Integral Homomorphisms

Recall that a polynomial
$$p(t) = \sum_{i=0}^{n} a_i t^i \in A[t]$$
is <u>monic</u> if $n \geq 1$ and $a_n = 1$.

<u>Def</u>. Let $A \to B$ be a ring homom. An element $b \in B$ is said to be <u>integral</u> over B if $p(b) = 0$ for some <u>monic</u> polynomial $p(t) \in A[t]$.

<u>Def.</u>

Let $A \xrightarrow{f} B$ be a ring hom, so B is an A-ring. We say that f is a finite type ring hom, and that B is a finitely generated A-ring, if there is a surjective A-ring hom.
$$\tilde{f} : A[t_1, \dots, t_n] \to B$$
for some n. In this case, letting $b_i := \tilde{f}(t_i) \in B$, we write
$$B = A[b_1, \dots, b_n].$$

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Theorem. TFAE for an A-ring B:

(i) B is finite over A (i.e. B is a fin. gen. A-module).

(ii) B is a finitely generated A-ring, and every $b \in B$ is integral over A.

(iii) $B = A[b_1, \dots, b_n]$, and each b_i is integral over A.

Pf. (i)$\Rightarrow$(ii): Say $B = \sum_{i=1}^{n} A \cdot b_i$. Then $B = A[b_1, \dots, b_n]$.
Take any $b \in B$.
There exist $a_{ij} \in A$ s.t. $b \cdot b_i = \sum_j a_{ij} \cdot b_j$.
By CH, the monic poly.
$p_{\underline{a}}(t)$ satisfies $p_{\underline{a}}(b) = 0$.

(ii)$\Rightarrow$(iii): Trivial.

(iii)$\Rightarrow$(i): Say b_i satisfies a monic poly.
$p_i(t) = t^{m_i} + q_i(t)$, $\deg(q_i(t)) < m_i$. Then

$$b_i^{m_i} = -q_i(b_i) \in \sum_{j=0}^{m_i - 1} b_i^{j} \cdot A.$$

Therefore

$$B = A[b_1, \dots, b_n] = \sum_{j_1=0}^{m_1-1} \cdots \sum_{j_n=0}^{m_n-1} A \cdot b_1^{j_1} \cdots b_n^{j_n}.$$

$\square$

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Exercise.

1) Given a ring hom. $A \to B$, define $C := \{ b \in B \mid b \text{ integral over } A \}$. Prove that C is a subring of B. (This is the integral closure of A in B.) (Hint: theorem.)

2) Let $A := \mathbb{C}[t]$ and $K := \mathbb{C}(t) = \mathrm{Frac}(A)$. Show that $p(t) \in K$ is integral over A iff it belongs to A. (Thus $\mathbb{C}[t]$ is integrally closed in $\mathbb{C}(t)$; so it is a normal ring.) (Hint: write $p(t) = \prod (t-\lambda_i)^{m_i}$, $\lambda_i \in \mathbb{C}$ distinct, $m_i \in \mathbb{Z}$, and using pole calculation show that $m_i \geq 0$.)

3) Prove that $\mathbb{Z}$ is normal (i.e. it is integ. closed in $\mathbb{Q}$.) (Hint: #2.)

(9.3)

Noetherian Rings

Def. An A-module M is <u>noetherian</u> if every submodule of M is finitely generated.

The ring A is <u>noetherian</u> if the A-module A is noetherian; i.e. if every ideal $\mathfrak{a} \subset A$ is f. gen.

Prop. An A-module M is noetherian iff it satisfies the <u>ascending chain condition</u>: if $\{N_i\}_{i \in \mathbb{N}}$ is an ascending chain of submodules of M, then $N_i = N_{i+1}$ for $i \gg 0$.

Pf. <u>Exercise</u>.

Prop. Let $0 \to N_0 \xrightarrow{\varphi_1} N_1 \xrightarrow{\varphi_2} N_2 \to 0$ be an exact seq. of A-modules. TFAE:

(i) N_1 is noetherian.

(ii) N_0 and N_2 are noetherian.

Pf. (i) $\Rightarrow$ (ii): Say $M_0 \subset N_0$ and $M_2 \subset N_2$ are submods. Then $M_0 \cong \varphi_1(M_0) \subset N_1$, so M_0 is f.gen. Since $\varphi_2^{-1}(M_2) \subset N_1$ is fin. gen., and $\varphi_2^{-1}(M_2) \to M_2$ is surj, we see that M_2 is f.g.

(ii) $\Rightarrow$ (i): Let $M_1 \subset N_2$. Choose finitely many elts. $\tilde{x}_1, \dots, \tilde{x}_r \in M_0 := \varphi_1^{-1}(M_1)$ that gen. it, and let $x_i := \varphi_1(x_i)$. Choose fin. many $\tilde{y}_1, \ldots, \tilde{y}_s \in M_2 := \varphi_2(M_1)$ that gen. it, and choose $y_i \in \varphi_2^{-1}(\tilde{y}_i)$. Then M_1 is gen. by $x_1, \ldots, x_r, y_1, \ldots, y_s$. $\square$

<u>Theorem</u>. If A is a noetherian ring, then any finitely generated A-module is noetherian.

<u>proof</u>. The free module A is noeth. by def. Now consider A^n for $n \geq 2$. There is an ex. seq.

$$0 \to A^{n-1} \to A^n \to A \to 0.$$

By induction and the prop. we see that A^n is noeth.

Finally let M be any f.g. module. There is an ex. seq.

$$0 \to N \to A^n \to M \to 0$$

for some n. Again using the prop, we conclude that M is noeth. $\square$

(95) <u>Theorem</u>. Let A be a noetherian ring, and let $S \subset A$ be a mult. closed subset. Then the ring A_S is noetherian.

We need a lemma first.

<u>Lemma.</u> Let M be an A-module. Consider the hom. $\lambda: M \to M_S$, $\lambda(m) = \frac{m}{1}$. Let $N \subset M_S$ be an A_S-submodule, and define $\tilde{N} := \lambda^{-1}(N)$, which is an A-submodule of M. Then

$$N = A_S \cdot \lambda(\tilde{N}).$$

<u>Pf.</u> Take $\frac{m}{s} \in N$. Then $\frac{s}{1} \cdot \frac{m}{s} = \frac{m}{1} = \lambda(m)$, and $\frac{s}{1} \cdot \frac{m}{s} \in N$. So $m \in \tilde{N}$. But $\frac{m}{s} = \frac{1}{s} \cdot \frac{m}{1} = \frac{1}{s} \cdot \lambda(m)$. $\square$

<u>Pf of thm.</u> Let $\mathfrak{b} \subset A_S$ be an ideal. Consider

$\mathfrak{a} := \lambda^{-1}(\mathfrak{b})$, which is an ideal of A.

By assumption $\mathfrak{a}$ is fin. gen., and by lemma $\mathfrak{b} = A_S \cdot \lambda(\mathfrak{a})$. So $\mathfrak{b}$ is fin. gen.

$\square$

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