Rigid Dualizing Complexes via Differential Graded Algebras

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Here is the plan of my lecture:

1. Dualizing Complexes: Overview

2. Rigid Complexes and DG Algebras

3. Properties of Rigid Complexes

4. Rigid Dualizing Complexes

5. Focus: Traces for Differential Forms

1 Dualizing Complexes: Overview

Let A be a noetherian commutative ring. Denote by $\mathsf{D}^{\mathrm{b}}_{\mathrm{f}}(\mathsf{Mod}\, A)$ the derived category of bounded complexes of A-modules with finitely generated cohomology modules.

Definition 1. (Grothendieck [RD])
A *dualizing complex* over A is a complex

$$R \in \mathsf{D}^{\mathrm{b}}_{\mathrm{f}}(\mathsf{Mod}\, A)$$

satisfying the two conditions:

(i) R has finite injective dimension.

(ii) The canonical morphism

$$A \to \mathrm{RHom}_A(R, R)$$

is an isomorphism.

Example 2. If $\mathbb{K}$ is a regular noetherian ring of finite Krull dimension (say a field, or the ring of integers $\mathbb{Z}$) then $R := \mathbb{K} \in \mathsf{D}^{\mathrm{b}}_{\mathrm{f}}(\mathsf{Mod}\,\mathbb{K})$ is a dualizing complex.

Dualizing complexes over commutative rings are part of Grothendieck's duality theory in algebraic geometry, which was developed in [RD]. This duality theory deals with dualizing complexes on schemes and relations between them.

Dualizing sheaves and complexes have for years been key ingredients in classification of varieties, construction of moduli spaces, resolution of singularities and arithmetic geometry.

In this lecture I will explain a new approach to dualizing complexes over commutative rings, due to James Zhang and myself (see [YZ4] and [YZ5]). Specifically, I'll talk about existence and uniqueness of *rigid dualizing complexes*.

The purpose of rigidity is to eliminate automorphisms. This idea is familiar from other areas of algebraic geometry (like level structures on elliptic curves, or marked points on higher genus curves).

In our context we want to endow a dualizing complex with added structure, i.e. a rigidifying isomorphism, that will not only get rid of all nontrivial automorphisms, but moreover will make the dualizing complex functorial.

In a sequel paper [Ye2] we use the technique of *perverse coherent sheaves* to construct rigid dualizing complexes on schemes, and we reproduce almost all of the geometric Grothendieck duality theory. But that's a subject for a separate lecture.

Related work in noncommutative algebraic geometry (where rigid dualizing complexes were first introduced) can be found in [VdB, YZ1, YZ2, YZ3]. Background on commutative duality theory, and recent work can be found in the other references.

2 Rigid Complexes and DG Algebras

Let me start with a discussion of rigidity for algebras over a field. By default all rings considered are commutative.

Suppose A is a field, B is an A-algebra, and $M \in \mathsf{D}(\mathsf{Mod}\, B)$. According to Van den Bergh [VdB] a *rigidifying isomorphism* for M is an isomorphism

$$\rho : M \xrightarrow{\simeq} \mathrm{RHom}_{B \otimes_A B}(B, M \otimes_A M) \tag{1}$$

in $\mathsf{D}(\mathsf{Mod}\, B)$.

Now suppose A is any ring. Then formula (1) does not make sense: instead of $M \otimes_A M$ we must take the derived tensor product $M \otimes^{\mathrm{L}}_A M$; but then there is no obvious way to make $M \otimes^{\mathrm{L}}_A M$ into a complex of $B \otimes_A B$ -modules. The problem is torsion: B might fail to be a flat A-algebra.

This is where *differential graded algebras* (DG algebras) enter the picture.

DG algebras have been around for many years, but these days they are again in vogue. The reason is perhaps interest in deformation theory and homological mirror symmetry. Anyhow, for us they are an indispensable tool.

A DG algebra is a graded ring $\tilde{A} = \bigoplus_{i \in \mathbb{Z}} \tilde{A}^i$, together with a graded derivation $\mathrm{d} : \tilde{A} \to \tilde{A}$ of degree 1, satisfying $\mathrm{d} \circ \mathrm{d} = 0$.

A DG algebra quasi-isomorphism is a homomorphism $f : \tilde{A} \to \tilde{B}$ respecting degrees, multiplications and differentials, and such that $\mathrm{H}(f) : \mathrm{H}\tilde{A} \to \mathrm{H}\tilde{B}$ is an isomorphism (of graded algebras).

We shall only consider *super-commutative non-positive* DG algebras.

Super-commutative means that $ab = (-1)^{ij} ba$ and $c^2 = 0$ for all $a \in \tilde{A}^i$, $b \in \tilde{A}^j$ and $c \in \tilde{A}^{2i+1}$. Non-positive means that $\tilde{A} = \bigoplus_{i \leq 0} \tilde{A}^i$.

We view a ring A as a DG algebra concentrated in degree 0. Given a DG algebra homomorphism $A \to \tilde{A}$ we say that $\tilde{A}$ is a DG A-algebra.

Suppose we are given a set X of variables, partitioned into $X = \coprod_{i \leq 0} X_i$. The variables in X_i are said to be of degree i. Let $X_{\text{ev}} := \coprod_{i \leq 0} X_{2i}$ and $X_{\text{odd}} := \coprod_{i \leq 0} X_{2i-1}$. The super-polynomial algebra over a ring A in the set of variables X is

$$A[X] := A[X_{\text{ev}}] \otimes_A A[X_{\text{odd}}],$$

where $A[X_{\text{ev}}]$ is a polynomial algebra and $A[X_{\text{odd}}]$ is an exterior algebra.

Let A be a ring. A *semi-free* DG A-algebra is a DG A-algebra $\tilde{A}$, such that after forgetting the differential $\tilde{A}$ is isomorphic, as graded A-algebra, to a super-polynomial algebra $A[X]$ for some graded set $X = \coprod_{i \leq 0} X_i$.

Definition 3. Let A be a ring and B an A-algebra. A *semi-free DG algebra resolution of B relative to A* is a quasi-isomorphism $\tilde{B} \to B$ of DG A-algebras, where $\tilde{B}$ is a semi-free DG A-algebra.

Such resolutions always exist, and they are unique up to quasi-isomorphism.

Example 4. Take $A := \mathbb{Z}$ and $B = \mathbb{Z}/(6)$. Define $\tilde{B}$ to be the super-polynomial algebra $\mathbb{Z}[\xi]$ on the variable ξ of degree -1. So $\tilde{B} = \mathbb{Z} \oplus \mathbb{Z}\xi$ as free $\mathbb{Z}$-module, and $\xi^2 = 0$. Let $\mathrm{d}(\xi) := 6$. Then $\tilde{B} \to \mathbb{Z}/(6)$ is a semi-free DG algebra resolution of $\mathbb{Z}/(6)$ relative to $\mathbb{Z}$.

For a DG algebra A one has the category $\mathsf{DGMod}\,\tilde{A}$ of DG $\tilde{A}$-modules. It is analogous to the category of complexes of modules over a ring, and by a similar process of inverting quasi-isomorphisms we obtain the derived category $\tilde{\mathsf{D}}(\mathsf{DGMod}\,\tilde{A})$; see [Ke], [Hi].

For a ring A (a DG algebra concentrated in degree 0) we have $\tilde{\mathsf{D}}(\mathsf{DGMod}\,A) = \mathsf{D}(\mathsf{Mod}\,A)$, the usual derived category.

It is possible to derive functors of DG modules, again in analogy to $\mathsf{D}(\mathsf{Mod}\,A)$.

An added feature is that for a quasi-isomorphism $\tilde{A} \to \tilde{B}$ the restriction of scalars functor

$$\tilde{\mathsf{D}}(\mathsf{DGMod}\,\tilde{B}) \to \tilde{\mathsf{D}}(\mathsf{DGMod}\,\tilde{A})$$

is an equivalence.

Getting back to our original problem, suppose A is a ring and B is an A-algebra. Choose a semi-free DG algebra resolution $\tilde{B} \to B$ relative to A. For $M \in \mathsf{D}(\mathsf{Mod}\, B)$ define

$$\operatorname{Sq}_{B/A} M := \operatorname{RHom}_{\tilde{B} \otimes_A \tilde{B}}(B, M \otimes^{\mathrm{L}}_A M)$$

in $\mathsf{D}(\mathsf{Mod}\, B)$.

Theorem 5. ([YZ4]) *The functor*

$$\operatorname{Sq}_{B/A} : \mathsf{D}(\mathsf{Mod}\, B) \to \mathsf{D}(\mathsf{Mod}\, B)$$

is independent of the resolution $\tilde{B} \to B$*.*

The functor $\operatorname{Sq}_{B/A}$, called the *squaring operation*, is nonlinear. In fact, given a morphism $\phi : M \to M$ in $\mathsf{D}(\mathsf{Mod}\, B)$ and an element $b \in B$ one has

$$\operatorname{Sq}_{B/A}(b\phi) = b^2 \operatorname{Sq}_{B/A}(\phi) \tag{2}$$

in $\operatorname{Hom}_{\mathsf{D}(\mathsf{Mod}\, B)}(\operatorname{Sq}_{B/A} M, \operatorname{Sq}_{B/A} M)$.

Definition 6. Let B be a noetherian A-algebra, and let M be a complex in $\mathsf{D}^{\mathrm{b}}_{\mathrm{f}}(\mathsf{Mod}\, B)$ that has finite flat dimension over A. Assume

$$\rho : M \xrightarrow{\simeq} \mathrm{Sq}_{B/A}\, M$$

is an isomorphism in $\mathsf{D}(\mathsf{Mod}\, B)$. Then the pair (M, ρ) is called a *rigid complex over B relative to A*.

Definition 7. Say (M, ρ) and (N, σ) are rigid complexes over B relative to A. A morphism $\phi : M \to N$ in $\mathsf{D}(\mathsf{Mod}\, B)$ is called a *rigid morphism relative to A* if the diagram

$$\begin{array}{ccc} M & \xrightarrow{\rho} & \mathrm{Sq}_{B/A}\, M \\ {\scriptstyle \phi}\downarrow & & \downarrow{\scriptstyle \mathrm{Sq}_{B/A}(\phi)} \\ N & \xrightarrow{\sigma} & \mathrm{Sq}_{B/A}\, N \end{array}$$

is commutative.

We denote by $\mathsf{D}^{\mathrm{b}}_{\mathrm{f}}(\mathsf{Mod}\, B)_{\mathrm{rig}/A}$ the category of rigid complexes over B relative to A.

Example 8. Take $M = B := A$. Then

$$\mathrm{Sq}_{A/A}\, A = \mathrm{RHom}_{A \otimes_A A}(A, A \otimes_A A) = A,$$

and we interpret this as the tautological rigidifying isomorphism $\rho^{\mathrm{tau}} : A \xrightarrow{\simeq} \mathrm{Sq}_{A/A}\, A$. The *tautological rigid complex* is

$$(A, \rho^{\mathrm{tau}}) \in \mathsf{D}^{\mathrm{b}}_{\mathrm{f}}(\mathsf{Mod}\, A)_{\mathrm{rig}/A}.$$

3 Properties of Rigid Complexes

The first property of rigid complexes explains their name.

Theorem 9. ([YZ4]) *Let A be a ring, B a noetherian A-algebra, and*

$$(M, \rho) \in \mathsf{D}^{\mathrm{b}}_{\mathrm{f}}(\mathsf{Mod}\, B)_{\mathrm{rig}/A}.$$

Assume the canonical homomorphism

$$B \to \mathrm{Hom}_{\mathsf{D}(\mathsf{Mod}\, B)}(M, M)$$

is bijective. Then the only automorphism of (M, ρ) in $\mathsf{D}^{\mathrm{b}}_{\mathrm{f}}(\mathsf{Mod}\, B)_{\mathrm{rig}/A}$ is the identity $\mathbf{1}_M$.

The proof is very easy: an automorphism ϕ of M has to be of the form $\phi = b\mathbf{1}_M$ for some invertible element $b \in B$. If ϕ is rigid then $b = b^2$ (cf. formula (2)), and hence $b = 1$.

We find it convenient to denote ring homomorphisms by f^* etc. Thus a ring homomorphism $f^* : A \to B$ corresponds to the morphism of schemes $f : \operatorname{Spec} B \to \operatorname{Spec} A$.

Let A be a noetherian ring. Recall that an A-algebra B is called essentially finite type if it is a localization of some finitely generated A-algebra.

We say that B is *essentially smooth* (resp. *essentially étale*) over A if it is essentially finite type and formally smooth (resp. formally étale).

Example 10. If A' is a localization of A then $A \to A'$ is essentially étale. If $B = A[t_1, \ldots, t_n]$ is a polynomial algebra then $A \to B$ is smooth, and hence also essentially smooth.

Let A be a noetherian ring and $f^* : A \to B$ an essentially smooth homomorphism. Then $\Omega^1_{B/A}$ is a finitely generated projective B-module.

Let

$$\operatorname{Spec} B = \coprod_i \operatorname{Spec} B_i$$

be the decomposition into connected components, and for every i let n_i be the rank of $\Omega^1_{B_i/A}$. We define a functor

$$f^\sharp : \mathsf{D}(\mathsf{Mod}\, A) \to \mathsf{D}(\mathsf{Mod}\, B)$$

by

$$f^\sharp M := \bigoplus_i \Omega^{n_i}_{B_i/A} \otimes_A M.$$

Recall that a ring homomorphism $f^* : A \to B$ is called finite if B is a finitely generated A-module. Given such a finite homomorphism we define a functor

$$f^\flat : \mathsf{D}(\mathsf{Mod}\, A) \to \mathsf{D}(\mathsf{Mod}\, B)$$

by

$$f^\flat M := \mathrm{RHom}_A(B, M).$$

Theorem 11. ([YZ4]) *Let A be a noetherian ring, let B, C be essentially finite type A-algebras, let $f^* : B \to C$ be an A-algebra homomorphism, and let*

$$(M, \rho) \in \mathsf{D}^{\mathrm{b}}_{\mathrm{f}}(\mathsf{Mod}\, B)_{\mathrm{rig}/A}.$$

(1) *If f^* is finite and $f^\flat M$ has finite flat dimension over A, then $f^\flat M$ has an induced rigidifying isomorphism*

$$f^\flat(\rho) : f^\flat M \xrightarrow{\simeq} \mathrm{Sq}_{C/A}\, f^\flat M.$$

(2) *If f^* is essentially smooth then $f^\sharp M$ has an induced rigidifying isomorphism*

$$f^\sharp(\rho) : f^\sharp M \xrightarrow{\simeq} \mathrm{Sq}_{C/A}\, f^\sharp M.$$

The idea behind (2) is explained in the next example.

Example 12. Suppose $A = B$, and C is smooth over B of relative dimension n. It's well known that there are canonical isomorphisms

$$\mathrm{Ext}^i_{C \otimes_B C}(C, \Omega^n_{C/B} \otimes_B \Omega^n_{C/B}) \cong \begin{cases} \Omega^n_{C/B} & \text{if } i = n \\ 0 & \text{if } i \neq n. \end{cases}$$

These are interpreted as a rigidifying isomorphism

$$\rho : \Omega^n_{C/B}[n] \xrightarrow{\simeq} \mathrm{Sq}_{C/B}\, \Omega^n_{C/B}[n].$$

Then

$$(\Omega^n_{C/B}[n], \rho) = f^\sharp(B, \rho^{\mathrm{tau}})$$

as rigid complexes relative to B, where (B, ρ^{tau}) is the tautological rigid complex.

4 Rigid Dualizing Complexes

From now on $\mathbb{K}$ is a fixed base ring, which is noetherian regular of finite Krull dimension. Let us denote by $\mathsf{EFTAlg}\,/\mathbb{K}$ the category of essentially finite type $\mathbb{K}$-algebras.

Definition 13. A *rigid dualizing complex* over A relative to $\mathbb{K}$ is a rigid complex (R_A, ρ_A) such that R_A is a dualizing complex.

Theorem 14. ([YZ5]) *Let $\mathbb{K}$ be a regular finite dimensional noetherian ring, and let A be an essentially finite type $\mathbb{K}$-algebra.*

(1) *The algebra A has a rigid dualizing complex (R_A, ρ_A), which is unique up to a unique rigid isomorphism.*

(2) *Given a finite homomorphism $f^* : A \to B$, there is a unique rigid isomorphism*

$$f^\flat(R_A, \rho_A) \xrightarrow{\simeq} (R_B, \rho_B).$$

(3) *Given an essentially smooth homomorphism $f^* : A \to B$, there is a unique rigid isomorphism*

$$f^\sharp(R_A, \rho_A) \xrightarrow{\simeq} (R_B, \rho_B).$$

Here is how the rigid dualizing complex (R_A, ρ_A) is obtained. We begin with the tautological rigid complex

$$(\mathbb{K}, \rho^{\mathrm{tau}}) \in \mathsf{D}^{\mathrm{b}}_{\mathrm{f}}(\mathsf{Mod}\,\mathbb{K})_{\mathrm{rig}/\mathbb{K}}.$$

Now the structural homomorphism $\mathbb{K} \to A$ can be factored into

$$\mathbb{K} \xrightarrow{f^*} B \xrightarrow{g^*} C \xrightarrow{h^*} A,$$

where f^* is smooth (B is a polynomial algebra over $\mathbb{K}$); g^* is finite (a surjection); and h^* is also smooth (a localization).

Then

$$(R_A, \rho_A) := h^{\sharp}\, g^{\flat}\, f^{\sharp}(\mathbb{K}, \rho^{\mathrm{tau}}) \in \mathsf{D}^{\mathrm{b}}_{\mathrm{f}}(\mathsf{Mod}\, A)_{\mathrm{rig}/\mathbb{K}}.$$

Definition 15. Given a homomorphism $f^* : A \to B$ in $\mathsf{EFTAlg} /\mathbb{K}$ define the *twisted inverse image functor*

$$f^! : \mathsf{D}^+_{\mathrm{f}}(\mathsf{Mod}\, A) \to \mathsf{D}^+_{\mathrm{f}}(\mathsf{Mod}\, B)$$

by the formula

$$f^! M := \mathrm{RHom}_B\big(B \otimes^{\mathrm{L}}_A \mathrm{RHom}_A(M, R_A), R_B\big).$$

Let Cat be the 2-category of all categories. The next result is easy to prove:

Proposition 16. *The assignment $f^* \mapsto f^!$ is a 2-functor from $\mathsf{EFTAlg} /\mathbb{K}$ to Cat. Namely for composable homomorphisms $A \xrightarrow{f^*} B \xrightarrow{g^*} C$ in $\mathsf{EFTAlg} /\mathbb{K}$ one has an isomorphism*

$$\phi_{f,g} : (f \circ g)^! \xrightarrow{\simeq} g^! \circ f^!,$$

and the $\phi_{f,g}$ satisfy a suitable compatibility condition.

Here is an important corollary of Theorem 14:

Corollary 17. *Let $f^* : A \to B$ be a homomorphism in* $\mathsf{EFTAlg} / \mathbb{K}$*. Suppose f^* is finite (resp. essentially smooth). Then there is an isomorphism $\psi^{\flat}_f : f^{\flat} \xrightarrow{\simeq} f^!$ (resp. $\psi^{\sharp}_f : f^{\sharp} \xrightarrow{\simeq} f^!$) of functors*

$$\mathsf{D}^+_{\mathrm{f}}(\mathsf{Mod}\, A) \to \mathsf{D}^+_{\mathrm{f}}(\mathsf{Mod}\, B).$$

The isomorphisms $\psi^{\flat}_f$ (resp. $\psi^{\sharp}_f$) are 2-functorial for finite (resp. essentially smooth) homomorphisms.

The next result is surprising. We still do not understand its significance.

Theorem 18. ([YZ5]) *Let* $A \in \mathsf{EFTAlg}\,/\mathbb{K}$*, and assume* $\operatorname{Spec} A$ *is connected and nonempty. Then there is exactly one (up to isomorphism) nonzero object in* $\mathsf{D}^{\mathrm{b}}_{\mathrm{f}}(\mathsf{Mod}\,A)_{\mathrm{rig}/\mathbb{K}}$*, namely the rigid dualizing complex* R_A*.*

5 Focus: Traces for Differential Forms

One of the most useful features of Grothendieck duality theory is that it gives rise to traces of differential forms. Such traces are usually quite hard to construct directly (cf. [Li] and [Hu]).

From Corollary 17 we immediately get:

Theorem 19. ([YZ5]) *Suppose* $A \to B \to C$ *are homomorphisms in* $\mathsf{EFTAlg}\,/\mathbb{K}$*, with* $A \to B$ *and* $A \to C$ *essentially smooth of relative dimension* n*, and* $B \to C$ *finite. Then there is a nondegenerate trace map*

$$\mathrm{Tr}_{C/B/A} : \Omega^n_{C/A} \to \Omega^n_{B/A}.$$

The trace maps $\mathrm{Tr}_{-/-/A}$ *are functorial for such finite homomorphisms* $B \to C$*.*

The trace map can actually be characterized using rigidity (cf. Example 12).

It turns out that in the situation of the theorem the homomorphism $B \to C$ is flat. The proof of this nice little fact will be published later.

For a finite flat homomorphism $B \to C$ let $\mathrm{tr}_{C/B} : C \to B$ be the usual trace map, i.e. $\mathrm{tr}_{C/B}(c) \in B$ is the trace of the operator c acting of the locally free B-module C.

Let me end with a result which shows the trace map has the expected behavior:

Proposition 20. *In the setup of Theorem* 19, *assume A is reduced. Then for any $\beta \in \Omega^n_{B/A}$ and $c \in C$ one has*

$$\mathrm{Tr}_{C/B/A}(c\beta) = \mathrm{tr}_{C/B}(c) \cdot \beta \in \Omega^n_{B/A}.$$

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