Weak Proregularity, Weak Stability, and the Noncommutative MGM Equivalence

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0. Outline of the Lecture

This is joint work with Rishi Vyas.

1. Weak Proregularity

In this section A is a commutative ring.

Let $\boldsymbol{a} = (a_1, \ldots, a_n)$ be a sequence of elements in A.

Recall that the *Koszul complex* $\mathrm{K}(A; \boldsymbol{a})$ associated to $\boldsymbol{a}$ is a complex of finitely generated free A-modules, concentrated in degrees $-n, \ldots, 0$.

For $n = 1$ it looks like this:

$$\mathrm{K}(A; a) = \big(\cdots 0 \to A \xrightarrow{a \,\cdot} A \to 0 \to \cdots \big).$$

For $n \geq 2$ the Koszul complex is a tensor product:

$$\mathrm{K}(A; \boldsymbol{a}) = \mathrm{K}(A; a_1) \otimes_A \cdots \otimes_A \mathrm{K}(A; a_n).$$

For any $i \geq 1$ let us consider the sequence $\boldsymbol{a}^i := (a_1^i, \ldots, a_n^i)$.

There is a corresponding Koszul complex $\mathrm{K}(A; \boldsymbol{a}^i)$.

For $j \geq i$ there is a homomorphism of complexes

$$\mathrm{K}(A; \boldsymbol{a}^j) \to \mathrm{K}(A; \boldsymbol{a}^i).$$

When $n = 1$ this homomorphism is described by the following commutative diagram:

(1.1)
$$\begin{array}{ccc} \mathrm{K}(A; a^j) & \qquad & A \xrightarrow{\;a^j \cdot\;} A \\ \big\downarrow & & {\scriptstyle a^{j-i} \cdot}\big\downarrow \qquad \big\downarrow{\scriptstyle \mathrm{id}} \\ \mathrm{K}(A; a^i) & & A \xrightarrow{\;a^i \cdot\;} A \end{array}$$

When $n \geq 2$ it is gotten by applying the tensor product.

In this way the collection of Koszul complexes

$$\{\mathrm{K}(A;\boldsymbol{a}^i)\}_{i\geq 1}$$

is an inverse system.

An inverse system of modules $\{M_i\}_{i\geq 1}$ is called pro-zero if for each i there is some $j \geq i$ such that the homomorphism $M_j \to M_i$ is zero.

Definition 1.2. The sequence $\boldsymbol{a}$ is called *weakly proregular* if for every $p < 0$, the inverse system of A-modules

$$\{\mathrm{H}^p(\mathrm{K}(A;\boldsymbol{a}^i))\}_{i\geq 1}$$

is pro-zero.

For $p = 0$ we do not expect any vanishing, since

$$\lim_{\leftarrow i} \mathrm{H}^0(\mathrm{K}(A;\boldsymbol{a}^i)) = \widehat{A},$$

the $\mathfrak{a}$-adic completion of A, where $\mathfrak{a}$ is the ideal generated by $\boldsymbol{a}$.

If $\boldsymbol{a}$ is a regular sequence, then $\mathrm{H}^p(\mathrm{K}(A;\boldsymbol{a}^i)) = 0$ for any $p < 0$ and $i \geq 1$. So $\boldsymbol{a}$ is a weakly proregular sequence.

But from the opposite extremity, if $\boldsymbol{a}$ is a sequence of nilpotent elements, then it is also is also weakly proregular, as can be seen from (1.1).

Anyhow, what does the definition mean?

Weak proregularity is a mysterious property.

Definition 1.2 was first considered in 1961 by Grothendieck in [LC]. But then it was forgotten for several decades.

The name "weakly proregular" was given around 2000 by Lipman et al. in [AJL, Correction].

Grothendieck had already proved this:

Theorem 1.3. *([LC]) If the ring A is noetherian, then any finite sequence $\boldsymbol{a}$ in A is weakly proregular.*

Definition 1.4. An ideal $\mathfrak{a} \subseteq A$ is called a *weakly proregular ideal* if it is generated by some weakly proregular sequence $\boldsymbol{a}$.

Weak proregularity turns out to be a property of the $\mathfrak{a}$-adic topology. To be precise:

Theorem 1.5. *([PSY1]) Let $\boldsymbol{a}$ and $\boldsymbol{b}$ be finite sequences in A, that generate ideals $\mathfrak{a}$ and $\mathfrak{b}$ respectively, and assume that $\sqrt{\mathfrak{a}} = \sqrt{\mathfrak{b}}$.*

Then $\boldsymbol{a}$ is weakly proregular iff $\boldsymbol{b}$ is weakly proregular.

In the next two sections we will present results that will clarify the significance of weak proregularity.

2. MGM Equivalence

The results of this section are the culmination of work by Matlis, Grothendieck, Greenlees, May, Alonso, Jeremias, Lipman, Schenzel, Porta, Shaul and myself. See the references.

We are still dealing with a commutative ring A. The category of A-modules is $\mathsf{M}(A)$, and the (unbounded) derived category is $\mathsf{D}(A)$.

I am assuming that the audience is familiar with derived categories. All the material I will use is explained briefly in [Ye3], and in full detail in the book [Ye6].

Let $\mathfrak{a} \subseteq A$ be a finitely generated ideal.

The $\mathfrak{a}$-*torsion submodule* of an A-module M is

$$\Gamma_{\mathfrak{a}}(M) := \lim_{i \to} \operatorname{Hom}_A(A/\mathfrak{a}^i, M).$$

The $\mathfrak{a}$-*adic completion* of M is the module

$$\Lambda_{\mathfrak{a}}(M) := \lim_{\leftarrow i} (M/\mathfrak{a}^i \cdot M).$$

These are additive functors

$$\Gamma_{\mathfrak{a}}, \Lambda_{\mathfrak{a}} : \mathsf{M}(A) \to \mathsf{M}(A).$$

The functors $\Gamma_{\mathfrak{a}}$ and $\Lambda_{\mathfrak{a}}$ seem as if they are adjoint to each other; but this is false.

The torsion functor $\Gamma_{\mathfrak{a}}$ is left exact. It is also idempotent: $\Gamma_{\mathfrak{a}} \circ \Gamma_{\mathfrak{a}} \cong \Gamma_{\mathfrak{a}}$.

The completion functor $\Lambda_{\mathfrak{a}}$ is neither left exact nor right exact. But it is an idempotent functor.

The non-exactness of $\Lambda_{\mathfrak{a}}$ is very different from its familiar behavior on the category of finitely generated modules over a noetherian ring.

The additive functors $\Gamma_{\mathfrak{a}}$ and $\Lambda_{\mathfrak{a}}$ can be derived, giving rise to triangulated functors

$$\mathrm{R}\Gamma_{\mathfrak{a}}, \mathrm{L}\Lambda_{\mathfrak{a}} : \mathsf{D}(A) \to \mathsf{D}(A).$$

Let us define the full subcategories

$$\mathsf{D}(A)_{\mathfrak{a}\text{-tor}}, \mathsf{D}(A)_{\mathfrak{a}\text{-com}} \subseteq \mathsf{D}(A)$$

to be the essential images of the functors $\mathrm{R}\Gamma_{\mathfrak{a}}$ and $\mathrm{L}\Lambda_{\mathfrak{a}}$ respectively.

Theorem 2.1. *(MGM Equivalence, [PSY1])*

Let A be a commutative ring, and let $\mathfrak{a} \subseteq A$ be a weakly proregular ideal.

Then:

1. The functor $\mathrm{L}\Lambda_{\mathfrak{a}}$ is right adjoint to $\mathrm{R}\Gamma_{\mathfrak{a}}$.
2. The functors $\mathrm{R}\Gamma_{\mathfrak{a}}$ and $\mathrm{L}\Lambda_{\mathfrak{a}}$ are idempotent.
3. The categories $\mathsf{D}(A)_{\mathfrak{a}\text{-tor}}$ and $\mathsf{D}(A)_{\mathfrak{a}\text{-com}}$ are full triangulated subcategories of $\mathsf{D}(A)$.
4. The functor
$$\mathrm{R}\Gamma_{\mathfrak{a}} : \mathsf{D}(A)_{\mathfrak{a}\text{-com}} \to \mathsf{D}(A)_{\mathfrak{a}\text{-tor}}$$
is an equivalence of triangulated categories, with quasi-inverse $\mathrm{L}\Lambda_{\mathfrak{a}}$.

The letters “MGM” stand for Matlis, Greenlees and May.

The objects of $\mathsf{D}(A)_{\mathfrak{a}\text{-com}}$ are called *cohomologically complete complexes*.

The objects of $\mathsf{D}(A)_{\mathfrak{a}\text{-tor}}$ are called *cohomologically torsion complexes*.

Here is a list of conditions on the pair $(A, \mathfrak{a})$, each one implying the next. The distinguishing features between conditions are in brackets.

- A is noetherian. [The completion $\widehat{A} = \Lambda_{\mathfrak{a}}(A)$ is flat over A].
- $\mathfrak{a}$ is weakly proregular. [MGM Equivalence holds.]
- $\mathfrak{a}$ is finitely generated. [The functor $\Lambda_{\mathfrak{a}}$ is idempotent.]
- No condition.

For more information on this hierarchy see [Ye2], [PSY1] and [Ye4].

MGM Equivalence is a powerful tool. Here are two examples to demonstrate this.

Example 2.2. Consider a field $\mathbb{K}$, and an $\mathfrak{a}$-adically complete noetherian $\mathbb{K}$-ring A, such that $\mathbb{K} \to A/\mathfrak{a}$ is of finite type.

For instance $A = \mathbb{K}[[t]]$, the power series ring in a variable t, and $\mathfrak{a} = (t)$.

Define the ring $B := A \otimes_{\mathbb{K}} A$ and the ideal

$$\mathfrak{b} := \mathfrak{a} \otimes_{\mathbb{K}} A + A \otimes_{\mathbb{K}} \mathfrak{a} \subseteq B.$$

The ring B is usually not noetherian; but the ideal $\mathfrak{b}$ is always weakly proregular, so the MGM Equivalence applies.

Also, the completion $\widehat{B} = \Lambda_{\mathfrak{b}}(B)$ is a noetherian ring.

These facts allowed Shaul [Sh] to prove that Hochschild cohomology commutes with adic completion, to calculate it in many previously unknown cases, and to answer a question of Buchweitz and Flenner that was open for 10 years.

Example 2.3. Let $\mathbb{K}$ be a field, and let A be an $\mathfrak{a}$-adically complete noetherian $\mathbb{K}$-ring, such that $\mathbb{K} \to A/\mathfrak{a}$ is finite.

For instance $A = \mathbb{K}[[t]]$ and $\mathfrak{a} = (t)$.

Define the A-module

$$A^* := \operatorname{Hom}^{\mathrm{cont}}_{\mathbb{K}}(A, \mathbb{K}),$$

where continuity is for the $\mathfrak{a}$-adic topology.

The module A^* is not quite a *dualizing complex* over A, in the original sense of Grothendieck in [RD].

Recall that a complex R is called dualizing if it has finitely generated cohomology modules, finite injective dimension, and the canonical morphism

$$A \to \operatorname{RHom}_A(R, R)$$

in $\mathsf{D}(A)$ is an isomorphism.

(cont.) The problem with A^* is that it is not a finitely generated A-module.

Indeed,

$$A^* = \lim_{i \to} \operatorname{Hom}_{\mathbb{K}}(A/\mathfrak{a}^i, \mathbb{K}),$$

so it is an artinian module of infinite length.

In the terminology of [AJL], A^* is a *t-dualizing complex*, where “t” stands for torsion.

However, by [AJL] and [PSY2], the complex

$$R := \mathrm{L}\Lambda_{\mathfrak{a}}(A^*) \in \mathsf{D}(A)$$

is a dualizing complex over A in the usual sense.

Remark 2.4. Positselski has recently done a lot of work that’s connected to the MGM Equivalence.

Here are two relations to his work:

- Let M be an A-module that is cohomologically $\mathfrak{a}$-adically complete as a complex. Then, in the terminology of [Po1], M is *contramodule*.
- The complex $\mathrm{R}\Gamma_{\mathfrak{a}}(A)$ is a *dedualizing complex* in the sense of [Po2].

Remark 2.5. The name “cohomologically complete” was introduced by Kashiwara and Schapira in [KS], for the case $A = \mathbb{K}[[t]]$ and $\mathfrak{a} = (t)$.

Their definition is different, but equivalent, to the definition given above. See [PSY1].

3. Torsion Classes and Weak Stability

As we saw earlier, weak proregularity – besides being mysterious – is a commutative condition. It is almost never possible to form Koszul complexes over noncommutative rings.

On the other hand, as Example 2.3 shows, having some sort of MGM Equivalence for noncommutative rings could be useful for producing dualizing complexes.

Indeed, the Van den Bergh Existence Theorem (see [VdB]) can be viewed as a noncommutative graded variant of Example 2.3. We shall return to this idea at the end of the talk.

In this section we are going to present a new characterization of weak proregularity, that can be applied to noncommutative rings.

Before going on, we have to recall a few ideas from the abstract theory of torsion.

Now A is a noncommutative ring, and $\mathsf{M}(A)$ is the category of left A-modules.

A (hereditary) *torsion class* in $\mathsf{M}(A)$ is a class of objects $\mathsf{T} \subseteq \mathsf{M}(A)$ that is closed under taking quotients, subobjects, extensions and infinite direct sums.

The torsion class T gives rise to the T-torsion functor Γ_{T}, which is an additive functor from $\mathsf{M}(A)$ to itself.

The formula for the functor Γ_{T} is this: $\Gamma_{\mathsf{T}}(M)$ is the largest submodule of M that belongs to T.

It is quite easy to see that the functor Γ_{T} is left exact and idempotent.

It is standard terminology (see [St]) to call T a *stable torsion class* if the functor Γ_{T} sends injectives to injectives.

Example 3.1. Suppose A is commutative noetherian, and $\mathfrak{a} \subseteq A$ is an ideal.

The $\mathfrak{a}$-torsion modules form the torsion class $\mathsf{T}_{\mathfrak{a}} \subseteq \mathsf{M}(A)$.

It is well-known that $\mathsf{T}_{\mathfrak{a}}$ is a stable torsion class.

The torsion functor Γ_{T} has a right derived functor

$$\mathrm{R}\Gamma_{\mathsf{T}} : \mathsf{D}(A) \to \mathsf{D}(A).$$

For any $q \geq 0$ there is equality $\mathrm{H}^q(\mathrm{R}\Gamma_{\mathsf{T}}) = \mathrm{R}^q\Gamma_{\mathsf{T}}$, where

$$\mathrm{R}^q\Gamma_{\mathsf{T}} : \mathsf{M}(A) \to \mathsf{M}(A)$$

is the classical q-th right derived functor of Γ_{T}.

Definition 3.2. ([YZ]) An A-module I is called T-*flasque* if $\mathrm{R}^q\Gamma_{\mathsf{T}}(I) = 0$ for all $q > 0$.

Of course any injective module is T-flasque, but usually there are many more, as the next example shows.

Example 3.3. If T is a stable torsion class (as in Example 3.1 for instance), then any module $M \in \mathsf{T}$ is T-flasque.

We now come to the main definition of this talk.

Definition 3.4. A torsion class $\mathsf{T} \subseteq \mathsf{M}(A)$ is called *weakly stable* if for any injective module I, the module $\Gamma_{\mathsf{T}}(I)$ is T-flasque.

It turns out that this property is indeed a noncommutative, or categorical, characterization of weak proregularity:

Theorem 3.5. *([VY1]) Let A be a commutative ring,* $\boldsymbol{a}$ *a finite sequence in A, and* $\mathfrak{a}$ *the ideal generated by* $\boldsymbol{a}$.

The following two conditions are equivalent:

(i) The sequence $\boldsymbol{a}$ is weakly proregular.

(ii) The torsion class $\mathsf{T}_{\mathfrak{a}}$ is weakly stable.

4. Noncommutative MGM Equivalence

Now that we have identified what weak proregularity ought to mean in the noncommutative setting, we can ask for a noncommutative version of Theorem 2.1.

As far as we can tell, in the noncommutative setting one must make more assumptions on the torsion class.

Definition 4.1. Let T be a torsion class in $\mathsf{M}(A)$.

1. We call T *finite dimensional* if the functors $\mathrm{R}^q\Gamma_{\mathsf{T}}$ vanish for $q \gg 0$.
2. We call T *quasi-compact* if the functors $\mathrm{R}^q\Gamma_{\mathsf{T}}$ commute with infinite direct sums.

From now on the rings we consider will be noncommutative, and central over a commutative base ring $\mathbb{K}$.

We shall also assume that these rings are *flat* over $\mathbb{K}$.

The flatness condition greatly simplifies the discussion. Most likely this condition is not essential, but the theory of derived categories of A-bimodules (see [Ye5]), that relies on flat DG ring resolutions, is not yet "fully operational".

The nonflat version of the subsequent theorems is predicted to be part of the upcoming paper [Vs1].

The enveloping ring of A is

$$A^{\mathrm{en}} := A \otimes_{\mathbb{K}} A^{\mathrm{op}}.$$

The category of A-bimodules is $\mathsf{M}(A^{\mathrm{en}})$, and the derived category is $\mathsf{D}(A^{\mathrm{en}})$.

All derived functors that we need exist in this setting.

The torsion class $\mathsf{T} \subseteq \mathsf{M}(A)$ extends to bimodules, as follows: a bimodule M is T-torsion if it is so as a left A-module.

Thus we get a bimodule torsion class $\mathsf{T} \subseteq \mathsf{M}(A^{\mathrm{en}})$.

There is a torsion functor

$$\Gamma_{\mathsf{T}} : \mathsf{M}(A^{\mathrm{en}}) \to \mathsf{M}(A^{\mathrm{en}}).$$

It has a derived functor

$$\mathrm{R}\Gamma_{\mathsf{T}} : \mathsf{D}(A^{\mathrm{en}}) \to \mathsf{D}(A^{\mathrm{en}}).$$

Theorem 4.2. *(Representabilty of Derived Torsion, [VY1])*

Let A be a flat central $\mathbb{K}$*-ring, and let* T *be a quasi-compact, finite dimensional, weakly stable torsion class in* $\mathsf{M}(A)$.

Define the object

$$P := \mathrm{R}\Gamma_{\mathsf{T}}(A) \in \mathsf{D}(A^{\mathrm{en}}).$$

Then there is an isomorphism

$$P \otimes^{\mathrm{L}}_{A} M \cong \mathrm{R}\Gamma_{\mathsf{T}}(M)$$

of triangulated functors from $\mathsf{D}(A)$ *to itself.*

Following Positselski, we call the complex of bimodules P a *noncommutative dedualizing complex*.

Special cases of this theorem were known before – e.g. [WZ, Lemma 3.4].

The category $\mathsf{D}(A^{\mathrm{en}})$ is monoidal, with monoidal product $- \otimes^{\mathrm{L}}_{A} -$ and unit object A.

There is a canonical morphism $\mathrm{R}\Gamma_{\mathsf{T}} \to \mathrm{Id}$ of triangulated functors from $\mathsf{D}(A^{\mathrm{en}})$ to itself. Applying it to object A gives a morphism $\rho : P \to A$ in $\mathsf{D}(A^{\mathrm{en}})$.

The two morphisms

$$P \otimes^{\mathrm{L}}_{A} P \to P$$

that are induced by ρ are isomorphisms.

We call such a pair (P, ρ) an *idempotent copointed object*.

This property of P is crucial for the proof of the Theorem 4.3 below.

Let us define the triangulated functor

$$G_{\mathsf{T}} : \mathsf{D}(A) \to \mathsf{D}(A)$$

by the formula

$$G_{\mathsf{T}} := \mathrm{RHom}_A(P, -).$$

This functor should be thought of as an abstract “derived completion functor”.

Next let us define the full subcategories

$$\mathsf{D}(A)_{\mathsf{T}\text{-tor}}\,,\ \mathsf{D}(A)_{\mathsf{T}\text{-com}} \subseteq \mathsf{D}(A)$$

to be the essential images of the functors $\mathrm{R}\Gamma_{\mathsf{T}}$ and G_{T} respectively.

Theorem 4.3. *(Noncommutative MGM Equivalence, [VY1])*

Let A be a flat central $\mathbb{K}$*-ring, and let* T *be a quasi-compact, weakly stable, finite dimensional torsion class in* $\mathsf{M}(A)$. *Then*:

1. The functor G_{T} is right adjoint to $\mathrm{R}\Gamma_{\mathsf{T}}$.
2. The functors $\mathrm{R}\Gamma_{\mathsf{T}}$ and G_{T} are idempotent.
3. The categories $\mathsf{D}(A)_{\mathsf{T}\text{-tor}}$ and $\mathsf{D}(A)_{\mathsf{T}\text{-com}}$ are full triangulated subcategories of $\mathsf{D}(A)$.
4. The functor
$$\mathrm{R}\Gamma_{\mathsf{T}} : \mathsf{D}(A)_{\mathsf{T}\text{-com}} \to \mathsf{D}(A)_{\mathsf{T}\text{-tor}}$$
is an equivalence of triangulated categories, with quasi-inverse G_{T}.

Recall that in the commutative situation, the right adjoint of the functor $\mathrm{R}\Gamma_{\mathsf{T}} = \mathrm{R}\Gamma_{\mathfrak{a}}$ was $G_{\mathsf{T}} = \mathrm{L}\Lambda_{\mathfrak{a}}$.

We therefore ask:

Question 4.4. In the situation of Theorem 4.3, under what assumption is there an additive functor $\Lambda : \mathsf{M}(A) \to \mathsf{M}(A)$, such that $G_{\mathsf{T}} = \mathrm{L}\Lambda$?

There are known counterexamples; see [Vs2].

Remark 4.5. The idempotence of the functor G_{T} means that there is a morphism of triangulated functors $\mathrm{Id} \to G_{\mathsf{T}}$, and the two induced morphisms $G_{\mathsf{T}} \to G_{\mathsf{T}} \circ G_{\mathsf{T}}$ are isomorphisms.

A functor with this property is often called an *idempotent monad* or a *Bousfield localization*; cf. [Kr].

Here are two of examples related to Theorem 4.3.

Example 4.6. Let A be a ring, and let S be a left denominator set in A, with left Ore localization $A_S = A[S^{-1}]$.

Define

$$\mathsf{T}_S := \{M \in \mathsf{M}(A) \mid A_S \otimes_A M = 0\}.$$

This is a torsion class in $\mathsf{M}(A)$.

It can be shown (see [Vs2]) that the torsion class T_S is weakly stable if and only if it has dimension ≤ 1.

Example 4.7. Consider the ring $A = \mathbb{Z}$, the multiplicatively closed set $S := \mathbb{Z} - \{0\}$, and the torsion class $\mathsf{T} = \mathsf{T}_S$ in $\mathsf{M}(\mathbb{Z})$, as in Example 4.6.

Thus for any abelian group M, $\Gamma_{\mathsf{T}}(M)$ is nothing but the torsion subgroup of M.

Because the ring $\mathbb{Z}$ is hereditary, we know that T is weakly stable. So Theorem 4.3 applies.

In this case the right adjoint to $\mathrm{R}\Gamma_{\mathsf{T}}$ is $G_{\mathsf{T}} = \mathrm{L}\Lambda_{\mathsf{T}}$, where

$$\Lambda_{\mathsf{T}} : \mathsf{M}(\mathbb{Z}) \to \mathsf{M}(\mathbb{Z})$$

is the “profinite completion” functor

$$\Lambda_{\mathsf{T}}(M) := \lim_{\leftarrow k} (M/k \cdot M).$$

Here k goes over the positive integers with their partial ordering by divisibilty.

See [Vs2] for details.

5. Dualizing Complexes in the Noncommutative Arithmetic Setting

To end the talk, let me sketch a conjectural strategy for proving existence of a *balanced dualizing complex* in the *arithmetic setting*, namely without a base field.

This strategy combines weak stability with some other noncommutative properties.

We can consider either a connected graded ring (like in [VdB]), or a complete semilocal ring (like in [WZ]).

I will talk about the complete case.

We work over a commutative base ring $\mathbb{K}$, and A is a noncommutative central $\mathbb{K}$-ring.

The ring $\mathbb{K}$ is noetherian, local and $\mathfrak{p}$-adically complete, where $\mathfrak{p} \subseteq \mathbb{K}$ is the maximal ideal.

The ring A is noetherian, semilocal, and $\mathfrak{a}$-adically complete, where $\mathfrak{a} \subseteq A$ is the Jacobson radical.

We further assume that A is flat over $\mathbb{K}$, and $A/\mathfrak{a}$ is a finite length $\mathbb{K}$-module.

As before, the flatness condition is probably not essential, but it greatly simplifies the discussion.

Here is a motivating example.

Example 5.1. Let G be a *compact p-adic Lie group*, and let $\mathbb{K}$ be either $\mathbb{F}_p$ or $\widehat{\mathbb{Z}}_p$. The *noncommutative Iwasawa algebra* $A := \mathbb{K}[[G]]$ has the required properties.

Inside $\mathsf{M}(A)$ we have the $\mathfrak{a}$-torsion class T, and inside $\mathsf{M}(A^{\mathrm{op}})$ we have the $\mathfrak{a}^{\mathrm{op}}$-torsion class T^{op}.

These torsion classes extend to bimodules as was explained before. So there are torsion classes

$$\mathsf{T}, \mathsf{T}^{\mathrm{op}} \subseteq \mathsf{M}(A^{\mathrm{en}}).$$

Let us define the ideal

$$\mathfrak{a}^{\mathrm{en}} := \mathfrak{a} \otimes_{\mathbb{K}} A^{\mathrm{op}} + A \otimes_{\mathbb{K}} \mathfrak{a}^{\mathrm{op}} \subseteq A^{\mathrm{en}}.$$

It is easy to see that the $\mathfrak{a}^{\mathrm{en}}$-torsion class T^{en} satisfies

$$\mathsf{T}^{\mathrm{en}} = \mathsf{T} \cap \mathsf{T}^{\mathrm{op}} \subseteq \mathsf{M}(A^{\mathrm{en}}).$$

The torsion functors on bimodules satisfy

$$\Gamma_{\mathsf{T}^{\mathrm{op}}} \circ \Gamma_{\mathsf{T}} = \Gamma_{\mathsf{T}} \circ \Gamma_{\mathsf{T}^{\mathrm{op}}} = \Gamma_{\mathsf{T}^{\mathrm{en}}}.$$

Definition 5.2. We say that a complex $M \in \mathsf{D}(A^{\mathrm{en}})$ has *symmetric derived* T-T^{op}*-torsion* if

$$\mathrm{R}^q\Gamma_{\mathsf{T}}(M) \in \mathsf{T}^{\mathrm{op}} \quad \text{and} \quad \mathrm{R}^q\Gamma_{\mathsf{T}^{\mathrm{op}}}(M) \in \mathsf{T}$$

for all q.

Theorem 5.3. *([VY1]) Under the assumptions above:*

1. The torsion classes T and T^{op} are stable, and the torsion class T^{en} is weakly stable.
2. Suppose $M \in \mathsf{D}^{+}(A^{\mathrm{en}})$ has symmetric derived T-T^{op}-torsion. Then the canonical morphisms

$$\mathrm{R}\Gamma_{\mathsf{T}}(M) \leftarrow \mathrm{R}\Gamma_{\mathsf{T}^{\mathrm{en}}}(M) \rightarrow \mathrm{R}\Gamma_{\mathsf{T}^{\mathrm{op}}}(M)$$

in $\mathsf{D}(A^{\mathrm{en}})$ are isomorphisms.

The next three definitions are following [Ye1] and [WZ].

Definition 5.4. A *noncommutative dualizing complex over A* is a complex $R \in \mathsf{D}^{\mathrm{b}}(A^{\mathrm{en}})$ with these properties:

- R has finite injective dimension on both sides.
- The cohomologies $\mathrm{H}^q(R)$ are finitely generated modules on both sides.
- The canonical morphisms

$$A \to \mathrm{RHom}_{A}(R, R) \quad \text{and} \quad A \to \mathrm{RHom}_{A^{\mathrm{op}}}(R, R)$$

in $\mathsf{D}(A^{\mathrm{en}})$ are isomorphisms.

Let $\mathbb{K}^*$ be an injective hull over $\mathbb{K}$ of the residue field $\mathbb{K}/\mathfrak{p}$.

Using it we define the A-bimodule

$$A^* := \operatorname{Hom}^{\text{cont}}_{\mathbb{K}}(A, \mathbb{K}^*).$$

It is an injective A-module on both sides.

We refer to A^* as a *noncommutative t-dualizing complex* over A.

Definition 5.5. A noncommutative dualizing complex R_A is said to be *balanced* if is has symmetric derived T-T^{op}-torsion, and there is an isomorphism

$$\beta : \mathrm{R}\Gamma_{\mathsf{T}^{\text{en}}}(R_A) \xrightarrow{\simeq} A^*$$

in $\mathsf{D}(A^{\text{en}})$.

A balanced dualizing complex (R_A, β) can be shown to be unique up to a unique isomorphism.

Definition 5.6. We say that A satisfies the *special χ condition* if the bimodule A has symmetric derived T-T^{op}-torsion, and the bimodules $\mathrm{R}^q\Gamma_{\mathsf{T}^{\text{en}}}(A)$ are artinian on both sides.

This is a special case of the χ condition of Artin and Zhang [AZ].

Conjecture 5.7. *Assume that the ring A also satisfies:*

- The special χ condition.
- The torsion classes T and T^{op} are finite dimensional.

Define the complexes

$$P_A := \mathrm{R}\Gamma_{\mathsf{T}^{\text{en}}}(A) \in \mathsf{D}(A^{\text{en}})$$

and

$$R_A := \operatorname{Hom}_{\mathbb{K}}(P_A, \mathbb{K}^*) \in \mathsf{D}(A^{\text{en}}).$$

Then R_A is a balanced dualizing complex over A.

We think that even more is true:

Conjecture 5.8. *With A as above, let $f : A \to B$ be a surjective ring homomorphism.*

Then the balanced dualizing complex R_B exists, and so does the balanced trace morphism

$$\operatorname{Tr}_{B/A} : R_B \to R_A$$

in $\mathsf{D}(A^{\text{en}})$.

The balanced trace morphism has this important property: the diagram

$$\begin{array}{ccc} \mathrm{R}\Gamma_{\mathsf{T}^{\text{en}}}(R_B) & \xrightarrow{\mathrm{R}\Gamma_{\mathsf{T}^{\text{en}}}(\operatorname{Tr}_{B/A})} & \mathrm{R}\Gamma_{\mathsf{T}^{\text{en}}}(R_A) \\ {\scriptstyle \beta_B}\downarrow & & \downarrow{\scriptstyle \beta_A} \\ B^* & \xrightarrow{f^*} & A^* \end{array}$$

in $\mathsf{D}(A^{\text{en}})$ is commutative.

We know that the Iwasawa algebra $A = \mathbb{K}[[G]]$ from Example 5.1 satisfies the assumptions of Conjectures 5.7 and 5.8.

The dualizing complex R_A from Conjecture 5.7 satisfies

$$R_A = \operatorname{Hom}_{\mathbb{K}}(P_A, \mathbb{K}^*) \cong \operatorname{Hom}_A(P_A, A^*) \cong \operatorname{Hom}_{A^{\text{op}}}(P_A, A^*). \tag{5.9}$$

There are three ways to interpret formula (5.9):

1. By definition R_A is the Matlis dual of the dedualizing complex P_A.
2. $R_A \cong G_{\mathsf{T}}(A^*)$, the derived completion of the t-dualizing complex A^* from the left side. Compare to Example 2.3.
3. $R_A \cong G_{\mathsf{T}^{\text{op}}}(A^*)$, the derived completion of A^* from the right side.

$\sim$ *END* $\sim$

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