

77

## Flatness of Localization

Let  $A$  be a ring,  $M \in \text{Mod } A$ , and  $S$  a m.c. subset of  $A$ . We proved that there is a ring  $A_S$ , with hom.  $\chi_{A,S}: A \rightarrow A_S$ ,  $a \mapsto \frac{a}{1}$ .

The elements of  $S$  became

invertible in  $A_S$ , and this is a universal property.

There is an  $A_S$ -module  $M_S$ , and an  $A$ -mod. hom.

$\chi_{M,S}: M \rightarrow M_S$ . Also  $M_S \cong A_S \otimes_A M$ ,  $\frac{m}{s} \leftrightarrow \frac{1}{s} \otimes m$ .

### Lemme

$$\text{Ker}(\chi_{M,S}) = \left\{ m \in M \mid sm = 0 \text{ for some } s \in S \right\}.$$

pf. By def. of  $\overset{\lambda \text{ end}}{M_S} = \frac{M \times S}{\sim}$ ,  $\chi(m) = \frac{m}{1} = \frac{0}{1}$  iff

$$\exists s \in S \text{ s.t. } s \cdot (m \cdot 1 - 1 \cdot 0) = 0.$$

$\underbrace{\hspace{1.5cm}}_{s \cdot m}$

□

Theorem Let  $S$  be a m.c. subset of  $A$ . Then  $A_S$  is flat over  $A$ .

pf. Suppose  $\psi: M \rightarrow N$  is an injective  $A$ -module hom. We have to show that  $\mathbb{1}_{A_S} \otimes \psi$  is injective.

Consider the diagram

$$\begin{array}{ccc} A_S \otimes_A M & \xrightarrow{\mathbb{1}_{A_S} \otimes \psi} & A_S \otimes_A N \\ \chi_M \downarrow \cong & & \cong \downarrow \chi_N \\ M_S & \xrightarrow{\psi_S} & N_S \end{array}$$

↘

(78) Where  $\chi_M$  is the map.  $\frac{a}{s} \otimes m \mapsto \frac{am}{s}$ ,  $\chi_N$  is likewise, and  $\psi_S(\frac{m}{s}) := \frac{\psi(m)}{s}$ . Checking what these maps do to  $\frac{a}{s} \otimes m \in A_S \otimes_A M$ , we see this is a comm. diag. So it suffices to prove that  $\psi_S$  is injective.

Let  $\frac{m}{s} \in \text{Ker}(\psi_S)$ . So  $\frac{\psi(m)}{s} = 0$ . Since  $\frac{1}{s}$  is invertible in  $A_S$ , and  $\frac{\psi(m)}{s} = \frac{1}{s} \cdot \frac{\psi(m)}{1}$ , it follows that  $\frac{\psi(m)}{1} = 0$ . But  $\frac{\psi(m)}{1} = \chi(\psi(m))$ , so by the lemma there is  $t \in \mathcal{D}$  st.  $t \cdot \psi(m) = 0$ . Now  $t \cdot \psi(m) = \psi(tm)$ , and  $\psi$  is injective. Hence  $tm = 0$ . Using the lemma again, we conclude that  $\chi(m) = \frac{m}{1} = 0$ . Hence  $\frac{m}{s} = \frac{1}{s} \cdot \frac{m}{1} = 0$ .  $\square$

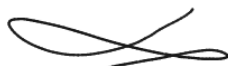


Cor. The induction functor

$$\text{ind}_S : \underline{\text{Mod}} A \rightarrow \underline{\text{Mod}} A_S$$

is exact.

Example  $\mathbb{Q}$  is a flat  $\mathbb{Z}$ -module.



Exercise Let  $G = \{1, \sigma\}$  be the Galois group of  $\mathbb{R} \rightarrow \mathbb{C}$ ;  $\sigma$  is conjugation.  $G$  acts on the rings  $\mathbb{C} \otimes_{\mathbb{R}} \mathbb{C}$  and  $\mathbb{C} \times \mathbb{C}$  by  $\sigma(a \otimes b) := a \otimes \sigma(b)$  and  $\sigma(a, b) := (b, a)$ . Show that there is a ring isomorphism  $\mathbb{C} \otimes_{\mathbb{R}} \mathbb{C} \cong \mathbb{C} \times \mathbb{C}$  which is compatible with the action of  $G$ .

79

## Prime Ideals

Def. An ideal  $\mathfrak{P}$  of  $A$  is called prime if it satisfies these conditions:

(i)  $\mathfrak{P} \neq A$ .

(ii) Let  $a, b \in A$ . If  $a \cdot b \in \mathfrak{P}$  then  $a \in \mathfrak{P}$  or  $b \in \mathfrak{P}$ .

Prop.  $\mathfrak{P}$  is a prime ideal of  $A$  iff the ring  $A/\mathfrak{P}$  is an integral domain.

(The proof is easy.)

Cor. Any maximal ideal is prime.

If  $\mathfrak{P}$  is a prime ideal of  $A$ , then

$$S := A - \mathfrak{P}$$

is a m.c. set.

Notation The localizing ring  $A_S$ , for  $S := A - \mathfrak{P}$ , is denoted by  $A_{\mathfrak{P}}$ . For an  $A$ -mod.  $M$  we write  $M_{\mathfrak{P}} := M_S$ .

Exercise. (1) List all the prime ideals of  $\mathbb{Z}$ .  
(2) What is the localiz. of  $\mathbb{Z}$  at the prime ideal  $0$ ?

(20)

## Local Rings

Def A ring  $A$  is called local if it has exactly one maximal ideal. If  $\mathfrak{m}$  is that maximal ideal, then  $A/\mathfrak{m}$  is called the residue field of  $A$ .

Lemma. Let  $A$  be a ring and  $\mathfrak{m}$  a maximal ideal.

IF AE: (i)  $A$  is local (with unique max. ideal  $\mathfrak{m}$ ).  
(ii) For every  $a \in \mathfrak{m}$  the element  $1+a$  is invertible.

pr. (i)  $\Rightarrow$  (ii): If  $1+a$  is not invertible, then ideal  $aA$  it generates is not all of  $A$ . So  $\exists$  max. ideal  $\mathfrak{m}_1$ ,  $a \in \mathfrak{m}_1$ . But since  $\mathfrak{m}$  is the unique max, we get  $\mathfrak{m} = \mathfrak{m}_1$ ; so  $1+a \in \mathfrak{m}$ . But then  $1 \in \mathfrak{m}$ , which is impossible.

(ii)  $\Rightarrow$  (i): If  $\mathfrak{m}$  is a max. ideal distinct from  $\mathfrak{m}$ , then  $\mathfrak{m} + \mathfrak{m} = A$ . So  $\exists a \in \mathfrak{m}$  and  $b \in \mathfrak{m}$  s.t.  $1 = a+b$ . But then  $1-a = b$  is invertible and in  $\mathfrak{m}$ . Impossible.  $\square$

Exercise. Let  $K$  be a field and  $t$  a variable.

(1) Show that the ring of formal power series  $A := K[[t]]$  is local. ( $K[[t]]$  is the completion of  $K[t]$  in the  $t$ -adic metric.)

(2) Show that the ring  $B := K[t]/(t^n)$  where  $n$  is any positive integer, is local.

(81) (3) What is the nec. & suff. condition on a polynomial  $p(t)$  for  $G := \mathbb{K}[t]/(p(t))$  to be a local ring?



Theorem Let  $\mathfrak{p}$  be a prime ideal of  $A$ .

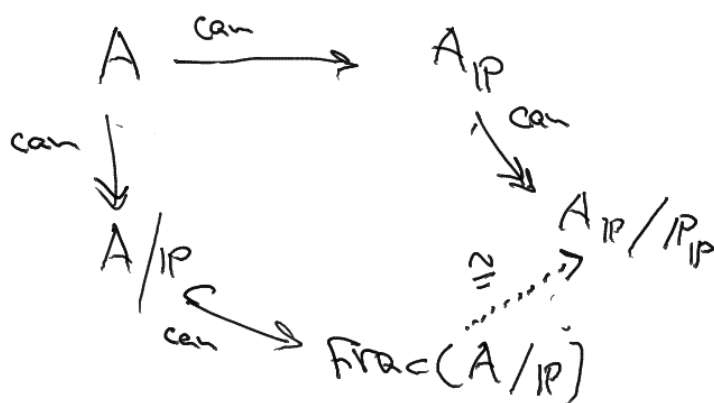
(1) The ring  $A_{\mathfrak{p}}$  is local, with maximal ideal  $\mathfrak{p}_{\mathfrak{p}}$ .

(2) There is a unique  $A$ -ring isomorphism

$$A_{\mathfrak{p}}/\mathfrak{p}_{\mathfrak{p}} \cong \text{Frac}(A/\mathfrak{p}).$$

Explanation. Since  $A_{\mathfrak{p}}$  is flat over  $A$ , the inclusion  $\mathfrak{p} \subset A$  induces an injective  $A_{\mathfrak{p}}$ -mod. hom.  $\mathfrak{p}_{\mathfrak{p}} \hookrightarrow A_{\mathfrak{p}}$ . So we can identify  $\mathfrak{p}_{\mathfrak{p}}$  with an ideal of  $A_{\mathfrak{p}}$ .

By "A-ring" we mean a ring  $B$  with a ring hom.  $A \rightarrow B$ . What item 2 says is that there is a

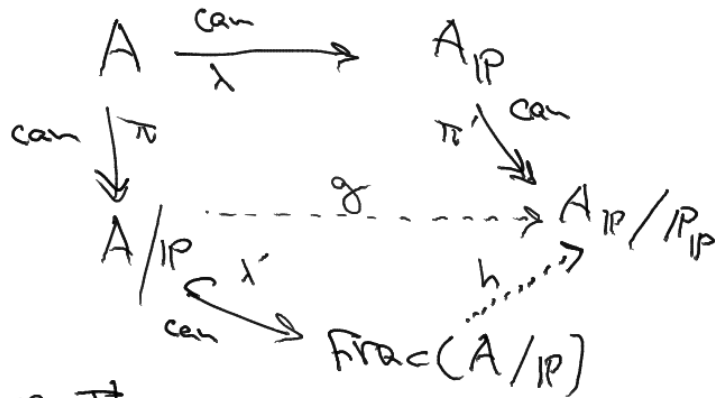


unique  $\cong$  making this diagram commutative

The field  $A_{\mathfrak{p}}/\mathfrak{p}_{\mathfrak{p}} \cong \text{Frac}(A/\mathfrak{p})$  is usually denoted by  $\mathbb{k}(\mathfrak{p})$ .

(82)

proof. We begin by showing that there is a hom. of making the diag. comm.



Take  $a \in \mathbb{P}$ . Then

$\lambda(a) = \frac{a}{1} \in \mathbb{P}_{\mathbb{P}}$ , so  $(\pi' \circ \lambda)(a) = 0$ . Hence  $\exists g$  s.t.

$g \circ \pi = \pi' \circ \lambda$ . Next we show that  $\exists h$  s.t. the

diag. is commut. Take  $b \in A/\mathbb{P}$ ,  $b \neq 0$ . So  $b = \pi(s)$ ,

$s \notin \mathbb{P}$ ,  $\lambda(s)$  is invert. in  $A_{\mathbb{P}}$ , and

$g(b) = \pi'(\lambda(s))$  is invertible in  $A_{\mathbb{P}}/\mathbb{P}_{\mathbb{P}}$ . By the univ.

prop. of localiz. there is a hom.  $h$  s.t.  $h \circ \lambda' = g$ .

The formula is

$$h\left(\frac{\pi(a)}{\pi(s)}\right) = \pi'\left(\frac{a}{s}\right), \quad a \in A, s \in A - \mathbb{P}.$$

This proves that  $h$  is unique.

Since  $\text{Frac}(A/\mathbb{P})$  is a field we see that  $h$  is inject.

And since any elt. of  $A_{\mathbb{P}}/\mathbb{P}_{\mathbb{P}}$  can be written as  $\pi'(a/s)$ , formula (4) shows that  $h$  is surject.

The fact that  $h$  is an isom. tells us that  $A_{\mathbb{P}}/\mathbb{P}_{\mathbb{P}}$  is a field, and thus  $\mathbb{P}_{\mathbb{P}}$  is maximal.

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2/12

Next we prove that  $A_{\mathbb{P}}/\mathbb{P}_{\mathbb{P}} \neq 0$ , i.e. that  $\mathbb{P}_{\mathbb{P}} \subsetneq A_{\mathbb{P}}$ . Otherwise we would have  $\frac{1}{1} \in \mathbb{P}_{\mathbb{P}}$ , so  $\frac{1}{1} = \frac{a}{s}$  for some  $a \in \mathbb{P}$  and  $s \in A - \mathbb{P}$ . Then  $\exists t \in A - \mathbb{P}$  s.t.  $t \cdot (s - a) = 0$ . This implies  $ts = ta \in \mathbb{P}$ . But  $\mathbb{P}$  is a prime ideal, and we have a contradiction.