ADDENDUM TO TALK ON PYTHAGOREAN TRIPLES

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As in our talk [Ye], for any integer $c > 1$, let us denote by $\mathrm{PT}(c)$ the set of reduced ordered pythagorean triples with hypotenuse c. And let

$$\mathrm{PT} := \bigcup_{c} \mathrm{PT}(c)$$

be the set of all reduced ordered pythagorean triples. In this way we conveniently chose exactly one representative from each equivalence class of pythagorean triples. This choice is not really important, and we could just as well define

$$\mathrm{PT} := \frac{\{\text{pythagorean triples}\}}{\text{equivalence}}.$$

Recall that in the talk, we made use of the group $G(\mathbb{Q})$ of rational points on the unit circle $G(\mathbb{R}) \subseteq \mathbf{A}^2(\mathbb{R})$. (For the algebraic geometers, the linear algebraic group G is the group scheme $G = \mathrm{SO}_{2,\mathbb{Q}}$ over $\mathbb{Q}$, and it is the nonsplit 1-dimensional torus over $\mathbb{Q}$. The split torus is $\mathrm{GL}_{1,\mathbb{Q}}$, sitting as a hyperbola inside $\mathbf{A}^2(\mathbb{R})$.) The easy observation was that each non-torsion element ζ in the abelian group $G(\mathbb{Q})$ represents a reduced ordered pythagorean triple $\mathrm{pt}(\zeta)$, and the resulting function

$$\mathrm{pt} : \big(G(\mathbb{Q}) - T\big) \to \mathrm{PT},$$

whete $T = \{\pm 1, \pm \boldsymbol{i}\}$ is the torsion subsgroup of $G(\mathbb{Q})$, is surjective and 8 to 1.

Our main result (Theorem 5.2) was that

$$G(\mathbb{Q}) = T \times F,$$

where F is a free abelian group, with basis indexed by the primes in $\mathbb{Z}$ that are congruent to 1 modulo 4. Combined with the symmetries of the function pt, this led to a full description of the sets $\mathrm{PT}(c)$ for all c.

After the talk, M. Newman, a member of the audience, mentioned to me the work of O. Taussky on a group structure of the set of pythagorean triples. Indeed, such a structure was suggested in [Ta]. It was later studied be several authors, e.g. [Ec].

I read these older references. Unfortunately, the idea of Taussky seems to be a somewhat misguided one. The set PT does not seem to have any useful group structure. It appears to me that in [Ta] and [Ec] the group structure of the free group $G(\mathbb{Q})/T$ was "forced" on the set PT in an artificial manner. This amounted to a treatment that was less than elegant.

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References

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