Flatness of Localization

Let A be a ring, $M \in \underline{\mathrm{Mod}}\, A$, and S a m.c. subset of M. We proved that there is a ring A_S, with hom. $\lambda_{A,S} : A \to A_S$; $a \mapsto \frac{a}{1}$.

The elements of S become invertible in A_S, and this is a universal property. There is an A_S-module M_S, and an A-mod. hom. $\lambda_{M,S} : M \to M_S$. Also $M_S \cong A_S \otimes_A M$, $\frac{m}{s} \leftrightarrow \frac{1}{s} \otimes m$.

__Lemma__

$$\mathrm{Ker}(\lambda_{M,S}) = \{ m \in M \mid sm = 0 \text{ for some } s \in S \}.$$

Pf. By def. of $\overset{\lambda \text{ and}}{\vee}$ $M_S = \frac{M \times S}{\sim}$, $\lambda(m) = \frac{m}{1} = \frac{0}{1}$ iff $\exists\, s \in S$ s.t. $s \cdot (\underset{\underset{s \cdot m}{\parallel}}{m \cdot 1} - 1 \cdot 0) = 0$. $\square$

__Theorem__ Let S be a m.c. subset of A. Then A_S is __flat__ over A.

Pf. Suppose $\varphi : M \to N$ is an injective A-module hom. We have to show that $1\!\!1_{A_S} \otimes \varphi$ is injective. Consider the diagram

$$\begin{tikzcd} A_S \otimes_A M \arrow[r, "1\!\!1_{A_S} \otimes \varphi"] \arrow[d, "\chi_M"', "\cong"] & A_S \otimes_A N \arrow[d, "\chi_N", "\cong"'] \\ M_S \arrow[r, "\varphi_S"] & N_S \end{tikzcd}$$

78 where χ_M is the isom. $\frac{a}{s}\otimes m \mapsto \frac{am}{s}$, χ_N is likewise, and $\varphi_S\left(\frac{m}{s}\right) := \frac{\varphi(m)}{s}$. Checking what these hom's do to

$\frac{a}{s}\otimes m \in A_S \otimes_A M$, we see this is a comm. diag. So it suffices to prove that φ_S is injective.

Let $\frac{m}{s} \in \operatorname{Ker}(\varphi_S)$. So $\frac{\varphi(m)}{s} = 0$. Since $\frac{1}{s}$ is invertible in A_S, and $\frac{\varphi(m)}{s} = \frac{1}{s}\cdot\frac{\varphi(m)}{1}$, it follows that $\frac{\varphi(m)}{1} = 0$. But $\frac{\varphi(m)}{1} = \lambda(\varphi(m))$, so by the lemma there is $t \in S$ s.t. $t\cdot\varphi(m) = 0$. Now $t\cdot\varphi(m) = \varphi(t\cdot m)$, and φ is injective. Hence $tm = 0$. Using the lemma again, we conclude that $\lambda(m) = \frac{m}{1} = 0$. Hence $\frac{m}{s} = \frac{1}{s}\cdot\frac{m}{1} = 0$. $\square$

<u>Cor</u>. The induction functor

$$\operatorname{ind}_\lambda : \underline{\mathrm{Mod}}\, A \to \underline{\mathrm{Mod}}\, A_S$$

is exact.

<u>Example</u> $\mathbb{Q}$ is a flat $\mathbb{Z}$-module.

<u>Exercise</u> Let $G = \{1, \sigma\}$ be the Galois group of $\mathbb{R} \to \mathbb{C}$; σ is conjugation. G acts on the rings $\mathbb{C}\otimes_{\mathbb{R}}\mathbb{C}$ and $\mathbb{C}\times\mathbb{C}$ by $\sigma(a\otimes b) := a\otimes\sigma(b)$ and $\sigma(a,b) := (b,a)$. Show that there is a ring isomorphism $\mathbb{C}\otimes_{\mathbb{R}}\mathbb{C} \cong \mathbb{C}\times\mathbb{C}$ which is compatible with the action of G.

Prime Ideals

Def. An ideal $\mathbb{P}$ of A is called prime if it satisfies these conditions:

(i) $\mathbb{P} \neq A$.

(ii) Let $a, b \in A$. If $a \cdot b \in \mathbb{P}$ then $a \in \mathbb{P}$ or $b \in \mathbb{P}$.

Prop. $\mathbb{P}$ is a prime ideal of A iff the ring $A/\mathbb{P}$ is an integral domain.

(The proof is easy)

Cor. Any maximal ideal is prime.

If $\mathbb{P}$ is a prime ideal of A, then

$$S := A - \mathbb{P}$$

is a m.c. set.

Notation The localizing ring A_S, for $S := A - \mathbb{P}$, is denoted by $A_{\mathbb{P}}$. For an A-mod. M we write $M_{\mathbb{P}} := M_S$.

Exercise. (1) List all the prime ideals of $\mathbb{Z}$.

(2) What is the localiz. of $\mathbb{Z}$ at the prime ideal 0?

Local Rings

Def A ring A is called local if it has exactly one maximal ideal. If $\mathfrak{m}$ is that maximal ideal, then $A/\mathfrak{m}$ is called the residue field of A.

Lemma. Let A be a ring and $\mathfrak{m}$ a maximal ideal. TFAE: (i) A is local (with unique max. ideal $\mathfrak{m}$). (ii) For every $a \in \mathfrak{m}$ the element $1+a$ is invertible.

pf. (i) $\Rightarrow$ (ii): If $1+a$ is not invertible, then ideal $\mathfrak{a}$ it generates is not all of A. So $\exists$ max. ideal $\mathfrak{n}$, $\mathfrak{a} \subset \mathfrak{n}$. But since $\mathfrak{m}$ is the unique max, we get $\mathfrak{n} = \mathfrak{m}$; so $1+a \in \mathfrak{m}$. But then $1 \in \mathfrak{m}$, which is impossible.

(ii) $\Rightarrow$ (i): If $\mathfrak{n}$ is a max. ideal distinct from $\mathfrak{m}$, then $\mathfrak{m}+\mathfrak{n} = A$. So $\exists$ $a \in \mathfrak{m}$ and $b \in \mathfrak{n}$ s.t. $1 = a+b$. But then $1-a = b$ is invertible and in $\mathfrak{n}$. Impossible. $\square$

Exercise. Let $\mathbb{k}$ be a field and t a variable.

hard ① Show that the ring of formal power series $A := \mathbb{k}[[t]]$ is local. ($\mathbb{k}[[t]]$ is the completion of $\mathbb{k}[t]$ in the t-adic metric.)

② Show that the ring $B := \mathbb{k}[t]/(t^n)$ where n is any positive integer, is local.

(3) What is the nec. & suff. condition on a polynomial $p(t)$ for $G := \Bbbk[t]/(p(t))$ to be a local ring?

<u>Theorem</u> Let $\mathfrak{p}$ be a prime ideal of A.

(1) The ring $A_{\mathfrak{p}}$ is <u>local</u>, with maximal ideal $\mathfrak{p}_{\mathfrak{p}}$.

(2) There is a unique A-ring isomorphism
$$A_{\mathfrak{p}}/\mathfrak{p}_{\mathfrak{p}} \cong \mathrm{Frac}(A/\mathfrak{p}).$$

<u>Explanation</u>. Since $A_{\mathfrak{p}}$ is flat over A, the inclusion $\mathfrak{p} \subset A$ induces an injective $A_{\mathfrak{p}}$-mod. hom. $\mathfrak{p}_{\mathfrak{p}} \hookrightarrow A_{\mathfrak{p}}$. So we can identify $\mathfrak{p}_{\mathfrak{p}}$ with an ideal of $A_{\mathfrak{p}}$.

By "A-ring" we mean a ring B with a ring hom. $A \to B$. What item 2 says is that there is a unique $\cong$ making this diagram commutative

$$\begin{tikzcd}
A \arrow[r, "\mathrm{can}"] \arrow[d, "\mathrm{can}"'] & A_{\mathfrak{p}} \arrow[d, "\mathrm{can}"] \\
A/\mathfrak{p} \arrow[dr, "\mathrm{can}"'] & A_{\mathfrak{p}}/\mathfrak{p}_{\mathfrak{p}} \\
& \mathrm{Frac}(A/\mathfrak{p}) \arrow[u, dotted, "\cong"']
\end{tikzcd}$$

The field $A_{\mathfrak{p}}/\mathfrak{p}_{\mathfrak{p}} \cong \mathrm{Frac}(A/\mathfrak{p})$ is usually denoted by $\underline{\underline{k}}(\mathfrak{p})$.

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proof. We begin by showing that there is a hom. g making the diag. comm.

$$\begin{tikzcd}
A \arrow[rr, "\text{can}", "\lambda"'] \arrow[d, "\text{can}"', "\pi"] & & A_{\mathbb{P}} \arrow[d, "\pi'"', "\text{can}"] \\
A/\mathbb{P} \arrow[rr, dashed, "g"] \arrow[dr, "\lambda'", "\text{can}"'] & & A_{\mathbb{P}}/\mathbb{P}_{\mathbb{P}} \\
& \mathrm{Frac}(A/\mathbb{P}) \arrow[ur, dotted, "h"] &
\end{tikzcd}$$

Take $a \in \mathbb{P}$. Then $\lambda(a) = \frac{a}{1} \in \mathbb{P}_{\mathbb{P}}$, so $(\pi' \circ \lambda)(a) = 0$. Hence $\exists$ g s.t. $g \circ \pi = \pi' \circ \lambda$. Next we show that $\exists$ h s.t. the diag. is commut. Take $b \in A/\mathbb{P}$, $b \neq 0$. So $b = \pi(s)$, $s \notin \mathbb{P}$, $\lambda(s)$ is invert. in $A_{\mathbb{P}}$, and $g(b) = \pi'(\lambda(s))$ is invertible in $A_{\mathbb{P}}/\mathbb{P}_{\mathbb{P}}$. By the univ. prop. of localiz. there is a hom. h s.t. $h \circ \lambda' = g$. The formula is

$$(\mathbb{P}) \qquad \boxed{h\left(\frac{\pi(a)}{\pi(s)}\right) = \pi'\left(\frac{a}{s}\right), \quad a \in A,\ s \in A - \mathbb{P}.}$$

This proves that h is unique.

Since $\mathrm{Frac}(A/\mathbb{P})$ is a field we see that h is inject. And since any elt. of $A_{\mathbb{P}}/\mathbb{P}_{\mathbb{P}}$ can be written as $\pi'(a/s)$, formula $(\mathbb{P})$ shows that h is surject.

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The fact that h is an isom. tells us that $A_{\mathbb{P}}/\mathbb{P}_{\mathbb{P}}$ is a field; and thus $\mathbb{P}_{\mathbb{P}}$ is maximal.

Next we prove that $A_{\mathbb{P}}/\mathbb{P}_{\mathbb{P}} \neq 0$, i.e. that $\mathbb{P}_{\mathbb{P}} \subsetneq A_{\mathbb{P}}$. Otherwise we would have $\frac{1}{1} \in \mathbb{P}_{\mathbb{P}}$, so $\frac{1}{1} = \frac{a}{s}$ for some $a \in \mathbb{P}$ and $s \in A - \mathbb{P}$. Then $\exists$ $t \in A - \mathbb{P}$ s.t. $t \cdot (s - a) = 0$. This implies $ts = ta \in \mathbb{P}$. But $\mathbb{P}$ is a prime ideal, and we have a contradiction.