Flatness and Completion Revisited

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1. Preliminaries

All rings in this talk are commutative.

Let A be a ring and let $\mathfrak{a}$ be an ideal in it.

Given an A-module M, its $\mathfrak{a}$-*adic completion* is the A-module

$$\widehat{M} := \lim_{\leftarrow k} (M / \mathfrak{a}^k \cdot M).$$

There is a canonical homomorphism

$$\tau_M : M \to \widehat{M} .$$

The A-module M is called $\mathfrak{a}$-*adically complete* if τ_M is bijective.

Older texts define $\mathfrak{a}$-adic completion in terms of metric topology; but this is not a good idea, as the next example shows.

Example 1.1. Take the ring

$$A = \mathbb{K}[t_0, t_1, t_2, \ldots]$$

of polynomials in countably many variables over a field $\mathbb{K}$.

Let $\mathfrak{a}$ be the maximal ideal generated by the variables.

Consider the module $M = A$.

The $\mathfrak{a}$-*adic completion* $\widehat{M}$ *is not* $\mathfrak{a}$-*adically complete*.

In other words, the canonical homomorphism

$$\tau_{\widehat{M}} : \widehat{M} \to \widehat{\widehat{M}}$$

is not bijective.

See [Ye1] for details.

The problem is that the ideal $\mathfrak{a}$ in this example is not finitely generated.

Indeed:

Theorem 1.2. *If the ideal* $\mathfrak{a}$ *is finitely generated, then for every* A*-module* M *the* $\mathfrak{a}$*-adic completion* $\widehat{M}$ *is* $\mathfrak{a}$*-adically complete.*

Oddly, this basic result, and the counterexample before it, are absent from all standard textbooks, and are virtually unknown.

For proofs of this theorem see [St], [Ye1] or [SzSi].

2. Flatness of Completion – the Noetherian Case

The completion $\widehat{A}$ is itself a ring, and $\tau_A : A \to \widehat{A}$ is a ring homomorphism.

When the ring A is *noetherian*, there are two important facts about completion:

1. The ring $\widehat{A}$ is flat as an A-module.
2. For every *finitely generated* A-module M, the canonical homomorphism
$$\widehat{A} \otimes_A M \to \widehat{M}$$
is bijective.

See any standard textbook on commutative algebra, e.g. [AtMc], [Bo], [Ma], [Ei] or [AlKl].

An immediate consequence is this: if M is a flat finitely generated A-module, then its completion $\widehat{M}$ is also flat.

If M is infinitely generated, then the canonical homomorphism $\widehat{A} \otimes_A M \to \widehat{M}$ is usually not bijective.

Example 2.1. Let $A = \mathbb{K}[[t]]$, the ring of power series over a field $\mathbb{K}$, with the ideal $\mathfrak{a} = (t)$. The ring A is noetherian and $\mathfrak{a}$-adically complete.

Since $A = \widehat{A}$, we have $\widehat{A} \otimes_A M = M$ for every module M.

Let Z be an infinite set. The module of *finitely supported functions* $f : Z \to A$ is denoted by $\mathrm{F_{fin}}(Z, A)$. This is a free A-module, with basis the delta functions $\delta_z : Z \to A$.

It turns out (see [Ye1]) that the $\mathfrak{a}$-adic completion of the A-module $M = \mathrm{F_{fin}}(Z, A)$ is the *module of decaying functions* $\widehat{M} = \mathrm{F_{dec}}(Z, A)$, and this is strictly bigger than $\mathrm{F_{fin}}(Z, A)$.

To be concrete, take $Z = \mathbb{N}$. The function $f : \mathbb{N} \to A$, $f(i) := t^i$, is decaying, and its support is $\mathbb{N}$.

Still, it might be true that completion preserves flatness even for infinitely generated modules, for more complicated reasons... And indeed we have:

Theorem 2.2. *If A is a noetherian ring, and M is a flat A-module, then the $\mathfrak{a}$-adic completion $\widehat{M}$ is flat.*

This theorem was proved in special cases by Bartijn and Strooker [BaSt] (1983), Enochs [En] (1995) and Schoutens [Sn] (2007).

I was under the impression that the general statement was not known. A literature search, and email conversations with a few experts in commutative algebra, reinforced this perception.

In 2016 I found a proof of Theorem 2.2.

Before making my proof public, I sent private copies to more experts. Most responses were encouraging.

However, Brian Conrad wrote back to say that there might be a proof in the book [GaRa] by Ofer Gabber and Lorenzo Ramero (published in 2003).

He was almost correct. I wrote to them, and they replied that they have a proof of a very similar statement in their book. Moreover, their proof can be modified to yield a very easy proof of Theorem 2.2.

Johan de Jong also wrote back, and pointed out that he has a proof of Theorem 2.2 in the Stacks Project [SP], from 2013.

This information certainly took the wind out of my sails!

But then I realized that my methods still have some value – I can prove a few theorems of secondary importance, that are truly new.

These other results will be explained in the rest of the lecture. I will also indicate how my proof of Theorem 2.2 goes.

3. Adic Systems and their Limits

As before, A is a ring and $\mathfrak{a} \subseteq A$ is an ideal.

For each $k \in \mathbb{N}$ we let $A_k := A/\mathfrak{a}^{k+1}$.

We get an inverse system of rings $\{A_k\}_{k\in\mathbb{N}}$.

Definition 3.1. An $\mathfrak{a}$-*adic system* of A-modules is an inverse system $\{M_k\}_{k\in\mathbb{N}}$,

such that:

- Each M_k is an A_k-module.
- The homomorphisms
$$A_k \otimes_{A_{k+1}} M_{k+1} \to M_k$$
are bijective.

Given an $\mathfrak{a}$-adic system $\{M_k\}_{k\in\mathbb{N}}$, its *limit* is the A-module
$$\widehat{M} := \lim_{\leftarrow k} M_k .$$

Example 3.2. Let M be an A-module. The $\mathfrak{a}$-adic system *induced* by M is
$$M_k := A_k \otimes_A M .$$

In this case the limit of the system $\{M_k\}_{k\in\mathbb{N}}$ is just the $\mathfrak{a}$-adic completion of the module M.

Two obvious questions come to mind:

Question 3.3. Given an $\mathfrak{a}$-adic system $\{M_k\}_{k\in\mathbb{N}}$, is its limit $\widehat{M}$ an $\mathfrak{a}$-adically complete module?

Question 3.4. Is every $\mathfrak{a}$-adic system $\{M_k\}_{k\in\mathbb{N}}$ induced by a module M ?

A negative answer to Question 3.3 is provided by Example 1.1.

I do not know the answer to Question 3.4.

On the other hand, here is a recent theorem, that gives a positive answer to both questions when the ideal is finitely generated.

Theorem 3.5. *([Ye2]) Assume the ideal $\mathfrak{a}$ is finitely generated. Let $\{M_k\}_{k\in\mathbb{N}}$ be an $\mathfrak{a}$-adic system. Then*:

1. The limit $\widehat{M} = \lim_{\leftarrow k} M_k$ is an $\mathfrak{a}$-adically complete module.
2. For each k, the canonical homomorphism
$$A_k \otimes_A \widehat{M} \to M_k$$
is bijective.

The proof of this theorem is elementary, using some facts from [Ye1].

Let $\mathsf{Mod}_{\mathfrak{a}\text{-com}} A$ be the category of $\mathfrak{a}$-adically complete A-modules.

And let $\mathsf{Sys}(A, \mathfrak{a})$ be the category of $\mathfrak{a}$-adic systems.

Theorem 3.5, together with another result from [Ye1], show that:

Corollary 3.6. *If the ideal $\mathfrak{a}$ is finitely generated, then the functor*
$$\lim : \mathsf{Sys}(A, \mathfrak{a}) \to \mathsf{Mod}_{\mathfrak{a}\text{-com}} A$$
is an equivalence.

4. Weakly Proregular Ideals

Let a be an element of the ring A.

The *Koszul complex* associated to a is

$$\mathrm{K}(A;a) := \big(\cdots \to 0 \to A \xrightarrow{\mathrm{d}} A \to 0 \to \cdots\big),$$

concentrated in degrees -1 and 0.

The differential d is multiplication by a.

For $j \geq i$ there is a homomorphism of complexes

$$\mathrm{K}(A;a^i) \to \mathrm{K}(A;a^j), \tag{4.1}$$

which is the identity in degree 0, and multiplication by a^{j-i} in degree -1.

Now consider a sequence of elements $\boldsymbol{a} = (a_1, \ldots, a_n)$ in A.

The associated Koszul complex is

$$\mathrm{K}(A;\boldsymbol{a}) := \mathrm{K}(A;a_1) \otimes_A \cdots \otimes_A \mathrm{K}(A;a_n).$$

This is a complex concentrated in degrees $-n, \ldots, 0$.

For $i \in \mathbb{N}$ let

$$\boldsymbol{a}^i := (a_1^i, \ldots, a_n^i).$$

The canonical homomorphism (4.1) makes the collection of Koszul complexes

$$\big\{\mathrm{K}(A;\boldsymbol{a}^i)\big\}_{i\in\mathbb{N}}$$

into an inverse system of complexes.

An inverse system of modules $\{N_i\}_{i\in\mathbb{N}}$ is called *pro-zero* if for every i there is some $j \geq i$ such that the homomorphism $N_j \to N_i$ is zero.

Definition 4.2. A finite sequence $\boldsymbol{a}$ in A is called *weakly proregular* (WPR) if for every $q < 0$ the inverse system of A-modules

$$\big\{\mathrm{H}^q(\mathrm{K}(A;\boldsymbol{a}^i))\big\}_{i\in\mathbb{N}}$$

is pro-zero.

Here is a fun exercise:

Exercise 4.3. What is

$$\lim_{\leftarrow i} \mathrm{H}^0(\mathrm{K}(A;\boldsymbol{a}^i)) \ ?$$

There is another complex we can associate to an element $a \in A$.

It is the complex

$$\mathrm{K}^{\vee}_{\infty}(A;a) := \big(\cdots \to 0 \to A \xrightarrow{\mathrm{d}} A[a^{-1}] \to 0 \to \cdots\big),$$

concentrated in degrees 0 and 1.

The differential d is the ring homomorphism $A \to A[a^{-1}]$.

Given a sequence of elements $\boldsymbol{a} = (a_1, \ldots, a_n)$ in A, we let

$$\mathrm{K}^{\vee}_{\infty}(A;\boldsymbol{a}) := \mathrm{K}^{\vee}_{\infty}(A;a_1) \otimes_A \cdots \otimes_A \mathrm{K}^{\vee}_{\infty}(A;a_n).$$

The complex $\mathrm{K}^{\vee}_{\infty}(A;\boldsymbol{a})$ has a few names in the literature.

I prefer calling it the *infinite dual Koszul complex*.

Another good name is the *augmented Čech complex*.

Here is another nice exercise:

Exercise 4.4. Find the relation between the infinite dual Koszul complex $\mathrm{K}^{\vee}_{\infty}(A; \boldsymbol{a})$ and the inverse system of Koszul complexes $\{\mathrm{K}(A; \boldsymbol{a}^i)\}_{i \in \mathbb{N}}$.

Theorem 4.5. *Let $\boldsymbol{a}$ be a finite sequence in the ring A. TFAE:*

(i) The sequence $\boldsymbol{a}$ is weakly proregular.
(ii) For every injective A-module I and every $q > 0$, the cohomology module

$$\mathrm{H}^q\big(\mathrm{K}^{\vee}_{\infty}(A; \boldsymbol{a}) \otimes_A I\big)$$

is zero.

The condition of weak proregularity was first considered (without a name) by Grothendieck in [LC] (1961). Theorem 4.5 was already proved there.

The name “weakly proregular” was given by Lipman around 2003; see [AlJeLi, Correction] and [Sz]. It is explained by the next example.

Example 4.6. Recall that a sequence $\boldsymbol{a} = (a_1, \ldots, a_n)$ is called a *regular sequence* if a_1 is not a zero-divisor in A, a_2 is not a zero-divisor in the quotient ring $A/(a_1)$, etc., and the ring $A/(a_1, \ldots, a_n)$ is nonzero.

It well known that if $\boldsymbol{a}$ is a regular sequence, then

$$\mathrm{H}^q(\mathrm{K}(A; \boldsymbol{a}^i)) = 0$$

for all $q < 0$ and $i \geq 1$.

Therefore $\boldsymbol{a}$ is weakly proregular.

Here is an important fact about WPR, also proved in [LC]:

Theorem 4.7. *([LC]) If the ring A is noetherian, then every finite sequence in it is WPR.*

Definition 4.8. Let $\mathfrak{a}$ be an ideal in a ring A.

We say that $\mathfrak{a}$ is a *weakly proregular ideal* if it is generated by some weakly proregular sequence.

It is known that if $\mathfrak{a}$ is a WPR ideal, then every finite sequence that generates $\mathfrak{a}$ is WPR.

Furthermore, if $\mathfrak{a}, \mathfrak{b} \subseteq A$ are finitely generated ideals such that $\sqrt{\mathfrak{a}} = \sqrt{\mathfrak{b}}$, then $\mathfrak{a}$ is WPR iff $\mathfrak{b}$ is WPR. See [PoShYe1].

Theorem 4.7 implies that when A is noetherian, every ideal in it is WPR.

An illuminating example of a WPR ideal in a non-noetherian ring will be provided at the end of this talk.

As shown in the papers [PoShYe1] and [PoShYe2], weak proregularity is the correct condition for studying the left derived completion functor. In particular, it implies the *MGM equivalence*.

The work of Liran Shaul [Sh] on *complete Hochschild cohomology* relies on delicate properties of WPR ideals.

In [VyYe], Rishi Vyas and I describe a categorical formulation of the WPR condition, which makes sense also for noncommutative rings.

5. Adic Flatness

As before, A is a ring and $\mathfrak{a} \subseteq A$ is a finitely generated ideal. For each $k \geq 0$ we have the quotient ring $A_k = A/\mathfrak{a}^{k+1}$.

Let M be an A-module. An element $m \in M$ is called an $\mathfrak{a}$-*torsion element* if $\mathfrak{a}^k \cdot m = 0$ for $k \gg 0$.

The set of $\mathfrak{a}$-torsion elements forms a submodule $\Gamma_{\mathfrak{a}}(M)$ of M.

We say that M is an $\mathfrak{a}$-*torsion module* if $\Gamma_{\mathfrak{a}}(M) = M$.

The next exercise should help to connect ideas.

Exercise 5.1. Suppose $\mathfrak{a}$ is generated by a finite sequence $\boldsymbol{a}$.

Show that there is a canonical isomorphism of A-modules

$$\Gamma_{\mathfrak{a}}(M) \cong \mathrm{H}^0\big(\mathrm{K}^{\vee}_{\infty}(A; \boldsymbol{a}) \otimes_A M\big).$$

5. Adic Flatness

As we all know, an A-module M is *flat* iff

$$\mathrm{Tor}^A_q(N, M) = 0$$

for all $q > 0$ and all A-modules N.

Here is a variation of this notion.

Definition 5.2. ([Ye2]) An A-module M is called $\mathfrak{a}$-*adically flat* if

$$\mathrm{Tor}^A_q(N, M) = 0$$

for all $q > 0$ and all $\mathfrak{a}$-*torsion* A-modules N.

Clearly if M is flat then it is $\mathfrak{a}$-adically flat.

In the next slide there is a useful characterization of $\mathfrak{a}$-adic flatness.

5. Adic Flatness

Theorem 5.3. *([Ye2]) Let M be an A-module, and write $M_k := A_k \otimes_A M$.*

The following conditions are equivalent.

(i) M is $\mathfrak{a}$-adically flat.
(ii) For every $k \geq 0$ and every $q > 0$ the module $\mathrm{Tor}^A_q(A_k, M)$ vanishes, and M_k is a flat A_k-module.
(iii) For every $q > 0$ the module $\mathrm{Tor}^A_q(A_0, M)$ vanishes, and M_0 is a flat A_0-module.

The proof of Theorem 5.3 uses a few standard properties of the *derived tensor functor* $(- \otimes^{\mathrm{L}}_A -)$, and it is very easy.

It is possible to give a "classical" proof of this theorem (without derived categories), see [SzSi, Section 2.6]; but it is more involved.

5. Adic Flatness

Recall that the limit of an $\mathfrak{a}$-adic system $\{M_k\}_{k \in \mathbb{N}}$ is the A-module $\widehat{M} = \lim_{\leftarrow k} M_k$.

We say that the $\mathfrak{a}$-adic system $\{M_k\}_{k \in \mathbb{N}}$ is *flat* if M_k is a flat A_k-module for all k.

Theorem 5.4. *([Ye2]) Let $\{M_k\}_{k \in \mathbb{N}}$ be a flat $\mathfrak{a}$-adic system of A-modules, with limit $\widehat{M}$.*

1. If the ideal $\mathfrak{a}$ is WPR, then $\widehat{M}$ is $\mathfrak{a}$-adically flat.
2. If the ring A is noetherian, then $\widehat{M}$ is flat.

The proof of this theorem uses the concept of *free resolutions of $\mathfrak{a}$-adic systems*.

Item (2) of Theorem 5.4 was already proved (using another method) by de Jong in [SP].

Here is my:

Proof of Theorem 2.2. Let M be a flat A-module.

Then the $\mathfrak{a}$-adic system $\{M_k\}_{k\in\mathbb{N}}$ induced by M, i.e. $M_k = A_k \otimes_A M$, is flat.

By Theorem 5.4(2) the module $\widehat{M} = \lim_{\leftarrow k} M_k$ is flat. □

We also have this new result:

Corollary 5.5. *([Ye2]) If the ideal $\mathfrak{a}$ is WPR, and if M is an $\mathfrak{a}$-adically flat A-module, then the $\mathfrak{a}$-adic completion $\widehat{M}$ is an $\mathfrak{a}$-adically flat A-module.*

Proof. Like the proof above, but now we use Theorem 5.3 and Theorem 5.4(1). □

Interestingly, as the next result shows, the distinction between flatness and adic flatness disappears when A is noetherian.

Corollary 5.6. *([Ye2]) If A is noetherian, and if M is an $\mathfrak{a}$-adically complete $\mathfrak{a}$-adically flat A-module, then M is a flat A-module.*

Proof. By Theorem 5.3 the the $\mathfrak{a}$-adic system $\{M_k\}_{k\in\mathbb{N}}$ induced by M is flat.

Now use Theorem 5.4(2). □

At this point it is natural to ask: *Are the notions of flatness and adic flatness really distinct?*

The example in the next section will demonstrate that they are indeed distinct.

Remark 5.7. In the new book [SzSi] the authors use the name *relatively $\mathfrak{a}$-flat* for what we call $\mathfrak{a}$-adically flat. This is studied in Section 2.6 of the book.

They prove our Theorem 2.2, Theorem 5.3 and Corollary 5.5. Their methods are very different from ours: they try to minimize the use of derived categories, and they do not discuss adic systems.

Our non-noetherian example from Section 6 is presented in this book, as Example 2.8.4.

Remark 5.8. In the recent paper [BhMoSc] the authors rediscovered the notion of $\mathfrak{a}$-adic flatness in the case of a *principal* WPR ideal $\mathfrak{a}$.

In fact, they were interested in the principal ideal $\mathfrak{a} = (p) \subseteq A$, for a prime number p.

For a sequence $\boldsymbol{a} = (p)$ of length 1 the WPR condition takes a much simpler form than Definition 4.2: it is called *bounded p-torsion*. See [SzSi, Example 7.3.2].

The authors of [BhMoSc] proved p-adic versions of Theorem 5.3 and Corollary 5.5.

Remark 5.9. In the more recent paper [BhSc], the authors introduce the concept of a *prism*. This is a pair (A, I), consisting of a $\mathbb{Z}_{(p)}$-ring A and a finitely generated ideal I, with some extra data and extra properties.

Let $\mathfrak{a} := (p) + I \subseteq A$.

One of the properties of the prism (A, I) is that the ring A is *derived* $\mathfrak{a}$*-adically complete*.

The prism (A, I) is called *bounded* if the element $p \in A/I$ is WPR, i.e. A/I has bounded p-torsion.

A morphism of prisms $(A, I) \to (B, J)$ is called *flat* if it is $\mathfrak{a}$-adically flat.

Presumably, *a better definition of prisms would require that the ideal* $\mathfrak{a}$ *is WPR*. Then the results mentioned earlier in this section would apply.

Remark 5.10. According to Positselski (private communication) there is a converse to Theorem 5.4(1).

Consider the A-module

$$P := \mathrm{F}_{\mathrm{fin}}(\mathbb{N}, A),$$

a free A-module of countable rank.

Its $\mathfrak{a}$-adic completion is

$$\widehat{P} = \mathrm{F}_{\mathrm{dec}}(\mathbb{N}, A).$$

Positselski claims that *if* $\widehat{P}$ *is* $\mathfrak{a}$*-adically flat, then the ideal* $\mathfrak{a}$ *must be WPR*.

The proof relies on some difficult results in his recent paper [Ps].

6. The Non-Noetherian Example

In this final section I present an example of a non-noetherian ring A, with a WPR ideal $\mathfrak{a}$, and a flat A-module M, such that $\mathfrak{a}$-adic completion $\widehat{M}$ is *not flat* over A.

Of course, according to Corollary 5.5, the module $\widehat{M}$ is $\mathfrak{a}$-adically flat over A.

This example is not exotic at all. It shows up often in papers on Hochschild cohomology, e.g. in [Sh].

The module M will be the ring A itself.

Theorem 6.1. *Let* $\mathbb{K}$ *be a field of characteristic* 0*, let* $\mathbb{K}[[t_1]]$ *and* $\mathbb{K}[[t_2]]$ *be the rings of power series in the variables* t_1 *and* t_2*, and let*

$$A := \mathbb{K}[[t_1]] \otimes_{\mathbb{K}} \mathbb{K}[[t_2]].$$

Consider the ideal $\mathfrak{a}$ *generated by* t_1 *and* t_2*, and let* $\widehat{A}$ *be the* $\mathfrak{a}$*-adic completion of* A*. Then:*

1. The ideal $\mathfrak{a}$ is weakly proregular.
2. The ring A is not noetherian.
3. The ring $\widehat{A}$ is noetherian.
4. The ring $\widehat{A}$ is $\mathfrak{a}$-adically flat over A.
5. The ring $\widehat{A}$ is not flat over A.

Parts (1-2) of the theorem were known to us for some time.

Part (3) is easy: $\widehat{A} \cong \mathbb{K}[[t_1, t_2]]$.

Part (4) follows from Corollary 5.5, as we already mentioned.

Part (5) of the theorem is new, and I want to outline the proof.

We need a lemma. Let $B := \mathbb{K}[[t]]$.

Lemma 6.2. *The module of differential* 1*-forms* $\Omega^1_{B/\mathbb{K}}$ *is not finitely generated.*

Proof. Let $L := \mathbb{K}((t))$, the field of fractions of L.

By standard facts of differential algebra (see [Ma]) there is an isomorphism

$$L \otimes_B \Omega^1_{B/\mathbb{K}} \cong \Omega^1_{L/\mathbb{K}}.$$

Thus, if $\Omega^1_{B/\mathbb{K}}$ were finitely generated over B, then $\Omega^1_{L/\mathbb{K}}$ would be finitely generated over L.

Because we are in characteristic 0, the rank of $\Omega^1_{L/\mathbb{K}}$ as an L-module equals the transcendence degree of L over $\mathbb{K}$.

But this transcendence degree is known to be infinite. □

Proof of Theorem 6.1(5).

There is a surjective ring homomorphism

$$f : A \to B, \quad f(t_1) = f(t_2) = t.$$

Letting I be its kernel, we get an exact sequence of A-modules

$$0 \to I \to A \xrightarrow{f} B \to 0. \tag{6.3}$$

There is a similar exact sequence of $\widehat{A}$-modules

$$0 \to J \to \widehat{A} \xrightarrow{\widehat{f}} B \to 0. \tag{6.4}$$

If $A \to \widehat{A}$ had been flat, then the sequence (6.4) would have been induced from (6.3), implying that

$$J \cong \widehat{A} \otimes_A I \tag{6.5}$$

as $\widehat{A}$-modules.

A calculation shows that if the isomorphism (6.5) holds, then

$$J/J^2 \cong I/I^2$$

as B-modules.

The ideal $J \subseteq \widehat{A}$ is finitely generated, since this ring is noetherian.

Therefore J/J^2 is finitely generated as a B-module.

Thus the assumption that $A \to \widehat{A}$ is flat leads to the conclusion that I/I^2 is a finitely generated B-module .

But

$$I/I^2 \cong \Omega^1_{B/\mathbb{K}}.$$

The lemma tells us this is not a finitely generated B-module. □

~ *END* ~

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