<u>Corollary</u> (Pre-Nakayama).

Let M be a finitely generated A-module, and let $\mathfrak{a}$ be an ideal in A. If $\mathfrak{a}\cdot M = M$, then there exists an element $a \in \mathfrak{a}$ s.t.

$$(1+a)\cdot M = 0.$$

<u>Proof.</u> We take $\varphi := 1\!\!1_M \in \mathrm{End}_A(M)$. By assumption there are $a_{ij} \in \mathfrak{a}$ s.t.

$$\varphi(m_i) = m_i = \sum_j a_{ij}\cdot m_j$$

for all i. The theorem says that the endomorphism $\boxed{P_{\underline{a}}(1\!\!1_M) = 0}$, where $\underline{a} := [a_{ij}]$. But we know that

$$\boxed{P_{\underline{a}}(1\!\!1_M) = 1\!\!1_M^n + \sum_{i=0}^{n-1} b_i\, 1\!\!1_M^i = \left(1 + \sum_{i=0}^{n-1} b_i\right)\cdot 1\!\!1_M} \quad \text{in } \mathrm{End}_A(M).$$

Taking $a := \sum_{i=0}^{n-1} b_i \in \mathfrak{a}$ we get

$$(1+a)\cdot m = P_{\underline{a}}(1\!\!1_M)(m) = 0$$

for all $m \in M$.

$\square$

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<u>Cor.</u> (Nakayama's Lemma)
Let A be a local ring, with maximal ideal $\mathfrak{m}$.
Let M be a fin. gen. A-mod.
If $\mathfrak{m}\cdot M = M$, then $M = 0$.

~~Pf~~. By the prev. cor, there is $a \in \mathfrak{m}$ s.t. $(1+a)\cdot M = 0$. But since A is local, we know that $1+a$ is invertible in it, so $(1+a)\cdot M = M$. Δ

<u>Cor.</u> Let $(A, \mathfrak{m}, K)$ be a local ring, and let M be a fin. gen. A-module. Let $(m_1, \dots, m_r)$ be a seq. of elts in M, with images $(\bar{m}_1, \dots, \bar{m}_r)$ in the K-module $\bar{M} := M/\mathfrak{m}M$. If the seq. $(\bar{m}_1, \dots, \bar{m}_r)$ generates $\bar{M}$, then the seq. $(m_1, \dots, m_r)$ gen's M.

<u>Pf.</u> There is an ex. seq.

$$A^{\oplus r} \xrightarrow{\varphi} M \xrightarrow{\psi} N \to 0,$$

where $\varphi(a_1, \dots, a_r) := \sum_i a_i m_i$, and

$N := \mathrm{Coker}(\varphi)$. We want to prove that $N = 0$. Applying $K \otimes_A -$ to this seq, we get an ex. seq.

$$K^{\oplus r} \xrightarrow{\bar{\varphi}} \bar{M} \xrightarrow{\bar{\psi}} K \otimes_A N \to 0,$$

and $\bar{\varphi}(\bar{a}_1, \dots, \bar{a}_r) := \sum_i \bar{a}_i \bar{m}_i$. We know $\bar{\varphi}$ is surjective, and hence $K \otimes_A N = 0$. But N is a fin. gen. A-mod, so by prev. cor. $N = 0$. Δ

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