get A-lin. hom.

$$\varphi : A_S \otimes_A M \longrightarrow M_S, \quad \varphi\left(\frac{a}{s}\otimes m\right) = \frac{am}{s}.$$

It is easy to see that φ is in fact A_S-linear. It remains to show that φ is bijective

Consider the function

$$\gamma : M \times S \to A_S \otimes_A M, \quad \gamma(m,s) := \frac{1}{s} \otimes m.$$

Suppose $(m,s) \sim (m',s')$. So $\exists\, r \in S$ s.t. $r \cdot (s'm - sm') = 0$.

Then

$$\frac{rss'}{1} \cdot \left(\frac{1}{s} \otimes m - \frac{1}{s'} \otimes m' \right) =$$

$$= \frac{rss'}{s} \otimes m - \frac{rss'}{s'} \otimes m' = \frac{1}{1} \otimes rs'm - \frac{1}{1} \otimes rsm'$$

$$= \frac{1}{1} \otimes r \cdot (s'm - sm') = 0.$$

Since $\frac{rss'}{1}$ is invertible, we see that

$$\gamma(m,s) = \frac{1}{s} \otimes m = \frac{1}{s'} \otimes m' = \gamma(m',s').$$

Thus γ descends to a function

$$\psi : M_S \to A_S \otimes_A M, \quad \psi\left(\frac{m}{s}\right) = \frac{1}{s} \otimes m.$$

Now $(\psi \circ \varphi)\left(\frac{a}{s} \otimes m\right) = \psi\left(\frac{am}{s}\right) = \frac{1}{s} \otimes am = \frac{a}{s} \otimes m$

and $(\varphi \circ \psi)\left(\frac{m}{s}\right) = \varphi\left(\frac{1}{s} \otimes m\right) = \frac{m}{s}$.

So φ is bijective.

$\square$

Exercise. Show that every element of $A_S \otimes_A M$ is a pure tensor.

Exercise Show that if S contains a nilpotent element, then $A_S = 0$.

Theorem (Universal Property of Localization)

Let $f : A \to B$ be a hom. in CRing, and let S be a m.c. subset of A. Assume that $f(S) \subset B^\times$, i.e. $f(s)$ is invertible in B for every $s \in S$. Then there is a unique ring hom. $f_S : A_S \to B$ s.t.

$$\boxed{(*) \quad f_S \circ \lambda_{A,S} = f.}$$

In a diagram:

$$\begin{tikzcd}
A \arrow[rr, "\lambda"] \arrow[dr] & & A_S \arrow[dl, "f_S"] \\
& B &
\end{tikzcd}$$

Pf. Consider the fun. $\gamma : A \times S \to B$, $\gamma(a,s) := f(a) \cdot f(s)^{-1}$. Say $(a,s) \sim (b,t)$. Then $\exists\, r \in S$ s.t. $r \cdot (ta - sb) = 0$. Then

$$f(r) \cdot \big(f(t) \cdot f(a) - f(s) \cdot f(b) \big) = 0.$$

But $f(r)$ is invertible, so $f(t) \cdot f(a) = f(s) \cdot f(b)$; and hence $\gamma(a,s) = \gamma(b,t)$.

Since γ respects the relation $\sim$, it descends to a function $f_S : A_S \to B$, $f_S\left(\frac{a}{s}\right) = f(a) \cdot f(s)^{-1}$.

It is easy to check that f_S is a ring hom, and $f_S \circ \lambda = f$.

Uniqueness is also easy to check. $\square$

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