* <u>Bounded</u> set $E$ is Lebesgue measurable if:

  $\forall \varepsilon > 0$ $\exists$ compact $K \subset E$ such that $m^*(E) \le m^*(K) + \varepsilon$

* If $E$ is any subset of $\mathbb{R}$, then $E$ is Lebesgue measurable if $E \cap (a,b)$ is measurable for any interval $(a,b)$.

* <u>Def:</u> $A, B \subset \mathbb{R}$ are seperated if $\operatorname{dist}(A,B) > 0$.

<u>Lemma:</u> If two sets $A, B$ are seperated then $\exists$ open sets $U_A, U_B$ such $A \subset U_A$; $B \subset U_B$ and $U_A \cap U_B = \phi$.

<u>Theorem:</u> If $A, B$ are seperated sets, then $\mu(A \uplus B) = \mu(A) + \mu(B)$.
($\mu = m^*$ is the Lebesgue measure).

<u>Proof:</u>

זה נכון לכל אוסף:

Always we have that $m^*(A \cup B) \overset{1}{\le} m^*(A) + m^*(B)$.

We need to show the opposite inequality:

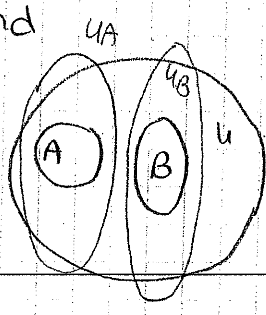
$m^*(A) + m^*(B) \le m^*(A \uplus B)$.

Fix $\varepsilon > 0$. Try to show $m^*(A) + m^*(B) \le m^*(A \uplus B) + \varepsilon$.

$m^*(A \uplus B) = \{\text{by definition}\} = \inf\{m(U) : A \uplus B \subset U, U \text{ is open}\}$.

There is open set $U$ such that $A \uplus B \subset U$ and

(*) $m(U) \le m^*(A \uplus B) + \varepsilon$.




$U_A \cap U$ - open set, $U_B \cap U$ - open set

$A \subset U_A \cap U \subset U$

$B \subset U_B \cap U \subset U$

Without loss of generality we can assume that $A \subset U_A \subset U$, $B \subset U_B \subset U$.

$A \uplus B \subset U_A \uplus U_B \subset U$

$$\underbrace{m^*(U_A \uplus U_B)}_{\substack{\parallel \\ \text{already proved} \\ \parallel \\ m^*(U_A) + m^*(U_B)}} \le m^*(U) \underset{\substack{(*) \\ m(U) = m^*(U)}}{\le} m^*(A \uplus B) + \varepsilon$$

So, $m^*(U_A) + m^*(U_B) \le m^*(A \uplus B) + \varepsilon$, for any $\varepsilon > 0$.

Take inf over $\varepsilon \Rightarrow m^*(U_A) + m^*(U_B) \le m^*(A \uplus B)$

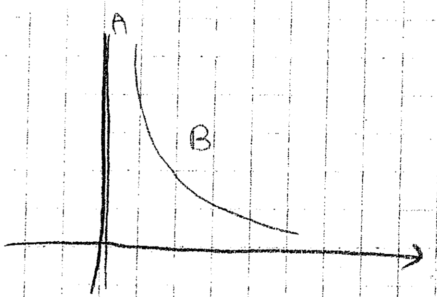
$m^*(A) \le m^*(U_A)$, $m^*(B) \le m^*(U_B)$ by monotonicity of $m^*$. (המשך)

$\Rightarrow m^*(A) + m^*(B) \underset{②}{\le} m^*(U_A) + m^*(U_B) \le m^*(A \uplus B)$

①+② $\Rightarrow$ equality.

Lemma: If A and B are disjoint compact sets then they are separated, i.e. $dis(A,B) > 0$.

# Example:




A, B are closed, not compact

$dist(A,B) = 0$

Proof:

1) Define $f: A \times B \to \mathbb{R}$, $f(x,y) = |x-y|$

This function is continuous, defined on a compact set $A \times B \subset \mathbb{R}^2$.

$f(x,y) > 0$ for any $(x,y)$. (כי A,B הם זרות)

By Weirestrass Theorem, there is minimal value –

$\exists \min\limits_{(x,y) \in A \times B} f(x,y) = r > 0$, So, $dist(A,B) = r > 0$

Corollary: Let $K_1, K_2, \dots, K_n$ a finite family of compact disjoint sets. Then $\mu(\biguplus_{i=1}^{n} K_i) = \sum_{i=1}^{n} \mu(K_i)$

Proof: induction over $n$, and by previous Lemma.

(איחוד סופי של קב' קומפקטיות הוא גם קומפקטי).

Theorem: Let $\{E_i\}_{i=1}^{\infty}$ be a finite of countable family of disjoint bounded measurable sets. Then:

1) $\mu(\biguplus_{i=1}^{\infty} E_i) = \sum_{i=1}^{\infty} \mu(E_i)$

This means that $\mu$ is $\sigma$-additive for bounded measurable set.

(המשך)

2) Denote $E=\biguplus_{i=1}^{\infty}E_i$. If $E$ is bounded then $E$ is measurable.

למה ש-$E$ לא תהיה חסומה?

<u>Proof:</u>

(a)

1) Always $\boxed{m^*\left(\bigcup_{i=1}^{\infty}E_i\right)\le\sum_{i=1}^{\infty}m^*(E_i)}$

זה נכון לכל סט ולאו רק לסט שהוא מדיד לבג

((נצטרך להראות את האי שוויון ההפוך ואז יהיה שוויון))

Fix any $\varepsilon>0$ and we will show that $\sum_{i=1}^{\infty}m^*(E_i)\le m^*\left(\biguplus_{i=1}^{\infty}E_i\right)+\varepsilon$.

$\forall i\in\mathbb{N}$ $\exists$compact $K_i\subset E_i$ such that $m^*(E_i)\le m^*(K_i)+\frac{\varepsilon}{2^i}$ (כי $E_i$ מדיד)

$E_i\cap E_j=\phi \Rightarrow K_i\cap K_j=\phi$ for every $i\neq j$.

Fix any finite $n$.

$m^*\left(\biguplus_{i=1}^{n}K_i\right)=\sum_{i=1}^{n}m^*(K_i)$ by our last corollary.

$\forall i=1,\dots,n$, $m^*(E_i)\le m^*(K_i)+\frac{\varepsilon}{2^i}$.

Take the sum $\sum_{i=1}^{n}m^*(E_i)\le\underbrace{\sum_{i=1}^{n}m^*(K_i)}_{=\,m^*\left(\biguplus_{i=1}^{n}K_i\right)}+\varepsilon\sum_{i=1}^{n}\frac{1}{2^i}$

Let $n\to\infty$: $\sum_{i=1}^{\infty}m^*(E_i)\le m^*\left(\biguplus_{i=1}^{\infty}K_i\right)+\varepsilon\overset{(*)}{\le}m^*\left(\biguplus_{i=1}^{\infty}E_i\right)+\varepsilon$

$\left((*)\ \biguplus_{i=1}^{\infty}K_i\subset\biguplus_{i=1}^{\infty}E_i\right)$

(b)

Let $\varepsilon\to0$: We obtained inequality $\boxed{\sum_{i=1}^{\infty}m^*(E_i)\le m^*\left(\biguplus_{i=1}^{\infty}E_i\right)}$

זה נכון רק ל-$E_i$ מדידים הכל!

~~$(a)+(b)\Longrightarrow \mu\left(\biguplus_{i=1}^{\infty}E_i\right)=\sum_{i=1}^{\infty}\mu(E_i)$~~

2) Fix any $\varepsilon>0$.

We need to find a compact $K\subset E$ such that $m^*(E)\le m^*(K)+\varepsilon$

By the first item (which we already proved)

$m^*(E)=\sum_{i=1}^{\infty}m^*(E_i)\underset{?}{\le}\infty$

$\exists n$ such that $\sum_{i=n+1}^{\infty}m^*(E_i)<\frac{\varepsilon}{2}$.

המשך←

for every $i=1,\dots,n$ $\exists K_i \subset E_i$ (compact) such that
$m^*(E_i) \le m^*(K_i) + \frac{\varepsilon}{2n}$

Define $K = \bigcup_{i=1}^{n} K_i$.

$K$ is compact because it is a finite union of compact subsets.

$K \subset \bigcup_{i=1}^{n} E_i \subset E$.

$$m^*(E) = \sum_{i=1}^{\infty} m^*(E_i) = \sum_{i=1}^{n} m^*(E_i) + \underbrace{\sum_{i=n+1}^{\infty} m^*(E_i)}_{<\varepsilon/2}$$

$m^*(E_i) \le m^*(K_i) + \frac{\varepsilon}{2n}$, $i=1,\dots,n$

$$\sum_{i=1}^{n} m^*(E_i) \le \sum_{i=1}^{n} m^*(K_i) + n\cdot\frac{\varepsilon}{2n} = \sum_{i=1}^{n} m^*(K_i) + \frac{\varepsilon}{2}$$

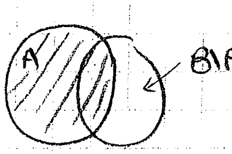
So, $m^*(E) \le m^*(K) + \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = m^*(K) + \varepsilon$

This is exactly what we need.

Theorem: Let $A, B$ are two bounded Lebesgue measurable sets. Then $A\cup B$, $A\cap B$, $A\setminus B$ are Lebesgue measurable sets.

Proof:

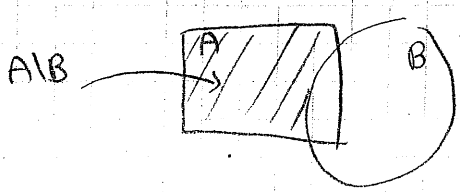
1. $A\cap B = A\setminus(A\setminus B)$
2. $A\cup B = A \uplus (B\setminus A)$

1. $x\in A\setminus(A\setminus B) \Rightarrow x\in A,\ x\notin A\setminus B \Rightarrow x\in B \Rightarrow x\in A\cap B$
2. 

, $A\cap(B\setminus A)=\emptyset \Rightarrow A\uplus(B\setminus A) = A\cup B$

---

Its enough to show that if $A,B$ are two bounded Lebesgue measurable sets then $A\setminus B$ is also measurable.




Take $\varepsilon>0$. We need to find a compact set $K\subset A\setminus B$ such that $m^*(A\setminus B) \le m^*(K) + \varepsilon$.

We can approximate $A$ with some compact, we can approximate $B$ with some compact. זהירות - חיסור של קומפקטים אינו קומפקטי!

$K \setminus$ open is compact!

$\exists K_1 \subset A$ compact, open $U_1 \supset A$ such that

$m^*(U_1) - \frac{\varepsilon}{4} \le m^*(A) \le m^*(K_1) + \frac{\varepsilon}{4}$

$\exists K_2$ compact, open $U_2 \supset B$ such that

$m^*(U_2) - \frac{\varepsilon}{4} \le m^*(B) \le m^*(K_2) + \frac{\varepsilon}{4}$

Define $K = K_1 \setminus U_2$ compact $\subset A$, $U = U_1 \setminus K_2$ - open

$\underbrace{K = K_1 \setminus U_2}_{\text{compact set}} \subset A \setminus B \subset \underbrace{U_1 \setminus K_2 = U}_{\text{open set}}$

$m^*(U \setminus K) \le 2\varepsilon$

$m(U_A \setminus K_B) - (m(K_A \setminus U_B)$

$m(U_A \setminus K_A) - m(U_B \setminus K_B) \leq \varepsilon$

$m(U_A) - m(K_A) + m(U_B) - m(K_B) \leq \varepsilon$

$m(K_A \setminus U_B) - m(U_A \setminus K_B) \leq \varepsilon$ $\underline{: \Gamma'' 3}$

$m(K_A) - m(U_B) - m(U_A) + m(K_B) \leq \varepsilon$

**Theorem:** If $A, B$ are two bounded measurable sets, then $A \cup B$, $A \cap B$, and $A \setminus B$ are measurable:

**Proof:**

$A \cup B = B \uplus (A \setminus B)$

$A \cap B = A \setminus (A \setminus B)$

So, it is enough to show that $A \setminus B$ is measurable.

Fix any $\varepsilon > 0$. $\exists \underset{\text{compact}}{K_A} \subset A \subset \underset{\text{open}}{U_A}$

אולי A-ש חסום?

$\Rightarrow m^*(U_A) - \varepsilon/4 \leq m^*(A) \leq m^*(K_A) + \varepsilon/4$

$\exists K_B \subset B \subset U_B$, $m^*(U_B) - \varepsilon/4 \leq m^*(B) \leq m^*(K_B) + \varepsilon/4$

$m^*(U_A \setminus K_A) \leq \frac{\varepsilon}{2}$, $m^*(U_B \setminus K_B) \leq \frac{\varepsilon}{2}$

We define $U = U_A \setminus K_B$ open, $K = K_A \setminus U_B$ compact.

$K = K_A \setminus U_B \subset A \setminus B \subset U_A \setminus K_B = U$.

We conclude that $m^*(U \setminus K) \leq \varepsilon$.

This means that the set $A \setminus B$ is measurable.

**Theorem:** The Triple $(\mathbb{R}, \mathcal{L}(\mathbb{R}), \mu)$, where $\mu = m^*$, is a measurable space: $\mathcal{L}(\mathbb{R})$ is a $\sigma$-algebra and $\mu$ is an $\sigma$-additive measure defined on $\mathcal{L}(\mathbb{R})$.

**Proof:**

כבר לא חסום!

We proved that if $\{E_i\}_{i=1}^{\infty}$ is a family of disjoint bounded measurable sets then $\mu(\biguplus_{i=1}^{\infty} E_i) = \sum_{i=1}^{\infty} \mu(E_i)$ and also we proved that if $E = \biguplus_{i=1}^{\infty} E_i$ is bounded (כבר לא חסום), then $E$ is measurable. $(\mu = m^*)$

By our definition $A \subset \mathbb{R}$ is measurable if $A \cap (a,b)$ is a bounded measurable set for any interval $(a,b)$.

$$\Big(\biguplus_{i=1}^{\infty} E_i\Big) \cap (a,b) = \biguplus_{i=1}^{\infty} \underbrace{(E_i \cap (a,b))}_{\text{bounded. measurable set}}.$$

כאשר $E_i$ לא חסום
המכפלה

כי $E_i$ מדידה ו-$(a,b)$ מדידה
וחיתוך של שני קב'
מדידות הם מדידה

So, $\mu\left(\biguplus_{i=1} E_i \cap (a,b)\right) = \sum_{i=1} \mu(E_i \cap (a,b))$.

Recall: $\mathcal{L}$ is $\sigma$-algebra if:

$\phi \in \mathcal{L}$ ; $A, B \in \mathcal{L} \Rightarrow A \setminus B \in \mathcal{L}$ ; $\rightarrow$ את 2 התכונות האלו הוכחנו

If $\{A_i\}_{i=1}^{\infty} \subset \mathcal{L} \Rightarrow \bigcup_{i=1}^{\infty} A_i \in \mathcal{L} \rightarrow$ צריך להוכיח

$\bigcup_{i=1}^{\infty} A_i = A_1 \uplus (A_2 \setminus A_1) \uplus A_3 \setminus (A_1 \cup A_2) \ldots \uplus (A_n \setminus (A_1 \cup A_2 \ldots \cup A_{n-1})) \uplus \ldots$

this is a disjoint countable union of measurable sets.

So, $\bigcup_{i=1}^{\infty} A_i$ is a measurable set. (ולכן $\bigcup_{i=1}^{\infty} A_i$ שייך ל-$\mathcal{L}(\mathbb{R})$)

* Note: $\mathcal{B}(\mathbb{R}) \subset \mathcal{L}(\mathbb{R}) \subset \mathcal{P}(\mathbb{R})$

| $\mathcal{B}(\mathbb{R})$ | $\mathcal{L}(\mathbb{R})$ | $\mathcal{P}(\mathbb{R})$ |
|---|---|---|
| $\sigma$-algebra of Borel set | $\sigma$-algebra Lebesgue measurable set | $\sigma$-algebra Of all subsets of $\mathbb{R}$ |

למה σ בורל מוכל ב-$\mathcal{L}(\mathbb{R})$? כי $\mathcal{B}(\mathbb{R})$ הוא ה-σ אלגברה המינימלית המכילה את כל הקבוצות הפתוחות.

Claim 1: $\mathcal{B}(\mathbb{R}) \neq \mathcal{L}(\mathbb{R})$

Claim 2: $\mathcal{L}(\mathbb{R}) \neq \mathcal{P}(\mathbb{R})$

$\mathcal{B}(\mathbb{R}) \neq \mathcal{L}(\mathbb{R}) \Longleftrightarrow \exists A \subset \mathbb{R}$ Lebesgue measurable set which is not Borel.

$\mathcal{L}(\mathbb{R}) \neq \mathcal{P}(\mathbb{R}) \Longleftrightarrow \exists A \subset \mathbb{R}$ which is not measurable.

Proof of Claim 1:

צריך להוכיח $|\mathcal{B}(\mathbb{R})| = \mathfrak{c}$ ; $|\mathcal{L}(\mathbb{R})| = 2^{\mathfrak{c}}$

We know by Cantor theorem that $\mathfrak{c} < 2^{\mathfrak{c}}$.

If $K$ is the Cantor set: $|K| = \mathfrak{c}$ ; $\mu(K) = 0 \Rightarrow$

$\Rightarrow$ any subset $A \subset K$ is also measurable and $\mu(A) = 0$.

$|\mathcal{P}(K)| = 2^{\mathfrak{c}}$

$\Rightarrow \mathcal{P}(K) \subset \mathcal{L}(\mathbb{R}) \subset \mathcal{P}(\mathbb{R})$

$$2^{\mathfrak{c}} \leq |\mathcal{L}(\mathbb{R})| \leq |\mathcal{P}(\mathbb{R})| = 2^{\mathfrak{c}} \Rightarrow \boxed{|\mathcal{L}(\mathbb{R})| = 2^{\mathfrak{c}}}$$

המשך ⟵

$|\mathcal{B}(\mathbb{R})| = ?$

Let $\mathcal{U}$ be the family of all open sets in $\mathbb{R}$. $|\mathcal{U}| = \mathfrak{c}$.

open set $= \biguplus$ open intervals.

open interval $(a,b)$, $a \in \mathbb{R}$, $|\mathbb{R}| = \mathfrak{c}$

$(a,b) \longleftrightarrow \mathbb{R}\times\mathbb{R}$ , $|\mathbb{R}\times\mathbb{R}| = \mathfrak{c}$

$\Rightarrow \mathcal{U} \longleftrightarrow \underbrace{\mathfrak{c}\times\mathfrak{c}\times\ldots\times\mathfrak{c}\times\ldots}_{\aleph_0 \text{ times}}$

$F$ family of all closed sets. $|F| = \mathfrak{c}$ (יש קשר חד ערכי בין קב' הפתוחות לקב' המשלימות שהן סגורות)

$$\left.\begin{array}{l} G_\delta \longleftrightarrow \underbrace{\mathfrak{c}\times\mathfrak{c}\times\ldots\times\mathfrak{c}\times\ldots}_{\aleph_0 \text{ times}} \Rightarrow |G_\delta| = \mathfrak{c} \\ |F_\sigma| = \mathfrak{c} \end{array}\right\} \Rightarrow \begin{array}{l} |G_{\delta\sigma}| = \underbrace{\mathfrak{c}\times\mathfrak{c}\times\ldots\times\mathfrak{c}\times\ldots}_{\aleph_0 \text{ times}} \\ |F_{\sigma\delta}| = \mathfrak{c} \end{array}$$

לכל היותר $\mathfrak{c}$, ולמה $\aleph_0$?

The union of $\mathfrak{c}$ many families, where each family has the size of $\mathfrak{c}$, is again a family of the size $\mathfrak{c}$.

<u>Proof of claim 2</u>:

<u>Theorem</u>: Vitali. $\exists$ non-measurable sets.

We will use the following property of $\mu$:

$\mu$ is translation-invariant. i.e. $A \subset \mathbb{R}$, $\underset{\text{fixed}}{x} \in \mathbb{R}$, $A+x = \{a+x : a \in A\}$,

$\mu(A) = \mu(A+x)$ for any $x$.

Define on $\mathbb{R}$ the following equivalence relation:

$x \sim y$ if $x-y$ is a rational number, $x-y \in \mathbb{Q}$.

Check this:

1) $x \sim x \Rightarrow x-x = 0 \in \mathbb{Q}$

2) $x \sim y \Rightarrow y \sim x \iff x-y \in \mathbb{Q} \iff y-x \in \mathbb{Q}$

3) $x \sim y, y \sim z \Rightarrow x \sim z$, because $x-z = \underbrace{x-y}_{\in\mathbb{Q}} + \underbrace{y-z}_{\in\mathbb{Q}} \in \mathbb{Q}$

So, $\mathbb{R}$ is a disjoint union of equivalence classes.

For any equivalence class pick one element.

המשך

Denote by $X_0$ obtaining set $\boxed{V = X_0 \cap [0,1].}$

Vitali set is not measurable.

Let us enumerate all rational numbers from the segment $[-1,1]$.

$\{q_1, \dots, q_n, \dots\} = \mathbb{Q} \cap [-1,1]$ (סט בר מנייה)

Denote $V_k = V + q_k$ for any $k \in \mathbb{N}$.

$\forall k \neq n \quad V_k \cap V_n = \emptyset$

איבר מ-$V_1$ איבר מ-$V_2$

If $x \in V_k \cap V_n \Rightarrow x = v_1 + q_k,\ x = v_2 + q_n \Rightarrow v_1 + q_k = v_2 + q_n \Rightarrow$

$\Rightarrow v_1 - v_2 = q_n - q_k \to$ rational

$\Rightarrow v_1, v_2 \in$ to the same equivalence class.

(זה לא יכול לקרות כי הנחנו שלקחנו פרט אחד מכל מחלקת שקילות
וכאן קיבלנו כי יש 2 פריטים ששייכים לאותה מחלקת שקילות)

Assume to the contrary that $V$ is measurable, then

$\mu(V) = m \leq 1$, because $V \subset [0,1]$.

$$[0,1] \subset \bigcup_{k \in \mathbb{N}} V_k \subset [-1,2]$$

$$\Rightarrow 1 \leq \underbrace{\mu\Big(\biguplus_{k \in \mathbb{N}} V_k\Big)}_{= \sum_{k \in \mathbb{N}} \mu(V_k)} \leq 3$$

translation

$\mu(V_k) = m$ for any $k$. (הזזה של $V$ לא משנה את המידה)

If $m = 0 \Rightarrow \sum \mu(V_k) = 0 \ngeq 1 \to$ סתירה!

If $m > 0$, $m \cdot \infty = \infty > 3 \to$ סתירה!

<u>Contradiction!</u>

Idea how to define Lebesgue measure in $\mathbb{R}^n$:

In $\mathbb{R}^1$: Any open $U = \biguplus_{i=1}^{\infty} U_i$, where each $U_i$ is an open interval. → this decomposition is unique

What about $U \subset \mathbb{R}^n$ open set?

We say $\overline{T} = \{(x_1, \dots, x_n) \in \mathbb{R}^n : a_i \le x_i \le b_i\}$ closed n-dimensional box,

$T = \{(x_1, \dots, x_n) \in \mathbb{R}^n : a_i < x_i < b_i\}$ open n-dimensional box.

For any $\Delta$ such that $T \subset \Delta \subset \overline{T}$ we actually define

$m(\Delta) = (b_1 - a_1)(b_2 - a_2) \cdots (b_n - a_n)$

We say $\Delta$ is a cell if $\Delta = \{(x_1, \dots, x_n) \in \mathbb{R}^n : a_i \le x < b_i\}$

$m(\Delta) = \prod_{i=1}^{n} (b_i - a_i)$ is defined.

Theorem: Any open set in $\mathbb{R}^n$ can be represented as a disjoint union of countably many cells.

(This decomposition is not unique, as in $\mathbb{R}^1$).

$\Rightarrow \mu(U) = \sum_{i=1}^{\infty} m(\Delta_i)$, where $U = \biguplus_{i=1}^{\infty} \Delta_i$, each $\Delta_i$ is a cell.

For any $A \subset \mathbb{R}^n$ we define $\mu^*(A) = \inf\{\mu(U) : A \subset U\}$ and $U$ is open.

* If $E \subset \mathbb{R}^n$ is bounded, then $E$ is called Lebesgue measurable set if $\forall \varepsilon > 0$ $\exists$ compact $K \subset E$ such that $\mu^*(E) \le \mu^*(K) + \varepsilon$.
* In general $E \subset \mathbb{R}^n$ is Lebesgue measurable if $\underbrace{E \cap B}_{\text{bounded}}$ is measurable for any ball $B \subset \mathbb{R}^n$.

* ניתן להכליל את הרעיון עבור מטריקות נוספות - ואז ה-cell מוגדר לפי המטריקה.

* כל ההוכחות מהשיעורים הקודמים תקפות גם במקרה זה, מכיוון שבהוכחות לא השתמשנו בכך שהסט שייך ל-$\mathbb{R}^1$.

# Measurable functions

$E \subset \mathbb{R}$ which is a measurable set.

**Definition:** $f: E \to \mathbb{R}$ is called a measurable function if

$\forall a \in \mathbb{R}$ $\underbrace{f^{-1}(a,\infty)}_{\text{is a measurable set}} \subset E$, $f^{-1}(a,\infty) = E(f>a)$, is a measurable set.

$f^{-1}(a,\infty) = E(f>a)$

$f^{-1}(-\infty,a) = E(f<a)$

$f^{-1}(-\infty,a] = E(f\leq a)$

$f^{-1}\{a\} = E(f=a)$

$* \; \underset{\text{מדיד}}{E} \setminus \underset{\text{מדיד}}{E(f>a)} = \underset{\text{ולכן גם זה מדיד}}{E(f\leq a)}$

**equivalent definition:** $f: E \to \mathbb{R}$ is measurable if $\forall a \in \mathbb{R}$ $E(f\leq a)$ is measurable.

**Proposition:** Assume $E = \bigcup_{k=1}^{\infty} E_k$, where each $E_k$ is measurable.

$f: E \to \mathbb{R}$.

If $f|_{E_k}: E_k \to \mathbb{R}$ is measurable for any $k$, then $f: E \to \mathbb{R}$ is measurable.

**Proof:**

Take real $a \in \mathbb{R}$.

$E(f>a) = \bigcup_{k=1}^{\infty} E_k(f>a)$.

By assumption, for every $k$ $E_k(f>a)$ is measurable.

$\Rightarrow \bigcup_{k=1}^{\infty} E_k(f>a)$ is also measurable.

**Definition:** $f: E \to \mathbb{R}$, $g: E \to \mathbb{R}$ two measurable functions.

We say that $f$ and $g$ are equivalent, if $\mu(\{E(f\neq g)\}) = 0$.

# **Example:** $f: \mathbb{R} \to \mathbb{R}$, $\underset{\text{מדידה}}{f(x)} = 0$, $g: \mathbb{R} \to \mathbb{R}$, $\underset{\text{פונק' דיריכלה}}{g(x)} = \begin{cases} 1, & x \in \mathbb{Q} \\ 0, & x \notin \mathbb{Q} \end{cases}$ → לא אינטגרבילית במובן רימן, במובן לבג האינטגרל על הפונק' הזו שווה ל-0

$\Rightarrow f \sim g$

מכיוון שהנק' בהן יש הבדל בין הפונק' הם המספרים הרציונליים, שהם קב' בת מניה ולכן המידה על הרציונליים שווה לאפס.

* Assume that $P$ is some property. We say that $f: E \to \mathbb{R}$ satisfies this property <u>almost everywhere</u>, if $\{x : \text{such that } f(x) \not\in P\}$ has measure zero.

(נניח לדוגמא f הומוגנית כמעט בכל מקום).

<u>Theorem</u>: Assume that $f: E \to \mathbb{R}$ a measurable function. Then:

(a) $f(x) + c$ is measurable function.

(b) $c \cdot f(x)$ — " —

(c) $f^2(x)$ — " —

(d) $|f(x)|$ — " —

(e) $\dfrac{1}{f(x)}$ $(f(x) \neq 0 \ \forall x)$ — " —

<u>Proof</u>:

(a) Take any $r \in \mathbb{R}$.

$E(f(x)+c > r) = E(f(x) > r-c)$ is measurable by assumption.

(c) $E(f^2(x) > r) = \begin{cases} E, & r < 0 \\ E(|f| > \sqrt{r}), & r \geq 0 \end{cases}$

למה? כי אז אפי׳ $f^2(x)$ שלילי
זה תמיד קורה!

צריך להוכיח את d

Now we need to prove that $|f|$ is measurable:

$E(|f| > r) = \begin{cases} E, & r < 0 \\ E(f > r) \cup \underbrace{E(f < -r)}_{\text{נראה שזה מדיד}}, & r \geq 0 \end{cases}$

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$E(f < -r) = E \setminus \underbrace{E(f \geq -r)}_{\underbrace{E(f > -r)}_{\text{מדיד}} \cup \underbrace{E(f = r)}_{\text{הוכחנו למטה שמדיד}}}$ $\Rightarrow$ $E(f < -r)$ מדיד!

$E(f = t) = ?$

נצטרך לייצג בצורה של $E(f > \text{משהו})$
(איחודים, חיתוכים, חיסור)

$\{a\} = \bigcap_{n=1}^{\infty} \left(a - \frac{1}{n}, a\right]$

$\{x : f(x) = a\} = \bigcap_{n=1}^{\infty} \{x : a - \frac{1}{n} < f(x) \leq a\}$ (*)

$\underbrace{\{x : a - \frac{1}{n} < f(x) \leq a\}}_{\text{מדיד כחיסור של 2 קבוצות מדידות}} = \underbrace{\{x : f(x) > a - \frac{1}{n}\}}_{\text{מדיד}} \setminus \underbrace{\{x : f(x) > a\}}_{\text{מדיד}}$

$\Rightarrow$ (*) is measurable as intersection of measurable sets.

Theorem: Any continuous function is measurable.

Proof:

$f(x): E \to \mathbb{R}$. fix any $a \in \mathbb{R}$.

$E(f>a) = \underbrace{f^{-1}(a,\infty)}_{\text{preimage of open is open}} = $ intersection of some $\underbrace{\text{open set}}_{\text{measurable}}$ with $E$ is

(פסוק רציפה?)

again measurable.

Theorem: Assume that $f(x), g(x): E \to \mathbb{R}$ are two measurable functions. Then $f(x)+g(x): E \to \mathbb{R}$ is also measurable.

Proof:

Fix any $a \in \mathbb{R}$.

$$E(f(x)+g(x) > a) = E(\underbrace{f(x)}_{\text{measurable}} > \underbrace{-g(x)+a}_{\text{measurable} + a \Rightarrow \text{measurable}})$$

we need another result.

Claim: Let $\alpha(x), \beta(x): E \to \mathbb{R}$ two measurable functions. $E(\alpha > \beta) = \{x : \alpha(x) > \beta(x)\}$ is measurable.

Proof:

$$\{x : \alpha(x) > \beta(x)\} = \bigcup_{q \in \mathbb{Q}} \overbrace{\left( \underbrace{\{x : \alpha(x) < q\}}_{\text{measurable because } \alpha(x) \text{ is measurable function}} \cap \underbrace{\{x : q < \beta(x)\}}_{\text{measurable}} \right)}^{\text{measurable}}$$

Theorem: Let $f(x), g(x)$ two measurable functions. $f(x) \cdot g(x)$ also is measurable.

Proof:

$$f(x) \cdot g(x) = \underbrace{\frac{1}{4}\underbrace{\left( \underbrace{(\underbrace{f+g}_{\text{מדיד}})^2}_{\text{מדיד}} - \underbrace{(\underbrace{f-g}_{\text{מדיד}})^2}_{\text{מדיד}} \right)}_{\text{מדיד}}}_{\text{מדיד}}$$

## Measurable functions:

$E \subset \mathbb{R}$ is a measurable set.

$f(x): E \to \mathbb{R}$ is called a <u>measurable function</u> if $\forall a \in \mathbb{R}$

$f^{-1}(a, \infty) = E(f > a)$ is measurable.

* $E(f > a) = \{x \in E : f(x) > a\}$.

Instead of such sets we consider:

$E(f \geq a) = \{x \in E : f(x) \geq a\}$

$E(f < a) = \{x \in E : f(x) < a\}$

$E(f \leq a) = \{x \in E : f(x) \leq a\}$

* $E(f > a) = \bigcup_{n=1}^{\infty} E(f \geq a + \frac{1}{n})$

$E(f \geq a) = \bigcap_{n=1}^{\infty} E(f > a - \frac{1}{n})$

$\Rightarrow$ $\boxed{\text{All possible definitions are equivalent.}}$

<u>Claim</u>: If $f(x): E \to \mathbb{R}$ is measurable function, then $|f(x)|$, $f^2(x)$, $\frac{1}{f(x)}$ $(f(x) \neq 0)$ are also measurable functions.

If $f(x), g(x): E \to \mathbb{R}$ are measurable functions, then $f(x) \pm g(x)$, $f(x) \cdot g(x)$, $\frac{f(x)}{g(x)}$ $(g(x) \neq 0)$ are measurable functions.

\# <u>Example</u>: Assume $M \subset E$ a subset.

פונק' אופיינית

$$\chi_M(x) = \begin{cases} 1, & \text{if } x \in M \\ 0, & \text{if } x \notin M \end{cases}, \quad \chi_M : E \to \mathbb{R} \text{ Characteristic function of } M.$$

<u>Claim</u>: $\chi_M(x)$ is a measurable function iff $M$ is a measurable set.

ברור ההוכחה כאן אך ב הכיוון אם E מ"ח אז M מדיד מ"ח

explanation: $E(f > a) = M$ for any $0 < a \leq 1$

$E(f > a) = E$ for any $a \leq 0$

Corollary: Not every function is measurable.

Take any Vitaly set $V$; $V$ is not a measurable set.

Define $\chi_V(x) = \begin{cases} 1, & x\in V \\ 0, & x\notin V \end{cases}$, $\chi_V : \mathbb{R} \to \mathbb{R}$ is not a measurable function.

#Example: $E = \mathbb{Q}$ - rational numbers.

$\chi_{\mathbb{Q}}(x) = \begin{cases} 1, & x\in \mathbb{Q} \\ 0, & x\notin \mathbb{Q} \end{cases}$ is called Dirichlet function.

It is a measurable function (because $\mathbb{Q}$ is measurable).

Theorem: Assume that $\{f_n(x)\}_{n=1}^{\infty}$ is a sequence of measurable functions from $E$ to $\mathbb{R}$, $f_n(x): E \to \mathbb{R}$.

Assume that $\forall x\in E \ \exists \lim\limits_{n\to\infty} f_n(x) = \varphi(x)$ (finite limit)

Then $\varphi(x): E \to \mathbb{R}$ also is a measurable function.

(התכנסות נקודתית וזהו) מהכלל המ"ש

אם הגבול של סדרת פונק' רציפות איננה בהכרח מדידה, אבל היא בהכרח מדידה

Proof:

Fix any $a\in\mathbb{R}$. We need to show that $E(\varphi > a) = \{x\in E : \varphi(x) > a\}$ is a measurable set.

צריך להציג את הסט באמצעות סדרת הפונקציות

For every natural numbers $k,n,m \in \mathbb{N}$ define:

$A_m^{(k)} = E(f_k > a + \frac{1}{m}) \longrightarrow$ is a measurable set

$B_m^{(n)} = \bigcap\limits_{k=n}^{\infty} A_m^{(k)} \longrightarrow$ is a measurable set

Claim: $E(\varphi > a) = \bigcup\limits_{m,n\in\mathbb{N}} B_m^{(n)}$ will be a measurable set as a countable union of measurable sets.

(ניקח כל נק' מאגף שמאל ונראה שהוא שייכת לאגף ימין ולהפך)

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Proof of claim:

Take: $\forall x_0 \in E(\varphi > a) \iff \varphi(x_0) > a$, (כיוון הכרחה)

then $\exists m\in\mathbb{N}$ $\varphi(x_0) > a + \frac{1}{m}$

$\varphi(x_0) = \lim\limits_{k\to\infty} f_k(x_0)$ by assumption, so $\exists n\in\mathbb{N}$ such that

$\forall k \geq n \ \ f_k(x_0) > a + \frac{1}{m} \iff x_0 \in E(f_k > a + \frac{1}{m}) \ \forall k\geq n \iff$

$x_0 \in A_m^{(k)} \ \forall k \geq n \iff x_0 \in \bigcap\limits_{k=n}^{\infty} A_m^{(k)} = B_m^{(n)} \longrightarrow$

$\Longrightarrow x_0 \in \bigcup\limits_{n,m\in\mathbb{N}} B_m^{(n)}$

To the opposite inclusion –

Take $x_0 \in \bigcup_{m,n\in\mathbb{N}} B_m^{(n)} \Rightarrow \exists m,n \in \mathbb{N}$ such that $x_0 \in B_m^{(n)}$

So $x_0 \in \bigcap_{k=n}^{\infty} A_m^{(k)} \Rightarrow \forall k \geq n \; x_0 \in A_m^{(k)}, \Leftrightarrow \forall k \geq n \; f_k(x_0) > a + \frac{1}{m} \Rightarrow$

$\Rightarrow \lim_{k\to\infty} f_k(x_0) \geq a + \frac{1}{m} > a$

$\Downarrow$

$\boxed{\varphi(x_0) > a} \Leftrightarrow x_0 \in E(\varphi > a)$

<u>Structure of any measurable function:</u>

<u>Theorem:</u> (Borel) Assume $f(x):[a,b] \to \mathbb{R}$ is a measurable function.
For any $\varepsilon > 0$, $\sigma > 0$ $\exists$ <u>continuous</u> $g(x):[a,b] \to \mathbb{R}$ such that $\mu(E(|f-g| \geq \sigma)) < \varepsilon$.

<u>Theorem:</u> (Luzin)

Assume that $f(x):[a,b] \to \mathbb{R}$ is a measurable function.
Then for any $\varepsilon > 0$ $\exists g(x):[a,b] \to \mathbb{R}$ <u>continuous</u>, such that $\mu(E(f \neq g)) < \varepsilon$. <u>f is nearly continuous.</u>

* <u>הערה</u>: $g(x)$ תלויה ב- $\varepsilon,\sigma$. אם נשנה את $\varepsilon,\sigma$ אז נשנה גם את $g(x)$.

<u>Definition of the integral according to Lebesgue:</u>

(R) $\int_a^b f(x)dx$ – Riemann

(L) $\int_E f(x)d\mu$ – Lebesgue

<u>Definition</u> (of Riemann):

Start with a partition of the segment $[a,b]$, $T: a = x_0 < x_1 < \ldots < x_n = b$.
For any index $i \in \{1,\ldots,n\}$ Choose any point $\xi_i \in [x_{i-1}, x_i]$ and define an integral $\boxed{I(T,\xi) = \sum_{i=1}^{n} f(\xi_i)\cdot\Delta x_i}$, where $\Delta x_i = x_i - x_{i-1}$.
$\lambda(T) = \max\{\Delta x_1, \Delta x_2, \ldots, \Delta x_n\}$ – a parameter of partition $T$.
(R) $\int_a^b f(x)dx = \lim_{\lambda(T)\to 0} \sum_{i=1}^{n} f(\xi_i)\Delta x_i = I$. In the following sense:
$\forall \varepsilon > 0 \; \exists \delta > 0 \; \forall T, \; \lambda(T) < \delta \Rightarrow \left|\sum_{i=1}^{n} f(\xi_i)\Delta x_i - I\right| < \varepsilon$.

רימן מוגדר
רק לקטעים
ולא לכל
סוגים של
קבוצות

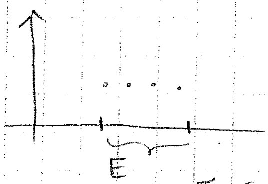
* הערה: אם $x(t) \to 0$ מתאפס מהר ש- $\infty$ [illegible]

<u>Necessary condition:</u> (כדי להיות אינטגרבילי רימן)

$f(x)$ is bounded.

<u>Sufficient conditions:</u>

* אינטגרל לבג הינו הכללה של אינטגרל רימן.



<u>אינטואיציה</u>: $E$ סט מדיד ואינו אינטגרבילי רימן:

נניח פונק' דיריכלה

אז השטח הינו גובה "המלבן" - 1 כפול המידה של הסט $E$.

<u>Definition:</u> Let $E \subset \mathbb{R}$ be a measurable set, $\mu(E) < \infty$, and $f(x): E \to \mathbb{R}$ be any <u>bounded</u> <u>measurable</u> function.

<u>Assume</u> $A \leq f(x) < B \quad \forall x \in [a,b]$

Take any partition of $[A,B]$.

$[A,B) = [y_0, y_1) \cup [y_1, y_2) \cup \ldots \cup [y_{n-1}, y_n)$

$y_0 = A, \quad y_n = B$

$E_i = \{x : y_{i-1} \leq f(x) < y_i\}$ are measurable sets.

$\mu(E_i) < \infty$ for any $i$, $\sum_{i=1}^{n} \mu(E_i) = \mu(E) \quad (\biguplus E_i = E)$

* $E \subset \mathbb{R}$ is a measurable set, $\mu(E) < \infty$.

$f(x): E \to \mathbb{R}$ is a measurable function, bounded.

$A < f(x) < B$ for any $x \in E$.

Let $T$ be a partition of the segment $[A,B]$.

$A = y_0 < y_1 < \dots < y_n = B$.

$\Delta y_k = y_k - y_{k-1}$ length of the segment $[y_{k-1}, y_k]$.

Look at $E_k = \{x \in E : y_{k-1} \leq f(x) < y_k\} = \underbrace{E(y_{k-1} \leq f < y_k)}_{\text{is a measurable set}}$, $k = 1, \dots, n$.

$E_k \cap E_{k'} = \emptyset \ \forall k \neq k'$, $\biguplus_{k=1}^{n} E_k = E. \Rightarrow \sum_{k=1}^{\infty} \mu(E_k) = \mu(E)$.

We define two integral sums:

$\underline{S}(T) = \sum_{k=1}^{n} y_{k-1} \cdot \mu(E_k)$ (סכום תחתון)

$\overline{S}(T) = \sum_{k=1}^{n} y_k \cdot \mu(E_k)$ (סכום עליון)

$\Rightarrow \underline{S}(T) \leq \overline{S}(T)$

$\overline{S}(T) - \underline{S}(T) = \sum_{k=1}^{n} (y_k - y_{k-1}) \mu(E_k) \leq \lambda(T) \cdot \underbrace{\mu(E)}_{\sum_{k=1}^{n} \mu(E_k)}$

Define parameter of $T$: $\lambda(T) = \max\{\Delta y_1, \Delta y_2, \dots, \Delta y_n\}$

<u>Claim:</u> Assume $T' = T \cup \{y^*\}$. $T'$ is a refinment of $T$. ($y^* \in [A,B]$)

$\underline{S}(T) \leq \underline{S}(T') \leq \overline{S}(T') \leq \overline{S}(T)$.

<u>Proof:</u>

$\exists [y_{k-1}, y_k]$ such that $y_{k-1} < y^* < y_k$.

$\underline{S}(T) = \sum_{\substack{i=1 \\ i \neq k}}^{n} y_{i-1} \mu(E_i) + \underbrace{y_{k-1} \mu(E_k)}_{1}$

$\underline{S}(T') = \sum_{\substack{i=1 \\ i \neq k}}^{n} y_{i-1} \mu(E_i) + \underbrace{y_{k-1} \cdot \mu(E_{k'}) + y^* \mu(E_{k''})}_{2}$

$E_{k'} = E(y_{k-1} \leq f < y^*)$ ; $E_{k''} = E(y^* \leq f < y_k)$

$\mu(E_k) = \mu(E_{k'} \uplus E_{k''}) = \mu(E_{k'}) + \mu(E_{k''})$

$y^* - y_{k-1} > 0 \Rightarrow y^* \mu(E_{k''}) - y_{k-1} \mu(E_{k''}) \geq 0$

$(2 - 1 = y^* \mu(E_{k''}) - y_{k-1} \mu(E_{k''}) \geq 0)$

<u>Claim:</u> Let $T_1, T_2$ be two partitions of $[A,B]$, Then $\underline{S}(T_1) \le \overline{S}(T_2)$.

<u>Proof:</u>

Define $T^* = T_1 \cup T_2$. $T^* \prec T_1$ ; $T^* \prec T_2$

העדנה הוא
מקום סיום של נקודות

$\underline{S}(T_1) \le \underline{S}(T^*) \le \overline{S}(T^*) \le \overline{S}(T_2)$ — מטענה קודמת

$\Rightarrow \underline{S}(T_1) \le \overline{S}(T_2)$

* We define: $U(f) = \inf\limits_{\substack{\text{over all} \\ \text{possible partitions} \\ T \text{ of } [A,B]}} \{\overline{S}(T)\}$

$V(f) = \sup\limits_{\substack{\text{over all} \\ \text{possible partitions} \\ T \text{ for } [A,B]}} \{\underline{S}(T)\}$

<u>Claim:</u> For any <u>bounded</u> <u>measurable</u> function $f(x)$ $U(f) = V(f)$.

<u>Proof:</u>

Because $\forall T$ $\overline{S}(T) - \underline{S}(T) \le \lambda(T) \cdot \mu(E)$

$\lambda(T) \to 0 \Rightarrow \overline{S}(T) - \underline{S}(T) \to 0$. It means that $U(f) = V(f)$.

By definition, <u>Lebesgue integral</u>: $\boxed{(L)\int_E f(x)\,d\mu = U(f) = V(f)}$

⇐ במובן לבג, כל פונק' <u>מדידה</u> הינה פונק' <u>אינטגרבילית</u>.

(ערך האינטגרל על קבוצה $E$ יהיה המידה על קב' $E$).

\# <u>דוגמא:</u> פונק' דיריכלה - האינטגרל במובן לבג שווה ל-0. (המונח הוא המס' הרציונליים להם יש מידה 0)
(המידה על הרציונליים הינה 0)
במובן רימן - האינטגרל לא מוגדר.

* <u>Claim:</u> If $E$ is a null-set, $\mu(E) = 0$, then any function $f: E \to \mathbb{R}$ is integrable and $\int_E f\,d\mu = 0$.

<u>Claim:</u> The definition of $\int_E f(x)\,d\mu$ does not depend on the segment $[A,B]$, $A < f(x) < B$.

Theorem: If $A \le f(x) \le B$ for any $x \in E$, then $A \cdot \mu(E) \le \int_E f(x)d\mu \le B \cdot \mu(E)$.

Proof:

Let take any $\varepsilon > 0$. Then $A - \varepsilon < f(x) < B + \varepsilon$.

Then $(A-\varepsilon)\mu(E) < \int_E f(x)d\mu < (B+\varepsilon)\mu(E)$

Now $\varepsilon \to 0$. (כאשר $\varepsilon \to 0$ (אי שוויון החזק יכול להפוך לשוויון)

$\Longrightarrow (A-\varepsilon)\mu(E) \le \int_E f(x)d\mu \le (B+\varepsilon)\mu(E)$

Proof of claim ※ :

If $A \le f(x) \le B$ then $A \cdot \underbrace{\mu(E)}_{0} \le \int_E f d\mu \le B \underbrace{\mu(E)}_{0} \Longrightarrow \int_E f d\mu = 0$

Theorem: If $E = \biguplus_{i=1}^{\infty} E_i$, where each $E_i$ is measurable, then:

$\int_E f(x)d\mu = \sum_{i=1}^{\infty} \int_{E_i} f(x)d\mu$.

Corollary: If $f \sim g$ $(\mu(E(f \neq g)) = 0)$, then $\int_E f(x)d\mu = \int_E g(x)d\mu$.

Proof:

$E_0 = \{x \in E : f(x) \neq g(x)\}$

$E_1 = \{x \in E : f(x) = g(x)\}$

$E_0 \uplus E_1 = E$.

By assumption, $\mu(E_0) = 0$.

$\int_E f(x)d\mu = \underbrace{\int_{E_0} f(x)d\mu}_{0} + \int_{E_1} f(x)d\mu$ (נובע מה- Theorem שלפני ה- Corollary)

$\int_E g(x)d\mu = \underbrace{\int_{E_0} g(x)d\mu}_{0} + \int_{E_1} g(x)d\mu$

$\Longrightarrow \int_E f(x)d\mu = \int_E g(x)d\mu$, because $f = g$ on $E_1$.

Corollary: Assume $f(x) \ge 0$ (measurable and bounded).

If $\int_E f(x)d\mu = 0 \Longrightarrow f \sim 0$, i.e. $\mu(E(f > 0)) = 0$.

המשך←

Proof:

By contradiction, assume that $\mu(E(f>0)) = m > 0$.

$E(f>0) = \{x \in E : f(x) > 0\} \underset{\text{ארכימדס}}{=} \bigcup_{n\in\mathbb{N}} \underbrace{\{x\in E : f(x) > \frac{1}{n}\}}_{=E_n}$

If $\forall \mu(E_n) = 0 \Longrightarrow \mu(\bigcup_{n=1}^{\infty} E_n) = 0$.

But $\mu(E) = m > 0$. Then $\exists n$ such that $\mu(E_n) > 0$.

$\int_E f(x)d\mu \geq \int_{E_n} f(x)d\mu \geq \frac{1}{n}\cdot\mu(E_n) > 0$. Contradiction!

כי הנתון $\int_E f(x)d\mu = 0$.

Properties:

① $\int_E (f+g)d\mu = \int_E f d\mu + \int_E g d\mu$

② $\int_E c\cdot f(x)d\mu = C\cdot\int_E f(x)d\mu$ for any constant $C$

③ If $f \leq g$ on $E$, then $\int_E f d\mu \leq \int_E g d\mu$

④ $\left|\int_E f(x)d\mu\right| \leq \int_E |f(x)|d\mu$

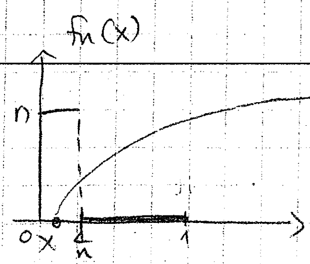
* What happens if $f_n(x) \xrightarrow[n\to\infty]{} f(x)$ for any $x\in E$.

$\Downarrow ?$

$\int_E f_n(x)d\mu \xrightarrow[n\to\infty]{} \int_E f(x)d\mu \Longrightarrow$ No!

#Counter Example:

$$f_n(x) = \begin{cases} 0, & \frac{1}{n} \leq x \leq 1 \\ n, & 0 < x < \frac{1}{n} \end{cases}$$




לכל נקודה x כאן,
יש n גדול מספיק
כך ש- $\frac{1}{n} < x$
אז $f_n(x) \to 0$

זו פונק' רציפה למקוטעין ובפרט מדידה.

$f_n(x) \xrightarrow[n\to\infty]{(x \text{ מקובע})} 0$

$\forall n \quad \int_0^1 f_n(x)d\mu = 1 = n\cdot\frac{1}{n} \not\longrightarrow \int 0 d\mu = 0$

* הערה: אם ההתכנסות היתה במ"ש זה היה יכול לעבוד.

**<u>Theorem</u>:** Dominated Convergence Theorem of Lebesgue

Assume that $f_n(x): E \to \mathbb{R}$ measurable functions, and $\forall x \in E$

$f_n(x) \xrightarrow[n\to\infty]{} f(x)$, and there is $g(x): E \to \mathbb{R}$ $|f_n(x)| \leq g(x)$ $\forall x \in E$.

Then $\int_E f_n(x)d\mu \xrightarrow[n\to\infty]{} \int_E f(x)d\mu$.

($g(x)$ integrable function)

* מה זה אומר אינטגרבילית במובן לבג?

פונק' שהינה מדידה לבג + אינטגרל לבג על הפונקציה הינו ערך סופי.

Fact: Any measurable bounded function defined on $E$ with $\mu(E)<\infty$ is integrable, $\int_E f(x)d\mu < \infty$.

Theorem: Lebesgue dominated convergence Th.

Assume $(X, \mathcal{L}, \mu)$ a measure space.

Let $\{f_n(x)\}_{n=1}^{\infty}$ be a sequence of integrable functions,

and assume $\forall x \in X$, $f_n(x) \xrightarrow[n\to\infty]{} f(x)$. (pointwise convergence)

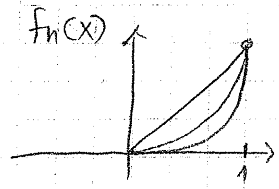
If $|f_n(x)| \leq g(x)$, $\forall x \in X$, $\forall n \in \mathbb{N}$ and $g(x)$ is integrable, then

$$\int_X f_n(x)d\mu \xrightarrow[n\to\infty]{} \int_X f(x)d\mu.$$

Extra assumption: $\mu(X) < \infty$.

* Particular case: $g(x)$ is bounded, i.e. $\exists$ constant $K$ such that $g(x) \leq K$.

\# Example: $f_n(x) = x^n$, $X = [0,1]$




$$f_n(x) \xrightarrow[n\to\infty]{} \underbrace{\begin{cases} 0, & \text{if } 0 \leq x < 1 \\ 1, & x = 1 \end{cases}}_{\text{not continuous!}} \Rightarrow \text{pointwise convergence.}$$

$$\Rightarrow \int_0^1 x^n dx = \frac{1}{n+1} \xrightarrow[n\to\infty]{} 0 = \int_0^1 0\, dx$$

⇐ כלומר המשפט מתקיים כשההתכנסות נקודתית וזו מ"ש בהוכחה.

Use in probability theory:

$f_X(t)$ density function of random variable $X$, $t \in (-\infty, \infty)$

Mean: $M[X] = \int_{-\infty}^{\infty} t \cdot f_X(t)dt$

Assume that $X_n \xrightarrow[n\to\infty]{} X$ in probability sense.

$f_n(t) \xrightarrow[n\to\infty]{} f(t)$ for almost all $t \in (-\infty, \infty)$, i.e. $\{t \in (-\infty,\infty) : f_{X_n}(t) \not\to f_X(t)\}$ has measure zero.

$\exists$ random variable $Y$ such that $P(|X_n| \leq Y) = 1$, $\forall n$. ($X_n$ חסומות ע"י $Y$)

Then $M[X_n] \xrightarrow[n\to\infty]{} M[X]$.

# Example: What happens if the measure is not finite?

$$f_n(x) = \begin{cases} 1, & x \in [n, n+1] \\ 0, & \text{otherwise} \end{cases}$$

$x \in [1, \infty)$, $\int_1^\infty f_n(x)\,dx = 1$ for any $n$

$\forall x \in [1, \infty)$, $f_n(x) \xrightarrow[n\to\infty]{} 0$

$$\int_1^\infty f_n(x)\,dx = 1 \not\longrightarrow \int_1^\infty 0\,dx = 0$$

למה המשפט לא עובד כאן?

$g(x) \equiv 1$ - שחוסמת את כל הפונקציות אינה אינטגרבילית - $\int_1^\infty 1\,dx = \underset{\uparrow}{\infty}$ הערך לא סופי

(אנו צריכים פונק' שחוסמת באופן אחיד את כל הפונק' $f_n(x)$)

If $g(x)$ is a dominating function for all elements $f_n(x)$, then $g(x) \geq 1$ $\forall x \in [1, \infty)$. $\int_1^\infty g(x)\,dx = \infty$

Proof (of the Theorem):

$\mu(X) < \infty$, $g(x) \leq K$, $\forall x \in X$. $\Rightarrow |f_n(x)| \leq K$, $|f(x)| \leq K \Rightarrow |f_n(x) - f(x)| \leq 2K$ (*)

Fix any $\sigma > 0$.

Define two sets:

$A_n(\sigma) = \{x \in X : |f_n(x) - f(x)| \geq \sigma\}$

$B_n(\sigma) = \{x \in X : |f_n(x) - f(x)| < \sigma\}$

$\forall n$, $X = A_n(\sigma) \uplus B_n(\sigma)$.

We need to prove that $\underbrace{\int_X f_n(x)\,d\mu - \int_X f(x)\,d\mu}_{= \int_X (f_n(x) - f(x))\,d\mu} \xrightarrow[n\to\infty]{} 0$

$$\left|\int_X (f_n(x) - f(x))\,d\mu\right| \leq \int_X |f_n(x) - f(x)|\,d\mu = \underbrace{\int_{A_n(\sigma)} |f_n(x) - f(x)|\,d\mu}_{2} + \underbrace{\int_{B_n(\sigma)} |f_n(x) - f(x)|\,d\mu}_{1}$$

$1 = \int_{B_n(\sigma)} |f_n(x) - f(x)|\,d\mu \leq \sigma \cdot \mu(B_n(\sigma))$

$2 = \int_{A_n(\sigma)} |f_n(x) - f(x)|\,d\mu \underset{(*)}{\leq} 2K \cdot \mu(A_n(\sigma))$

$\Rightarrow \left|\int_X (f_n(x) - f(x))\,d\mu\right| \leq 2K\,\mu(A_n(\sigma)) + \sigma\,\mu(B_n(\sigma))$

←המשך

Our plan: $\varepsilon > 0$ be arbitrary.

We need to show that $\exists n_0 \in \mathbb{N}\ \forall n > n_0,\ \underbrace{2K \cdot \mu(A_n(\sigma))}_{\text{נרצה } < \frac{\varepsilon}{2}} + \underbrace{\sigma \mu(B_n(\sigma))}_{\text{נרצה } < \frac{\varepsilon}{2}} < \varepsilon$

$\mu(B_n(\sigma)) \leq \mu(X) \underset{\text{הנחה}}{<} \infty$

Fix $\sigma > 0$ such that $\sigma \cdot \mu(X) < \frac{\varepsilon}{2} \iff \sigma < \frac{\varepsilon}{2\mu(X)}$

* על מנת להגיע למצב של $2K \cdot \mu(A_n(\sigma)) < \frac{\varepsilon}{2}$ נצטרך להשתמש ב-Lemma.

<u>Lemma:</u>

If $n \to \infty$ then $\mu(A_n(\sigma)) \xrightarrow[n\to\infty]{} 0$.

$\Downarrow$

$\exists n_0 \in \mathbb{N},\ \forall n > n_0,\ \mu(A_n(\sigma)) < \frac{\varepsilon}{4K}$

<u>Proof of lemma:</u>

Define $R_n(\sigma) = \bigcup_{k=n}^{\infty} A_k(\sigma)$.

Of course, $A_n(\sigma) \subset R_n(\sigma)$.

$R_n(\sigma) \supset R_{n+1}(\sigma)$

$R_1(\sigma) \supset R_2(\sigma) \supset R_3(\sigma) \supset \dots \supset R_n(\sigma) \supset \dots$

$X \supset R_1(\sigma) \implies \mu(R_1(\sigma)) \leq \mu(X) < \infty$.

Denote $M = \bigcap_{n=1}^{\infty} R_n(\sigma)$.

We have $\mu(R_n(\sigma)) \xrightarrow[n\to\infty]{} \mu(M)$. (by exercise from HW 3)

In fact, in our situation $M = \emptyset$.

~~We use the assumption: $f_n(x) \xrightarrow[n\to\infty]{} f(x)$.~~

On the contrary, assume that $M \neq \emptyset$, $\exists x \in M$.

So, $x \in R_n(\sigma)\ \forall n \implies \exists k \geq n,\ x \in A_k(\sigma) \iff |f_k(x) - f(x)| \geq \sigma \iff$

$\iff x \in X\ \forall n\ \exists k \geq n\ |f_k(x) - f(x)| \geq \sigma \iff$ exactly nagation

of $f_n(x) \xrightarrow[n\to\infty]{} f(x)$.

$\implies M = \emptyset$.

$A_n(\sigma) \subseteq R_n(\sigma)$

$\Downarrow$

$0 \leq \mu(A_n(\sigma)) \leq \underset{\substack{\downarrow n\to\infty \\ 0}}{\mu(R_n(\sigma))}$

So, by "sandwich", $\mu(A_n(\sigma)) \xrightarrow[n\to\infty]{} 0$.

Further generalization of Lebesgue integrable function:

If $\mu$ is a $\sigma$-finite measure, i.e. $X = \bigcup_{n=1}^{\infty} X_n$, $\mu(X_n) < \infty$,

Then $\int_X f(x)d\mu = \underbrace{\lim_{n\to\infty} \int_{\bigcup_{i=1}^{n} X_i} f(x)d\mu}_{*}$ ..

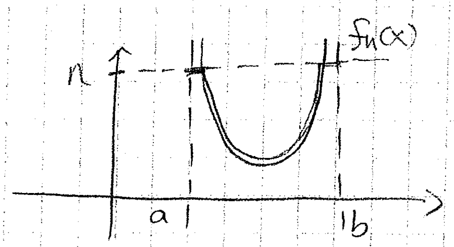
Definition: f(x) is Lebesgue integrable if lim (*) above is finite.

For example: $\int_1^{\infty} \frac{1}{x^2}dx = \lim_{n\to\infty} \int_1^{n} \frac{1}{x^2}dx = \lim_{n\to\infty}(1-\frac{1}{n}) = 1 \Rightarrow$ finite

$\int_1^{\infty} \frac{1}{x}dx = \lim_{n\to\infty} \int_1^{n} \frac{1}{x}dx = \lim_{n\to\infty}(\ln(n) - \ln(1)) = \infty$

$\Rightarrow f(x)$ is not integrable on $[1,\infty)$.

* How to define $\int_X f(x)d\mu$ for not necessarily bounded function?




First, define for $f(x) \geq 0$.

$\forall n \in \mathbb{N}$ define:

$$f_n(x) = \begin{cases} f(x), & \text{if } f(x) \leq n \\ n, & \text{if } f(x) > n \end{cases}$$

Function $f_n(x)$ is bounded from above by $n$.

$f_n(x) \leq f_{n+1}(x)$, $\forall x \in X$, $\forall n \in \mathbb{N}$

So, $\int_X f_n(x)d\mu \leq \int_X f_{n+1}(x)d\mu$ $(f(x) \geq 0)$

We have a monotone (increasing) sequence of numbers, $\left\{ \int_X f_n(x)d\mu \right\}_{n=1}^{\infty}$.

Definition: $\int_X f(x)d\mu = \lim_{n\to\infty} \int_X f_n(x)d\mu$ if this limit is finite.

Otherwise, f(x) is not integrable function.

*כעת צריכים להגדיר גם עבור f(x) שאינה אי שלילי, מהרכה.

Most general case:

$f(x)$ is any measurable function.

Define: $f_+(x) = \begin{cases} f(x), & \text{if } f(x) \geq 0 \\ 0, & \text{if } f(x) < 0 \end{cases}$ , $f_+(x) \geq 0$

$f_-(x) = \begin{cases} 0, & \text{if } f(x) \geq 0 \\ -f(x), & \text{if } f(x) < 0 \end{cases}$ , $f_-(x) \geq 0$

$f_+(x) - f_-(x) = f(x)$

$f_+(x) + f_-(x) = |f(x)|$

$\Rightarrow$ $f_+(x)$, $f_-(x)$ both are measurable functions.

$$\boxed{\int_X f(x)d\mu \underset{\text{by definition}}{=} \int_X f_+(x)d\mu - \int_X f_-(x)d\mu}$$ , when we say

function $f(x)$ is integrable if both $\int_X f_+(x)d\mu$ ; $\int_X f_-(x)d\mu$ are finite.

Corollary: $\int_X f(x)d\mu$ is finite $\Longleftrightarrow$ $\int_X |f(x)|d\mu$ is finite.

שעות קבלה עד המבחן: 21.1 , 28.1 , 4.2 ($18^{00}$-$19^{00}$) , 7.2 - ט.ט - $9^{00}$-$10^{00}$

## Basic properties of Lebesgue integral:

**Property 1:** If $\mu(E)=0$, then $\forall f(x): E \to \mathbb{R}$, $\int_E f(x)\,d\mu = 0$.

**Proof:**

* If $f(x)$ is a bounded function, this claim has been proved already.

$m \leq f(x) \leq M \quad , \forall x \in E$

$\Downarrow$

(✱) $$m \cdot \underbrace{\mu(E)}_{0} \leq \int_E f(x)\,d\mu \leq M \cdot \underbrace{\mu(E)}_{0} \Longrightarrow \boxed{\int_E f(x)\,d\mu = 0}$$

* Assume now that $f(x) \geq 0$, $\forall x \in E$

$\forall n \in \mathbb{N}$ we define: $f_n(x) = \begin{cases} f(x), & \text{if } f(x) \leq n \\ n, & \text{if } f(x) > n \end{cases}$

This $f_n(x)$ is a measurable function and bounded.

By definition: $$\int_E f(x)\,d\mu = \lim_{n\to\infty} \underbrace{\int_E f_n(x)\,d\mu}_{0}$$

לפי הגדרה עבור $f(x)$ שהיא לא חסומה

כי $f_n(x)$ חסומה ולכן זה מתקיים לפי (✱)

* The last step.

For any $f(x)$ we define two positive functions $f_+(x)$ and $f_-(x)$.

$f_+(x) = \begin{cases} f(x), & \text{if } f(x) \geq 0 \\ 0, & \text{otherwise} \end{cases}$ , $f_-(x) = \begin{cases} 0, & \text{if } f(x) > 0 \\ -f(x), & \text{if } f(x) \leq 0 \end{cases}$

$f(x) = f_+(x) - f_-(x)$

By definition, $$\int_E f(x)\,d\mu = \underbrace{\int_E f_+(x)\,d\mu}_{=0} - \underbrace{\int_E f_-(x)\,d\mu}_{=0} = 0 \quad (\infty - \infty \text{ is undefined})$$

כי $f_+(x)$ חיובית ← לפי *  כי $f_-(x)$ חיובית

Property 2: Assume that $\exists \int_E f d\mu$. ($E$ is a measurable set, $f$ is a measurable function, $\int_E f d\mu < \infty$).

Then for every measurable subset $E' \subset E$ the integral $\int_{E'} f d\mu$ also exists.

הערה: לכול תת סט של סט הוא מגיע!

Proof:

If $f \geq 0$ – $\underbrace{\int_{E'} f d\mu}_{<\infty} \leq \underbrace{\int_E f d\mu}_{<\infty}$

Property 3: Assume $f(x)$ and $F(x)$ are two measurable functions, and $|f(x)| \leq F(x)$, $\forall x \in E$.

If $\exists \int_E F(x) d\mu$, then also $\exists \int_E f(x) d\mu$.

Property 4: If $f \sim g$ on $E$, i.e. $\mu(E(f \neq g)) = 0$ and $\exists \int_E g d\mu$, then also $\exists \int_E f d\mu$ and $\int_E f \cdot d\mu = \int_E g \cdot d\mu$

Property 5: Assume that $E = \biguplus_{i=1}^{n} E_i$ and every $E_i$ is measurable.

Then, $\int_E f d\mu = \sum_{i=1}^{f} \int_{E_i} f d\mu$

(אם $E = \biguplus_{i=1}^{\infty} E_i$ אז יכול להיות $\sum_{i=1}^{\infty} \underbrace{\int_{E_i} f d\mu}_{<\infty}$, אבל הסכום יכול להיות שווה $\infty$!)

$\int_E f d\mu = \infty \rightarrow$ לכן!

Theorem:

Assume that $E = \biguplus_{i=1}^{\infty} E_i$ and every $E_i$ is a measurable set.

Assume also that $f$ is integrable over $E$, i.e $\int_E f d\mu < \infty$.

Then, $\int_E f d\mu = \sum_{i=1}^{\infty} \int_{E_i} f d\mu$

Property 6: Linearity

For any measurable set $E$ and any two integrable functions $f, g$ we have $\int_E (\alpha f + \beta g) d\mu = \alpha \underbrace{\int_E f d\mu}_{\text{חייב להיות סופי כי } f \text{ אינטגרבילי}} + \beta \underbrace{\int_E g d\mu}_{\text{חייב להיות סופי כי } g \text{ אינטגרבילי}}$, here $\alpha, \beta \in \mathbb{R}$.

## Relation between Riemann and Lebesgue integrals:

**Theorem:** (1)

A bounded function $f(x)$ is integrable in the sense of Riemann on $[a,b]$ if and only if $f(x)$ is equivalent (in the sense of Lebesgue) to a continous function, i.e. $\mu(A)=0$, where $A$ is the set of all points, where $f(x)$ is not continuous.

# **Example:** $f(x) = \begin{cases} 1, & x \in \text{Cantor set } C \\ 0, & \text{otherwise} \end{cases}$

$\mu(C) = 0,\ |C| = 2^{\aleph_0} \longrightarrow$ כלומר ל-$f$ יש $2^{\aleph_0}$ נק' אי-רציפות!
ולא מס' בן מנייה של
נק' אי-רציפות

$f(x)$ is continuous for any $x \notin C$.

$C$
$[0,1]\setminus C$ open.
$\downarrow$
לכן קיימת סביבה לכל נק' בה $f$ שווה ל-0,
סביבה בה $f$ שווה ל-0.

$\Leftarrow$ $f(x)$ אינטגרבילית במובן רימן לפי ה-Theorem.

**Theorem:** (2)

If $f(x)$ is Riemann integrable, then it is also integrable in the sense of Lebesgue, and $(R)\int_{[a,b]} f(x)dx = (L)\int_{[a,b]} f(x)d\mu$.

Definition of Riemann integral: $f(x)$ is bounded function.

~~A partition of the closed interval $[a,b]$.~~

$T: a = x_0 < x_1 < x_2 < \dots < x_n = b$

parameter $\lambda(T) = \max\{\Delta x_1, \Delta x_2, \dots, \Delta x_n\}$, where

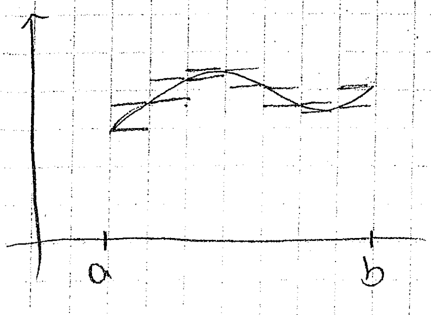
$\Delta x_i = x_i - x_{i-1},\quad i = 1, \dots, N$

$m_i = \inf_{x \in [x_{i-1}, x_i]} f(x)\ ;\quad M_i = \sup_{x \in [x_{i-1}, x_i]} f(x)$

$\underline{S}(T) = \sum_{i=1}^{n} m_i \Delta x_i$ – Lower Darboux sum

$\overline{S}(T) = \sum_{i=1}^{n} M_i \Delta x_i$ – Upper Darboux sum

Theorem: $f(x)$ is integrable (Riemann) if $\bar{S}(T) - \underline{S}(T) \xrightarrow[\lambda(T)\to 0]{} 0$.



- $m(x) = m_i \iff x \in (x_{i-1}, x_i)$, $m(x) = 0$ if $x = x_i$
  $m(x) \leq f(x)$ $\forall x$ – a step function
- $M(x) = M_i \iff x \in (x_{i-1}, x_i)$, $M(x) = 0$ if $x \neq x_i$
  $f(x) \leq M(x)$ – a step function

<u>claim</u>: $\bar{S}(T) - \underline{S}(T) \xrightarrow[\lambda(T)\to 0]{} 0 \iff \underbrace{m(x)}_{\text{מדרגה}} \xrightarrow[\lambda(T)\to 0]{} f(x)$ for $x \neq x_i$,

$\underbrace{M(x)}_{\text{מדרגה}} \xrightarrow[\lambda(T)\to 0]{} f(x)$ for $x \neq x_i$

Actually we have two sequences of bounded measurable functions which both converge to $f(x)$ almost for all $x \in [a,b]$.

$$(L)\int_{[a,b]} m(x)\,d\mu = (R)\int_{[a,b]} m(x)\,dx \xrightarrow[\lambda(T)\to 0]{} (L)\int_{[a,b]} f(x)\,d\mu$$

עבור פונקציות מדרגה
אינטגרל רימן שווה לאינטגרל
לבג - ניתן להוכיח בכלל.

by Lebesgue Theorem of dominated convergence

Similarly, $(L)\int_{[a,b]} M(x)\,d\mu = (R)\int_{[a,b]} M(x)\,d\mu \xrightarrow[\lambda(T)\to 0]{} (L)\int_{[a,b]} f(x)\,d\mu$.

The final conclusion will be:

$$(R)\int_{[a,b]} f(x)\,dx = (L)\int_{[a,b]} f(x)\,d\mu$$

<u>Idea of Proof (Th. 2)</u>:

If $f(x)$ is Riemann integrable $\Rightarrow$ $f(x) \sim$ continuous function.