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Fundamentals of Analysis for EE

Homework 1

Question 1. Let A be any infinite set. Prove that for every countable set B the following equality holds: $|A| = |A \cup B|$.

Question 2. Find an explicit formula for a bijective mapping $f : [0,1] \to R$. Is it possible to find such a mapping continuous?

Question 3. For every set $A \subset R$ and every number $r \in R$ define the set

$$A + r = \{x + r : x \in A\} \subset R.$$

Assume that A is a countable set. Prove that there exists a number $r \in R$ such that

$$A \cap (A + r) = \varnothing.$$

Question 4. Let $f : R \to R$ be a monotone real function. Denote by A the set of all points of discontinuity of f. Prove that A is a finite or a countable set.
Hint: Which kind of discontinuity may have any monotone function?

Question 5. Let (X, d) be a metric space. Prove that for any 3 points $x, y, z \in X$ the following inequality holds: $d(x, y) \geq |d(x, z) - d(z, y)|$.

Question 6. Let E be a finite set. Denote the power set of E by X: $X = \{A \subseteq E\}$.
For every two subsets $A \subseteq E$, $B \subseteq E$ define the number $d(A, B) = |A \triangle B|$.
(Here $A \triangle B = (A \setminus B) \cup (B \setminus A)$ denotes the symmetric difference).
Prove that (X, d) is a metric space.

Question 7. Let (X, d) be a metric space. Find all values of constant numbers C such that
(a) $C \cdot d$ defines a metric on the set X.
(b) $C + d$ defines a metric on the set X.

Question 8. Let d_1 and d_2 be two metrics defined on a set X. Find which formulas below necessarily define a metric on the same set X:

(a) $d_1 + d_2$.

(b) $\max\{d_1, d_2\}$.

(c) $\min\{d_1, d_2\}$.

Question 9. Let (X,d) be a metric space. Suppose that a sequence $\{x_n\}_{n=1}^{\infty} \subset X$ converges to $x \in X$ according to metric d. Prove that $\lim_{n\to\infty} d(x_n, y) = d(x, y)$ for each $y \in X$.

Question 10. Definition: Let (X,d) be a metric space. The following set

$$\overline{B}(P_0, r) = \{P \in X : d(P, P_0) \le r\}$$

is called a closed ball with the radius $r > 0$ in the metric space (X,d).

Assume that d_1 and d_2 are two metrics in the space R^3 defined as follows: $P_1 = P_1(x_1, y_1, z_1)$; $P_2 = P_2(x_2, y_2, z_2)$ are any two points and

$$d_1(P_1, P_2) = |z_2 - z_1| + |y_2 - y_1| + |x_2 - x_1|$$

$$d_2(P_1, P_2) = \max\{|z_2 - z_1|, |y_2 - y_1|, |x_2 - x_1|\}$$

Describe geometric form of the closed balls $\overline{B}(P_0, r)$, where $P_0 = (0,0,0), r = 1$ in the metric spaces (R^3, d_1) and (R^3, d_2).