Exam in „Fundamentals of Analysis", 22.02.2019

Question 1 Let $d_1, d_2, \ldots, d_4$ all are metrics defined on the same set $X \neq \emptyset$.

Define $d(x,y) = \max\{d_i(x,y) : i = 1, \ldots, 4\}$ for all $x, y \in X$.

(K) Prove that d defines a metric on X.

(a) Prove or disprove the following claim: if all d_i are complete metrics, then d is also a complete metric.

Question 2 Let $X \subset \mathbb{R}^n$ be any subset of an Euclidean space $\mathbb{R}^n$ with a standard metric. Assume that X satisfies to the following property: If $f(x) : X \to \mathbb{R}$ is any continuous function, then its image $f(X) = \{f(x) : x \in X\}$ is a closed subset of $\mathbb{R}$. Prove that X is a compact set.

Question 3 Let $E \subset \mathbb{R}$ be a Lebesgue measurable set. Prove that at least one of the sets E or $\mathbb{R} \setminus E$ contains a compact subset K such that $|K| > \aleph_0$.

<u>Question 4</u> Let $E \subset [a,b]$ be a Lebesgue measurable set. Denote by

$E_x = E \cap [a,x]$ for every $x \in [a,b]$.

Define function $f(x) = \mu(E_x)$ for every $x \in [a,b]$, where μ is the Lebesgue measure.

Prove that there exists $c \in [a,b]$ such that

$$f(c) = \frac{1}{2}\mu(E).$$