

Dualizing Complexes over Noncommutative Rings

Lecture Notes ¹

Amnon Yekutieli

Ben Gurion University, ISRAEL

<http://www.math.bgu.ac.il/~amyekut>

Here is the plan of my lecture:

1. Notation, and Review of Derived Categories
2. Dualizing Complexes
3. Existence of Dualizing Complexes
4. The Auslander Condition
5. Classification of Dualizing Complexes
6. Applications in Ring Theory

There will be a second talk about the geometric aspects of noncommutative duality.

Most of the work is joint with James Zhang (UW, Seattle).

1. NOTATION, AND REVIEW OF DERIVED CATEGORIES

Let A be a ring. We denote by $\text{Mod } A$ the category of left A -modules.

The objects of the derived category $D(\text{Mod } A)$ are complexes of A -modules

$$M = (\cdots \rightarrow M^{-1} \rightarrow M^0 \rightarrow M^1 \rightarrow \cdots).$$

Recall that a homomorphism of complexes $\phi : M \rightarrow N$ is a quasi-isomorphism if $H^i(\phi) : H^i M \rightarrow H^i N$ is an isomorphism for all i . The morphisms $\psi : M \rightarrow N$ in $D(\text{Mod } A)$ are of the form $\psi = \phi_2^{-1} \circ \phi_1$ where $\phi_1 : M \rightarrow L$ is a homomorphism of complexes and $\phi_2 : N \rightarrow L$ is a quasi-isomorphism.

There is a full embedding

$$\text{Mod } A \hookrightarrow D(\text{Mod } A)$$

which is gotten by viewing a module M as a complex concentrated in degree 0.

Of utmost importance for us is the derived functor RHom . Given complexes $M, N \in D(\text{Mod } A)$ there is a complex

$$\text{RHom}_A(M, N) \in D(\text{Mod } \mathbb{Z})$$

depending functorially on M and N . If N happens to be an A -bimodule then

$$\text{RHom}_A(M, N) \in D(\text{Mod } A^{\text{op}}),$$

where A^{op} is the opposite ring. There's a functorial isomorphism

$$H^i \text{RHom}_A(M, N) \cong \text{Hom}_{D(\text{Mod } A)}(M, N[i])$$

where $N[i]$ is the shifted complex. If $M, N \in \text{Mod } A$ then we recover the familiar Exts:

$$H^i \text{RHom}_A(M, N) = \text{Ext}_A^i(M, N).$$

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2. DUALIZING COMPLEXES

Dualizing complexes on (commutative) schemes were introduced by Grothendieck in the 1960's, in the book [RD]. Let us recall the definition of a dualizing complex over a commutative noetherian ring A . It is a complex $R \in D_f^b(\text{Mod } A)$ such that the contravariant functor

$$\text{RHom}_A(-, R) : D_f^b(\text{Mod } A) \rightarrow D_f^b(\text{Mod } A)$$

is a duality (i.e. a contravariant equivalence). (I am omitting some details.) Here $D_f^b(\text{Mod } A)$ is the derived category of bounded complexes with finitely generated cohomology modules.

Example 2.1. Let \mathbb{K} be a field. Then the complex $R := \mathbb{K}$ is a dualizing complex over \mathbb{K} . The duality $\text{RHom}_{\mathbb{K}}(-, \mathbb{K})$ extends the usual duality of linear algebra.

So far for the classical commutative picture. From now on \mathbb{K} will be a field, and A will be a noetherian, unital, associative \mathbb{K} -algebra (not necessarily commutative). We shall write $A^e := A \otimes_{\mathbb{K}} A^{\text{op}}$, where A^{op} is the opposite ring. So $\text{Mod } A^e$ is the category of A -bimodules.

Definition 2.2. ([Ye1]) A complex $R \in D^b(\text{Mod } A^e)$ is called *dualizing* if the functor

$$\text{RHom}_A(-, R) : D_f^b(\text{Mod } A) \rightarrow D_f^b(\text{Mod } A^{\text{op}})$$

is a duality, with adjoint $\text{RHom}_{A^{\text{op}}}(-, R)$. (Again I'm suppressing some details.)

Example 2.3. The complex $R := A$ is a dualizing complex over A iff A is a Gorenstein ring (i.e. A has finite injective dimension as left and right module over itself).

There is a graded version of dualizing complex. Suppose A is a connected graded algebra, namely $A = \bigoplus_{i \geq 0} A_i$, with $A_0 = \mathbb{K}$ and each A_i a finitely generated \mathbb{K} -module. Consider the category $\text{GrMod } A$ of graded left A -modules. Similarly to Definition 2.2 we may define a graded dualizing complex $R \in D^b(\text{GrMod } A^e)$.

The augmentation ideal of A is denoted by \mathfrak{m} , and the left (resp. right) \mathfrak{m} -torsion functor is denoted by $\Gamma_{\mathfrak{m}}$ (resp. $\Gamma_{\mathfrak{m}^{\text{op}}}$). We let $A^* := \text{Hom}_{\mathbb{K}}^{\text{gr}}(A, \mathbb{K})$, the graded dual of A .

Definition 2.4. ([Ye1]) Let A be a connected graded \mathbb{K} -algebra. A graded dualizing complex R is called *balanced* if

$$\text{R}\Gamma_{\mathfrak{m}} R \cong \text{R}\Gamma_{\mathfrak{m}^{\text{op}}} R \cong A^*$$

in $D^b(\text{GrMod } A^e)$.

It is known that a balanced dualizing complex is unique up to isomorphism.

Again A is any noetherian \mathbb{K} -algebra (not graded). Van den Bergh discovered the following condition on a dualizing complex R that turns out to be extremely powerful.

Definition 2.5. ([VdB]) Let R be a dualizing complex over A . Suppose there is an isomorphism

$$\rho : R \xrightarrow{\sim} \text{RHom}_{A^e}(A, R \otimes_{\mathbb{K}} R)$$

in $D(\text{Mod } A^e)$. Then R is called a *rigid dualizing complex* and ρ is a *rigidifying isomorphism*.

Theorem 2.6. ([VdB], [YZ1]) *A rigid dualizing complex (R, ρ) is unique up to a unique isomorphism in $D(\text{Mod } A^e)$.*

Example 2.7. If A is a commutative finitely generated \mathbb{K} -algebra, $X := \text{Spec } A$ and $\pi : X \rightarrow \text{Spec } \mathbb{K}$ is the structural morphism, then the dualizing complex $R := R\Gamma(X, \pi^! \mathbb{K})$ from [RD] is rigid.

Example 2.8. If A is finite over \mathbb{K} then the bimodule $A^* := \text{Hom}_{\mathbb{K}}(A, \mathbb{K})$ is a rigid dualizing complex over A .

3. EXISTENCE OF DUALIZING COMPLEXES

The question of existence of rigid dualizing complexes is quite hard. The best existence criterion we know is due to Van den Bergh.

Theorem 3.1. ([VdB]) *Suppose A admits a nonnegative exhaustive filtration $F = \{F_i A\}_{i \in \mathbb{Z}}$ such that the graded algebra $\bar{A} := \text{gr}^F A$ is a connected graded, commutative, finitely generated \mathbb{K} -algebra. Then A has a rigid dualizing complex.*

Here is an outline of the proof. Let

$$\tilde{A} := \bigoplus_i (F_i A) t^i \subset A[t]$$

be the Rees algebra, where t is a central indeterminate of degree 1. So $\bar{A} \cong \tilde{A}/(t)$ and $A \cong \tilde{A}/(t-1)$.

Since \bar{A} is commutative it follows that \tilde{A} satisfies the χ condition of [AZ]. This implies that the local duality functor $\tilde{M} \mapsto (R\Gamma_{\tilde{m}} \tilde{M})^*$ is represented by a balanced dualizing complex \tilde{R} over \tilde{A} . Then

$$R_A := A \otimes_{\tilde{A}} \tilde{R}[-1] \otimes_{\tilde{A}} A$$

is a rigid dualizing complex over A .

One should think of the filtration F as a “compactification of $\text{Spec } A$ ”. Indeed if A is commutative then $\text{Proj } \tilde{A}$ is a projective \mathbb{K} -scheme, $\{t = 0\}$ is an ample divisor, and its complement is isomorphic to $\text{Spec } A$.

In practice often an algebra A comes equipped with a filtration G that satisfies the conditions of the next definition, but is not connected (i.e. $\text{gr}^G A$ is not a connected graded \mathbb{K} -algebra).

Definition 3.2. A nonnegative exhaustive filtration $G = \{G_i A\}_{i \in \mathbb{Z}}$ such that $\text{gr}^G A$ is finite over its center $Z(\text{gr}^G A)$, and $Z(\text{gr}^G A)$ is a finitely generated \mathbb{K} -algebra, is called a *differential filtration of finite type*. If A admits such a filtration then it is called a *differential \mathbb{K} -algebra of finite type*.

We call the next result the “Theorem on the Two Filtrations”. A slightly weaker result appeared in [MS].

Theorem 3.3. ([YZ5]) *Assume the ring A has a differential filtration of finite type G . Then there exists a differential filtration of finite type F on A such that the graded algebra $\text{gr}^F A$ is connected and commutative.*

The prototypical example is:

Example 3.4. Let $\text{char } \mathbb{K} = 0$. Consider the first Weyl algebra

$$A := \mathbb{K}\langle x, y \rangle / (yx - xy - 1).$$

It is of course isomorphic to the ring of differential operators $\mathcal{D}(\mathbf{A}^1)$ on the affine line $\mathbf{A}^1 = \text{Spec } \mathbb{K}[x]$, via $y \mapsto \frac{\partial}{\partial x}$. The first filtration of A is the filtration G by order of operator, namely $\deg^G(x) = 0$ and $\deg^G(y) = 1$. The filtration G has the benefit of localizing to a filtration of the sheaf of differential operators $\mathcal{D}_{\mathbf{A}^1}$. However $\text{gr}_0^G A = \mathbb{K}[\bar{x}]$, so $\text{gr}^G A$ is not connected.

The second filtration of A is the filtration F in which $\deg^F(x) = \deg^F(y) = 1$. Here $\text{gr}^F A$ is a polynomial algebra in the variables \bar{x}, \bar{y} , both of degree 1, so it is connected.

More examples of differential \mathbb{K} -algebras of finite type are:

Example 3.5. The ring $\mathcal{D}(X)$ of differential operators on a smooth affine variety X in characteristic 0. The rigid dualizing complex is $\mathcal{D}(X)[2n]$ where $n := \dim X$.

Example 3.6. The universal enveloping algebra $U(\mathfrak{g})$ of a finite dimensional Lie algebra \mathfrak{g} . The rigid dualizing complex is $U(\mathfrak{g}) \otimes (\bigwedge^n \mathfrak{g})[n]$ where $n := \dim \mathfrak{g}$.

Example 3.7. Generalizing the previous two examples, the universal enveloping algebroid $U_C(L)$, where C is a f.g. commutative \mathbb{K} -algebra and L is a f.g. Lie algebroid over C .

Example 3.8. Any quotient ring A/I or any matrix ring $M_n(A)$ of a differential \mathbb{K} -algebra of finite type A .

By combining Van den Bergh's existence result with the Theorem on the Two Filtrations, and some more work, we get:

Theorem 3.9. ([YZ5]) *Let A be a differential \mathbb{K} -algebra of finite type.*

- (1) *A has a rigid dualizing complex R_A , which is unique up to a unique rigid isomorphism.*
- (2) *Suppose A' is a localization of A such that each bimodule $H^i R_A$ is evenly localizable to A' . Then A' has a rigid dualizing complex $R_{A'}$, and there is a unique rigid localization morphism $q_{A/A'} : R_A \rightarrow R_{A'}$.*
- (3) *Suppose $A \rightarrow B$ is a finite centralizing homomorphism. Then B has a rigid dualizing complex R_B , and there is a unique rigid trace morphism $\text{Tr}_{B/A} : R_B \rightarrow R_A$.*

“Evenly localizable” is a variant of the Ore condition. Part (2) basically says that

$$R_{A'} \cong A' \otimes_A R_A \otimes_A A'$$

in $D(\text{Mod } A'^e)$. And part (3) says that

$$R_B \cong \text{RHom}_A(B, R_A) \cong \text{RHom}_{A^{\text{op}}}(B, R_A)$$

$D(\text{Mod } A^e)$.

Remark 3.10. I wish to amplify the significance of part (3) of the theorem. Suppose $B = A/I$ and $M \in D(\text{Mod } A^e)$. Then $\text{Ext}_A^i(B, M)$ is a $B \otimes_{\mathbb{K}} A^{\text{op}}$ -module, but usually it is *not* a $B \otimes_{\mathbb{K}} B^{\text{op}}$ -module, i.e.

$$\text{Ext}_A^i(B, M) \cdot I \neq 0.$$

The existence of the rigid trace implies, among other things, that $\text{Ext}_A^i(B, R_A)$ is indeed a $B \otimes_{\mathbb{K}} B^{\text{op}}$ -module.

Applications of this theorem to ring theory will be discussed in Section 6. The geometric significance of part (2) will be explained in the second lecture.

4. THE AUSLANDER CONDITION

We continue with the hypothesis that A is a noetherian algebra over a field \mathbb{K} .

Definition 4.1. ([Ye2], [YZ1]) Let R be a dualizing complex over A . We say R is *Auslander* if the two conditions below hold.

- (i) For any finitely generated A -module M , any integers $p > q$ and any A^{op} -submodule $N \subset \text{Ext}_A^p(M, R)$ one has $\text{Ext}_{A^{\text{op}}}^q(M, R) = 0$.
- (ii) The same after exchanging A and A^{op} .

This is a generalization of the classical notion of *Auslander-Gorenstein* ring. Indeed, a \mathbb{K} -algebra A is called Auslander-Gorenstein precisely if it is Gorenstein, and the dualizing complex $R := A$ is Auslander in the sense of the definition above.

Auslander-Gorenstein rings were studied by Gabber, Levasseur and Björk, especially in the context of \mathcal{D} -modules. However the Gorenstein condition is very restrictive (recall that unlike the commutative situation, a noncommutative noetherian ring A is seldom a quotient of a “nice” noetherian ring). On the other hand, Auslander dualizing complexes are relatively easy to find:

Theorem 4.2. ([YZ5]) *Suppose A is a differential \mathbb{K} -algebra of finite type. Then its rigid dualizing complex R_A is Auslander.*

Applications of the theorem to ring theory will be discussed in Section 6. The geometric significance (the relation with perverse t-structures) will be explained in the second lecture.

5. CLASSIFICATION OF DUALIZING COMPLEXES

In the commutative case the dualizing complexes are classified by the Picard group. Namely, given two dualizing complexes R, R' over a commutative noetherian ring A , one has

$$R' \cong L[n] \otimes_A R$$

for some invertible A -module L and some integer n . See [RD].

The noncommutative picture is much more complicated. Again let A be a noetherian algebra over a field \mathbb{K} . A *two-sided tilting complex* over A is a complex $P \in \text{D}^b(\text{Mod } A^e)$ such there exists some $Q \in \text{D}^b(\text{Mod } A^e)$ and isomorphisms $P \otimes_A^L Q \cong Q \otimes_A^L P \cong A$ in $\text{D}(\text{Mod } A^e)$.

Definition 5.1. ([Ye3]) The derived Picard group of A is the group

$$\text{DPic}(A) := \frac{\{\text{two-sided tilting complexes over } A\}}{\text{isomorphism}}.$$

The derived Picard group classifies dualizing complexes in the following sense:

Theorem 5.2. ([Ye3]) *Assume A has at least one dualizing complex. Then the action of $\text{DPic}(A)$ on the set*

$$\frac{\{\text{dualizing complexes over } A\}}{\text{isomorphism}},$$

given by $(P, R) \mapsto P \otimes_A^L R$, is transitive with trivial stabilizers.

The group $\mathrm{DPic}(A)$ always contains the subgroup $\mathrm{Pic}(A) \times \mathbb{Z}$, where $\mathrm{Pic}(A)$ is the noncommutative Picard group of A (consisting of invertible bimodules), and \mathbb{Z} is generated by the shift σ . However when A is neither commutative nor local, often $\mathrm{DPic}(A)$ is bigger than $\mathrm{Pic}(A) \times \mathbb{Z}$.

Example 5.3. Let $A := \begin{bmatrix} \mathbb{K} & \mathbb{K} \\ 0 & \mathbb{K} \end{bmatrix}$, the algebra of upper triangular 2×2 matrices over \mathbb{K} . The rigid dualizing complex $R_A = A^*$ turns out to be a two-sided tilting complex. In fact the functor $R_A \otimes_A^L -$ is the Serre functor of $\mathrm{D}_f^b(\mathrm{Mod} A)$, in the sense of [BK]. Here the group $\mathrm{Pic}(A)$ is trivial, and $\mathrm{DPic}(A) \cong \mathbb{Z}$, generated by the class ν of R_A . The shift satisfies $\sigma = \nu^3$. Thus

$$\mathrm{Pic}(A) \times \mathbb{Z} \subsetneq \mathrm{DPic}(A).$$

The relation $\sigma = \nu^3$ says that A has “Calabi-Yau dimension $\frac{1}{3}$ ”, in the terminology of Kontsevich. See [MY] for details.

6. APPLICATIONS IN RING THEORY

Here are a few applications of the theory of dualizing complexes.

6.1. Left vs. Right Gorenstein. In [Jo1] Jørgensen used balanced dualizing complexes to prove that a connected graded algebra A is left Gorenstein iff it is right Gorenstein.

6.2. Free Resolutions. Jørgensen [Jo2] used balanced dualizing complexes (implicitly) to establish a noncommutative version of Castelnuovo-Mumford regularity. In [Jo3] he proceeded to show that if A is a Koszul connected graded algebra with balanced dualizing complex, then any finitely generated A -module M , possibly after truncating low degrees, will admit a linear free resolution.

6.3. Duals of Verma Modules. Consider the universal enveloping algebra $A := U(\mathfrak{g})$ of a finite dimensional Lie algebra \mathfrak{g} . In [Ye4] we described the structure of the rigid dualizing complex of A (this had been conjectured by Van den Bergh). As a consequence, and using the functoriality of rigid dualizing complexes (the rigid trace) we extended results of Duflo, Brown and Levasseur [BL] regarding the Ext duals of Verma modules.

6.4. Multiplicities of Injectives. In [YZ4] we obtained several results regarding multiplicities of indecomposable injectives in the minimal injective resolution of a ring A . These results extend work of previous authors (see Barou and Malliavin [BM], Brown and Levasseur [BL]). Of particular interest is the case $A = U(\mathfrak{g})$, the universal enveloping algebra of a finite dimensional Lie algebra \mathfrak{g} . Earlier papers on this topic tended to rely on localization; and this restricted their scope to solvable Lie algebras. Since Auslander rigid dualizing complexes were used in [YZ4], we were able to obtain similar results for any Lie algebra (solvable or not).

6.5. Homological Transcendence Degree. In the paper [YZ6] we introduced a new notion of transcendence degree for division rings, called the *homological transcendence degree*, and denoted by $\mathrm{Htr} D$. This invariant seems to be better-behaved than other noncommutative invariants meant to generalize the commutative transcendence degree. For instance, if D is the total ring of fractions of an Artin-Schelter regular algebra A of global dimension n , then $\mathrm{Htr} D = n$. This, and some other

good properties of the homological transcendence degree, were established with the aid of Auslander rigid dualizing complexes.

6.6. Catenarity. Recall that a noetherian ring A is called catenary if given two prime ideals $\mathfrak{p} \subset \mathfrak{q}$, any saturated chain of prime ideals

$$\mathfrak{p} = \mathfrak{p}_0 \subset \mathfrak{p}_1 \subset \cdots \subset \mathfrak{q}$$

has the same length. It is known that if A is commutative and admits some dualizing complex then it is catenary (see [RD]). In [YZ1] we proved that some rings of quantum type are catenary. This was extended by Goodearl-Zhang [GZ] to the case of the quantized coordinate rings $\mathcal{O}_q(G)$.

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