Commutative DG Rings and their Derived Categories

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21 January 2020

1. Introduction

This talk is about *commutative DG rings*.

As usual, "DG" is an abbreviation for "differential graded".

The commutative DG rings I will talk about are what most people would call "commutative associative unital nonpositive cochain DG algebras". A precise definition will come a bit later.

There are two kinds of *derived categories* that are related to commutative DG rings.

First, fixing a commutative DG ring A, there is the *derived category* $\mathsf{D}(A)$ *of DG A-modules*.

Second, we can look at all commutative DG rings (over some fixed commutative base ring $\mathbb{K}$). They form a category $\mathsf{DGRng}^{\leq 0}_{\mathrm{sc}} / \mathbb{K}$.

By inverting the quasi-isomorphisms in $\mathsf{DGRng}^{\leq 0}_{\mathrm{sc}} / \mathbb{K}$ we obtain the *derived category* $\mathsf{D}(\mathsf{DGRng}^{\leq 0}_{\mathrm{sc}} / \mathbb{K})$ *of commutative DG $\mathbb{K}$-rings*.

I will discuss both these kinds of derived categories later, and explain how they interact.

But before getting into that, I want to say a few words about how I got interested in this story. It is because of *rigid dualizing complexes*.

This concept was introduced by Van den Bergh [VdB] in 1997, in the context of noncommutative ring theory.

Around 2005, James Zhang and I tried to use rigid dualizing complexes in commutative algebraic geometry.

Suppose $\mathbb{K}$ is a field and A is a finite type commutative $\mathbb{K}$-ring.

Given a complex $M \in \mathsf{D}(A)$, its *square* relative to $\mathbb{K}$ is the complex

$$\mathrm{Sq}_{A/\mathbb{K}}(M) := \mathrm{RHom}_{A \otimes_{\mathbb{K}} A}(A, M \otimes_{\mathbb{K}} M) \in \mathsf{D}(A). \tag{1.1}$$

This is a *quadratic functor*

$$\mathrm{Sq}_{A/\mathbb{K}} : \mathsf{D}(A) \to \mathsf{D}(A).$$

A *rigidifying isomorphism* for M is an isomorphisms

$$\rho : M \xrightarrow{\simeq} \mathrm{Sq}_{A/\mathbb{K}}(M)$$

in $\mathsf{D}(A)$.

A *rigid dualizing complex* is a pair (M, ρ), where M is a dualizing complex over A, in the sense of Grothendieck [RD], and ρ is a rigidifying isomorphism for M.

Van den Bergh proved that a rigid dualizing complex (M, ρ) exists, and it is unique up to isomorphism.

Later Zhang and I, in [YZ1], [YZ2] and [YZ3], proved that the rigid dualizing complex (M, ρ) is unique up to a unique rigid isomorphism, and furthermore it has several important functorial properties.

All this is explained in Chapter 13 of the new book [Ye3].

However, in algebraic geometry we do not want to restrict ourselves to working over a base field. So Zhang and I tried to extend the notion of rigid dualizing complex to this setting:

$\mathbb{K}$ *is a regular noetherian ring, and* A *is an essentially finite type* $\mathbb{K}$*-ring*.

A very important case is $\mathbb{K} = \mathbb{Z}$ of course.

In this general setting *we can't expect* A *to be flat over* $\mathbb{K}$.

Thus the squaring operation (1.1) takes on this much more complicated, and even mysterious, form:

$$\mathrm{Sq}_{A/\mathbb{K}}(M) := \mathrm{RHom}_{A \otimes^{\mathrm{L}}_{\mathbb{K}} A}(A, M \otimes^{\mathrm{L}}_{\mathbb{K}} M) \in \mathsf{D}(A). \tag{1.2}$$

Formula (1.2) raises these questions:

What is $A \otimes^{\mathrm{L}}_{\mathbb{K}} A$?

The object $M \otimes^{\mathrm{L}}_{\mathbb{K}} M$ belongs to but what is this derived category?

I will try to explain these things in my talk, as much as time will permit.

Unfortunately, Zhang and I made several errors in our papers [YZ2] and [YZ3]. They were due to our poor understanding of the theory of DG rings. In fact, there was no reliable published account of the theory of DG rings at that time.

The errors in the papers [YZ2] and [YZ3] were discovered, and very partially repaired, in [AILN].

A full correction of [YZ2] and [YZ3] is in the published paper [Ye2], and the papers in preparation [OSY1] and [OSY2], which are joint with Mattia Ornaghi and Saurabh Singh.

The application in algebraic geometry we are aiming for (in the papers in preparation [Ye5] and [Ye6]) is *Rigidity, Residues and Duality for Deligne-Mumford Stacks*. See the lecture notes [Ye7] for a sketch.

Before being able to repair my work with Zhang, I felt I had to understand DG rings (commutative and noncommutative).

At a certain stage it became apparent to me that the existing approaches to *derived commutative algebra*, that were mostly subordinate to *derived algebraic geometry* (DAG), and relied too much on abstract homotopy theory, were not sufficient.

To be specific: the general homotopical methods (Quillen model structures and their variants) were *too coarse for my needs*.

For instance, these methods do not permit the delicate manipulations and interactions between *commutative DG* A*-rings* and *noncommutative central DG* A*-rings*, for some fixed commutative DG $\mathbb{K}$-ring A.

This is the reason I began my detailed study of DG rings.

Later I observed that the homotopical methods are also problematic when working with sheaves of DG rings. This is the topic of my next lecture.

2. Commutative DG Rings

A *DG ring* is a graded ring

$$A = \bigoplus_{i \in \mathbb{Z}} A^i$$

with a differential d of degree 1, satisfying the graded Leibniz rule

$$\mathrm{d}(a \cdot b) = \mathrm{d}(a) \cdot b + (-1)^{i \cdot j} \cdot a \cdot \mathrm{d}(b) \tag{2.1}$$

for $a \in A^i$ and $b \in A^j$.

Most people would call A a *unital associative cochain DG algebra*.

A DG ring homomorphism $f : A \to B$ must respect multiplications, units, gradings and differentials.

Definition 2.2. A DG ring A is called *commutative* if it has these two properties:

- (Nonpositivity) $A = \bigoplus_{i \leq 0} A^i$.
- (Strong Commutativity) $b \cdot a = (-1)^{i \cdot j} \cdot a \cdot b$ for all $a \in A^i$ and $b \in A^j$, and $a \cdot a = 0$ if i is odd.

Let me give some prototypical examples of commutative DG rings.

Example 2.3. A commutative ring A is a commutative DG ring, concentrated in degree 0, and with a trivial differential.

Example 2.4. Let $A \to B$ and $A \to C$ be commutative DG ring homomorphisms. Then $B \otimes_A C$ is a commutative DG ring.

Example 2.5. Let A be a commutative ring, and let $a \in A$ be an element.

The *Koszul complex* $B := \mathrm{K}(A; a)$ is a commutative DG ring, concentrated in degrees $-1, 0$.

In degree 0 we have $B^0 = A$.

In degree -1 we have $B^{-1} = A \cdot x$, a free A-module with basis element x.

The operations are $x \cdot x = 0$ and $\mathrm{d}(x) = a$, extended A-linearly.

Example 2.6. Let A be a commutative ring, and let

$$\boldsymbol{a} = (a_1, \ldots, a_n)$$

be a sequence of elements in A.

The Koszul complex

$$\mathrm{K}(A; \boldsymbol{a}) := \mathrm{K}(A; a_1) \otimes_A \cdots \otimes_A \mathrm{K}(A; a)$$

is a commutative DG ring, concentrated in degrees $-n, \ldots, 0$.

Let us fix a nonzero commutative base ring $\mathbb{K}$ (e.g. a field of the ring of integers $\mathbb{Z}$).

A commutative DG $\mathbb{K}$-ring is a commutative DG ring A, equipped with a DG ring homomorphism $\mathbb{K} \to A$.

The commutative DG $\mathbb{K}$-rings form a category, that we denote by $\mathsf{DGRng}^{\leq 0}_{\mathrm{sc}} / \mathbb{K}$.

The next convention will hold until the end of Section 7:

Convention 2.7. *All DG rings are commutative DG $\mathbb{K}$-rings.*

We shall use the shorthand

$$\mathsf{DGRng} := \mathsf{DGRng}^{\leq 0}_{\mathrm{sc}} / \mathbb{K}.$$

In the next two slides I will mention a few properties of commutative DG rings.

Given a DG ring A, its *cohomology*

(2.8) $$\mathrm{H}(A) := \bigoplus_{i \leq 0} \mathrm{H}^i(A)$$

is a commutative graded ring.

A DG ring homomorphism $f : A \to B$ is *quasi-isomorphism* if the graded ring homomorphism

$$\mathrm{H}(f) : \mathrm{H}(A) \to \mathrm{H}(B)$$

is an isomorphism.

We say that a DG ring A is *cohomologically bounded* if the graded ring $\mathrm{H}(A)$ is bounded, i.e. $\mathrm{H}^i(A) = 0$ for $i \ll 0$.

The DG ring A is *cohomologically pseudo-noetherian* if the ring $\bar{A} := \mathrm{H}^0(A)$ is noetherian, and each $\mathrm{H}^i(A)$ is a finitely generated $\bar{A}$-module.

A DG ring homomorphism $f : A \to B$ is *cohomologically pseudo-finite* if the ring homomorphism $\bar{f} : \bar{A} \to \bar{B}$ is finite.

The DG ring A is called *Gorenstein* if it is cohomologically pseudo-noetherian, and the DG module A has finite injective dimension. See [Ye1] or [Ye3] for details.

Very recently Liran Shaul defined *Cohen-Macaulay DG rings*, see [Sh].

A DG ring homomorphism $f : A \to B$ is called *derived étale* if B has flat dimention 0 as a DG A-module (see [Ye3]), and the ring homomorphism $\bar{f} : \bar{A} \to \bar{B}$ is étale.

One of the approaches to DAG is to view *derived stacks* as “sheaves of higher groupoids” on the “site” DGRng with its “étale topology”. Making precise sense of this requires a lot of homotopical machinery, and this was done mainly by Bertrand Toën and his collaborators [To].

3. The Derived Category of DG Modules

Almost all details about the content of this section can be found in the book [Ye3].

Let A be a DG ring (commutative, see Convention 2.7).

A *DG A-module* is a graded A-module $M = \bigoplus_{i \in \mathbb{Z}} M^i$, with a differential d of degree 1, satisfying the graded Leibniz rule: the obvious variant of formula (2.1).

Example 3.1. If A is a ring, then a DG A-module is the same as a complex of A-modules.

There are certain standard operations that can be performed of DG A-modules.

Given DG A-modules M and N, their tensor product $M \otimes_A N$ is also a DG A-module.

There is also the Hom DG module

(3.2) $$\mathrm{Hom}_A(M, N) = \bigoplus_{i \in \mathbb{Z}} \mathrm{Hom}_A(M, N)^i,$$

where $\mathrm{Hom}_A(M, N)^i$ is the $\mathbb{K}$-module of degree i homomorphisms $\phi : M \to N$ that commute with the elements of A (in the graded sense).

The DG A-modules form the *DG category* $\mathsf{C}(A)$, in which the DG module of morphisms is

$$\mathrm{Hom}_{\mathsf{C}(A)}(M, N) := \mathrm{Hom}_A(M, N).$$

The *strict category of DG A-modules* is the category $\mathsf{C}_{\mathrm{str}}(A) \subseteq \mathsf{C}(A)$ on all the objects, and whose morphisms are

(3.3) $$\mathrm{Hom}_{\mathsf{C}_{\mathrm{str}}(A)}(M, N) := \mathrm{Z}^0\big(\mathrm{Hom}_A(M, N)\big).$$

In other words, $\phi : M \to N$ is a strict homomorphism if it has degree 0 and it respects the differentials.

It is easy to see that $\mathsf{C}_{\mathrm{str}}(A)$ is an abelian category.

Proceeding just like Grothendieck and Verdier [RD], one has the *homotopy category* of A. This is the triangulated category $\mathsf{K}(A)$, which has the same objects as $\mathsf{C}(A)$, and

$$\mathrm{Hom}_{\mathsf{K}(A)}(M,N) := \mathrm{H}^0\big(\mathrm{Hom}_A(M,N)\big). \tag{3.4}$$

There is an obvious full functor

$$\mathrm{P} : \mathsf{C}_{\mathrm{str}}(A) \to \mathsf{K}(A) \tag{3.5}$$

that's the identity on objects.

Let $\mathsf{S}(A)$ be the set of quasi-isomorphisms in $\mathsf{C}_{\mathrm{str}}(A)$. The *derived category* $\mathsf{D}(A)$ of A is the categorical localization of $\mathsf{C}_{\mathrm{str}}(A)$ w.r.t. $\mathsf{S}(A)$. Namely, the quasi-isomorphisms are inverted by formally introducing inverses.

There is the categorical localization functor

$$\mathrm{Q} : \mathsf{C}_{\mathrm{str}}(A) \to \mathsf{D}(A) \tag{3.6}$$

that's the identity on objects.

In general the categorical localization is a horrible construction. But not in this case...

The functor Q factors as follows:

$$\mathsf{C}_{\mathrm{str}}(A) \xrightarrow{\ \mathrm{P}\ } \mathsf{K}(A) \xrightarrow{\ \bar{\mathrm{Q}}\ } \mathsf{D}(A), \qquad \mathrm{Q} = \bar{\mathrm{Q}} \circ \mathrm{P} \tag{3.7}$$

An important theorem in [RD] says that the *functor* $\bar{\mathrm{Q}}$ *is a left and right Ore localization.*

This implies that every morphism ψ in $\mathsf{D}(A)$ can be written as a simple left or right fraction:

$$\psi = \mathrm{Q}(\sigma)^{-1} \circ \mathrm{Q}(\alpha) = \mathrm{Q}(\beta) \circ \mathrm{Q}(\tau)^{-1} \tag{3.8}$$

for suitable homomorphisms α, β in $\mathsf{C}_{\mathrm{str}}(A)$ and quasi-isomorphisms σ, τ.

Derived functors work like in the setting of Grothendieck and Verdier, but more generally of course.

Suppose B is another DG ring, and

$$F : \mathsf{C}(A) \to \mathsf{C}(B)$$

is a *DG functor*.

There is an induced *triangulated functor*

$$F : \mathsf{K}(A) \to \mathsf{K}(B).$$

In $\mathsf{K}(A)$ there are enough resolutions (K-projective and K-injective). Therefore the *left and right derived functors*

$$\mathrm{L}F, \mathrm{R}F : \mathsf{D}(A) \to \mathsf{D}(B)$$

of F exist.

DG bifunctors such as $(- \otimes_A -)$ and $\mathrm{Hom}_A(-,-)$ can also be derived, giving the *standard triangulated bifunctors*

$$(- \otimes^{\mathrm{L}}_A -) : \mathsf{D}(A) \times \mathsf{D}(A) \to \mathsf{D}(A) \tag{3.9}$$

and

$$\mathrm{RHom}_A(-,-) : \mathsf{D}(A)^{\mathrm{op}} \times \mathsf{D}(A) \to \mathsf{D}(A). \tag{3.10}$$

Many of the definitions and constructions for rings hold also for DG rings, with few modifications.

One of these is *dualizing DG modules* over A. The definition is a mild modification of the one in [RD]. See [Ye1] for details.

Existence and classification of dualizing DG A-modules is very similar to the case of a ring.

Moreover, the derived coreduction functor

$$\mathrm{RHom}_A(\bar{A}, -) : \mathsf{D}(A) \to \mathsf{D}(\bar{A})$$

induces an equivalence between the categories of dualizing DG modules.

Theorem 3.11. *Let $A \to B$ be a DG ring quasi-isomorphism. Then the following hold:*

1. The restriction functor
$$\mathrm{Rest}_{B/A} : \mathsf{D}(B) \to \mathsf{D}(A)$$
is an equivalence of triangulated categories, with quasi-inverse $B \otimes^{\mathrm{L}}_{A} (-)$.
2. The equivalence $\mathrm{Rest}_{B/A}$ respects the standard derived bifunctors (3.9) and (3.10).

The moral of Theorem 3.11 is that *homological algebra is invariant under DG ring quasi-isomorphisms*.

Thus, for instance, if $M \in \mathsf{D}(B)$ is a dualizing DG B-module, then $\mathrm{Rest}_{B/A}(M) \in \mathsf{D}(A)$ is a dualizing DG A-module, and vice versa.

4. The Derived Category of Commutative DG Rings

Recall that there is a base ring $\mathbb{K}$ in the background. All DG rings are commutative DG $\mathbb{K}$-rings. The category of these DG rings is denoted by DGRng. See Definition 2.2 and Convention 2.7.

Let $\mathsf{S}(\mathsf{DGRng})$ be the set of quasi-isomorphisms in DGRng. It is a multiplicatively closed set of morphisms.

Definition 4.1. The *derived category of commutative DG rings* is the category
$$\mathsf{D}(\mathsf{DGRng}) := \mathsf{DGRng}_{\mathsf{S}(\mathsf{DGRng})},$$
the categorical localization of DGRng w.r.t. the quasi-isomorphisms in it.

There is the categorical localization functor

$$\mathrm{Q} : \mathsf{DGRng} \to \mathsf{D}(\mathsf{DGRng}) \tag{4.2}$$

that's the identity on objects.

Here is an important question:

Question 4.3. *Does the categorical localization functor* Q *in (4.2) admit a calculus of fractions?*

Namely, can every morphism in $\mathsf{D}(\mathsf{DGRng})$ *be expressed as a simple left or right fraction of homomorphisms in* DGRng *?*

We shall see that the answer to Question 4.3 is yes, partially: every morphism $\mathsf{D}(\mathsf{DGRng})$ be expressed as a simple right fraction of homomorphisms in DGRng.

This is Theorem 6.6.

Before stating this theorem we need to introduce some concepts. This will be done in Sections 5 and 6.

I should pause here to remark on terminology.

Among homotopy theorists it is customary to use the name *homotopy category*, with notation $\mathsf{Ho}(-)$, for what I call the derived category, with notation $\mathsf{D}(-)$.

The choice of terminology by the homotopy theorists is unfortunate, in my opinion, since it creates a lot of confusion.

I think the terminology of Grothendieck and Verdier, as presented in [RD], provides a much better description of the mathematical picture.

Indeed, there is a genuine homotopy category $\mathsf{K}(\mathsf{DGRng})$. It sits inside the commutative diagram (6.5), just like the homotopy category $\mathsf{K}(A)$ sits inside diagram (3.8) .

5. Semi-Free DG Rings

Recall that we work in DGRng, the category of commutative DG $\mathbb{K}$-rings.

Given a DG ring A, let $A^{\natural}$ be the graded ring gotten by forgetting the differential.

By a *nonpositive graded set* we mean a set X, partitioned into $X = \coprod_{i \leq 0} X^i$. The elements of X^i are said to have degree i.

The *commutative polynomial ring* $\mathbb{K}[X]$ on a nonpositive graded set X is the graded ring generated by X, modulo the strong commutativity relations in Definition 2.2.

Definition 5.1. Let A be a DG ring. We say that A is a *semi-free DG ring* if there is an isomorphism of graded $\mathbb{K}$-rings

$$A^{\natural} \cong \mathbb{K}[X]$$

for some nonpositive graded set X.

Definition 5.2. Let A be a DG ring. A *semi-free DG ring resolution of* A is a quasi-isomorphism $\tilde{A} \to A$ from a semi-free DG ring $\tilde{A}$.

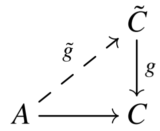
There are two important results about semi-free DG ring resolutions.

Theorem 5.3. *(Existence of Resolutions, [Ye1])*

Every $A \in$ DGRng *admits a semi-free resolution* $\tilde{A} \to A$.

Theorem 5.4. *(Lifting, [Ye1], [Ye4])*

Suppose we are given the solid diagram



in DGRng, *where A is semi-free and g is a quasi-isomorphism.*

Then there is a homomorphism $\tilde{g}$ *making the diagram commutative,.*

There are a couple of obvious questions that these theorems present:

Question 5.5. *How unique is the semi-free DG ring* $\tilde{A}$ *in Theorem 5.3 ?*

Question 5.6. *How unique is the lifting* $\tilde{g}$ *in Theorem 5.4 ?*

The answers will be given in Corollary 6.9.

⋄ ⋄ ⋄

Remark 5.7. All we do in Sections 5-7 over the base ring $\mathbb{K}$ can be done in a *relative manner*.

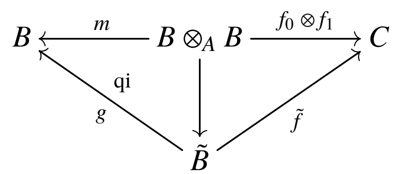
Namely we can fix $A \in$ DGRng, and consider the category DGRng$/A$ of DG A-rings.

Then we can talk about semi-free resolutions in DGRng$/A$, etc., and all the results will still hold.

6. The Quasi-Homotopy Relation

Definition 6.1. Let $f_0, f_1 : A \to B$ be homomorphisms in DGRng. A *homotopy from* f_0 *to* f_1 *in* DGRng is a commutative diagram

(γ)



in DGRng, where m is the multiplication homomorphism, and g is a quasi-isomorphism.

We denote this homotopy by $\gamma : f_0 \Rightarrow f_1$.

Definition 6.2. Let $f_0, f_1 : A \to B$ be homomorphisms in DGRng. A *quasi-homotopy from f_0 to f_1 in* DGRng consists of a quasi-isomorphism $g : \tilde{A} \to A$ in DGRng, and a homotopy

$$\gamma : f_0 \circ g \Rightarrow f_1 \circ g$$

in DGRng.

Here it is in a diagram, with $i = 0, 1$:

$$\tilde{A} \xrightarrow[\text{qi}]{g} A \xrightarrow{f_i} B, \qquad f_i \circ g : \tilde{A} \to B$$

Recall that a *congruence* on a category C is a collection of equivalence relations $\sim$ on the sets of morphisms $\mathrm{Hom}_{\mathsf{C}}(C_0, C_1)$, that defines a quotient category.

Theorem 6.3. *([Ye4]) The relation of quasi-homotopy is a congruence on the category* DGRng.

Definition 6.4. The *homotopy category of* DGRng is its quotient category modulo the quasi-homotopy congruence, and we denote it by $\mathsf{K}(\mathsf{DGRng})$.

In other words, there is a functor

$$\mathrm{P} : \mathsf{DGRng} \to \mathsf{K}(\mathsf{DGRng}),$$

which is the identity on objects, and surjective on morphisms.

It turns out that the categorical localization functor Q from (4.2) factors as follows:

(6.5) $$\mathsf{DGRng} \xrightarrow{\mathrm{P}} \mathsf{K}(\mathsf{DGRng}) \xrightarrow{\bar{\mathrm{Q}}} \mathsf{D}(\mathsf{DGRng}), \qquad \mathrm{Q} = \bar{\mathrm{Q}} \circ \mathrm{P}$$

Theorem 6.6. *([Ye4]) The functor $\bar{\mathrm{Q}}$ in (6.5) is a faithful right Ore localization of* $\mathsf{K}(\mathsf{DGRng})$*, with respect to the quasi-isomorphisms in it.*

This implies that every morphism u in $\mathsf{D}(\mathsf{DGRng})$ can be written as a simple right fraction:

(6.7) $$u = \mathrm{Q}(f) \circ \mathrm{Q}(g)^{-1}$$

for suitable homomorphisms f, g in DGRng with g a quasi-isomorphism.

We can say more when the source is semi-free.

Theorem 6.8. *([Ye4]) Let $A, B \in \mathsf{DGRng}$, and assume that A is semi-free. Then the function*

$$\bar{\mathrm{Q}} : \mathrm{Hom}_{\mathsf{K}(\mathsf{DGRng})}(A, B) \to \mathrm{Hom}_{\mathsf{D}(\mathsf{DGRng})}(A, B)$$

is bijective.

Corollary 6.9.

1. The semi-free DG ring $\tilde{A}$ in Question 5.5 is unique, up to a unique isomorphism in $\mathsf{K}(\mathsf{DGRng})$.
2. The lifting $\tilde{g}$ in Question 5.6 is unique in $\mathsf{K}(\mathsf{DGRng})$.

7. What Can Be Done with Commutative DG Rings

In this section we will see two nice features of commutative DG rings.

Let's fix $A \in \mathsf{DGRng}$, and consider the category DGRng/A of DG A-rings.

We know that given $B, C \in \mathsf{DGRng}/A$, their tensor product $B \otimes_A C$ is also in DGRng/A.

In this way we get a bifunctor

$$(7.1) \qquad (- \otimes_A -) : (\mathsf{DGRng}/A) \times (\mathsf{DGRng}/A) \to \mathsf{DGRng}/A.$$

Theorem 7.2. *The bifunctor $(- \otimes_A -)$ in (7.1) has a left derived bifunctor*

$$(- \otimes^{\mathrm{L}}_A -) : \mathsf{D}(\mathsf{DGRng}/A) \times \mathsf{D}(\mathsf{DGRng}/A) \to \mathsf{D}(\mathsf{DGRng}/A).$$

If either B of C is K-flat over A, then the canonical morphism

$$B \otimes^{\mathrm{L}}_A C \to B \otimes_A C$$

in $\mathsf{D}(\mathsf{DGRng}/A)$ is an isomorphism.

Let us denote by TrCat the 2-category of $\mathbb{K}$-linear triangulated categories.

The objects of TrCat are the $\mathbb{K}$-linear triangulated categories D.

The 1-morphisms are the triangulated functors $F : \mathsf{D} \to \mathsf{E}$.

And the 2-morphisms are the morphisms of triangulated functors $\eta : F \Rightarrow G$..

The next theorem shows how the derived category $\mathsf{D}(\mathsf{DGRng})$ and the derived categories $\mathsf{D}(A)$ are related. It is an upgrade of Theorem 3.11.

Theorem 7.3. *([Ye4]) There is a pseudofunctor*

$$\mathrm{Der} : \mathsf{D}(\mathsf{DGRng}) \to \mathsf{TrCat}$$

that on an object $A \in \mathsf{DGRng}$ it takes the value

$$\mathrm{Der}(A) = \mathsf{D}(A) \in \mathsf{TrCat},$$

and on homomorphism $f : A \to B$ in DGRng it takes the value

$$\mathrm{Der}(f) = B \otimes^{\mathrm{L}}_A (-) : \mathsf{D}(A) \to \mathsf{D}(B).$$

Theorem 7.2 explains, finally, what is the object $A \otimes^{\mathrm{L}}_{\mathbb{K}} A$ appearing in formula (1.2).

And Theorem 7.3 says – indirectly – what is the triangulated category $\mathsf{D}(A \otimes^{\mathrm{L}}_{\mathbb{K}} A)$.

But the squaring operation requires more! This will be addressed in the last section of the talk.

8. What Cannot Be Done with Commutative DG Rings

In this last section we need to include noncommutative (NC) DG rings in the discussion.

Therefore we must abandon the convenient shorthand DGRng that we have used since Convention 2.7, and revert to the full expression $\mathsf{DGRng}^{\leq 0}_{\mathrm{sc}}/\mathbb{K}$ for the commutative DG $\mathbb{K}$-rings.

We also must introduce a new sort of DG ring.

Definition 8.1. A *central DG $\mathbb{K}$-ring* is a DG ring A, possibly noncommutative, equipped with a central DG ring homomorphism $\mathbb{K} \to A$.

Saying that $\mathbb{K} \to A$ is central means that the image of every element of $\mathbb{K}$ commutes with all the elements of A.

The category of central DG $\mathbb{K}$-rings is denoted by $\mathsf{DGRng}/_{\mathrm{c}}\, \mathbb{K}$.

Because a commutative DG $\mathbb{K}$-ring A is also a central DG $\mathbb{K}$-ring, there is a fully faithful embedding

$$\mathsf{DGRng}^{\leq 0}_{\mathrm{sc}} / \mathbb{K} \to \mathsf{DGRng} /_{\mathrm{c}} \mathbb{K}. \tag{8.2}$$

The proof, in [Ye2], that the squaring operation $\mathrm{Sq}_{A/\mathbb{K}}(M)$ exists and has the expected properties, for $A \in \mathsf{DGRng}^{\leq 0}_{\mathrm{sc}} / \mathbb{K}$ and $M \in \mathsf{D}(A)$, relies on the embedding (8.2).

The way it is done is this: we define the *rectangle operation*

$$\mathrm{Rect}_{A/\mathbb{K}}(M_1, M_2) \in \mathsf{D}(\mathrm{Cent}(A)), \tag{8.3}$$

where $A \in \mathsf{DGRng} /_{\mathrm{c}} \mathbb{K}$, $M_1 \in \mathsf{D}(A)$, $M_2 \in \mathsf{D}(A^{\mathrm{op}})$, and $\mathrm{Cent}(A)$ is the center of A (in the graded sense).

The construction goes as follows. We choose a K-flat DG ring resolution $g : \tilde{A} \to A$; namely a quasi-isomorphism g in $\mathsf{DGRng} /_{\mathrm{c}} \mathbb{K}$, where $\tilde{A}$ is a K-flat DG $\mathbb{K}$-module.

We then define

$$\mathrm{Rect}_{A/\mathbb{K}}(M_1, M_2) := \mathrm{RHom}_{\tilde{A} \otimes_{\mathbb{K}} \tilde{A}^{\mathrm{op}}}(A, M_1 \otimes^{\mathrm{L}}_{\mathbb{K}} M_2) \in \mathsf{D}(\mathrm{Cent}(A)). \tag{8.4}$$

The proof that (8.4) does not depend on the DG ring resolution $g : \tilde{A} \to A$ uses special properties of *NC semi-free DG rings*.

Finally we specialize to $A \in \mathsf{DGRng}^{\leq 0}_{\mathrm{sc}} / \mathbb{K}$, for which $\mathrm{Cent}(A) = A$. Taking $M_1 = M_1 := M$, we get

$$\mathrm{Sq}_{A/\mathbb{K}}(M) := \mathrm{Rect}_{A/\mathbb{K}}(M, M) \in \mathsf{D}(A),$$

and this is independent on the DG ring resolution.

The last thing I wanted to say – but for lack of time I will just "tweet", is about *cotangent complexes*.

I have a feeling that to a homomorphism $A \to B$ of commutative DG $\mathbb{K}$-rings, the cotangent DG module $\mathrm{L}_{B/A} \in \mathsf{D}(B)$ can be calculated using *NC semi-free DG ring resolutions*.

But so far this is only science fiction...

~ *END* ~

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8. What Cannot Be Done with Commutative DG Rings

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Amnon Yekutieli (BGU) Commutative DG Rings 39 / 39