Rigid Dualizing Complexes and Perverse Coherent Sheaves

Lecture Notes
30 Oct 2007

Amnon Yekutieli
Ben Gurion University, ISRAEL
http://www.math.bgu.ac.il/∼amyekut/lectures

Here is the plan of my lecture:

1. Background on Dualizing Complexes
2. Rigid Complexes over Rings
3. Rigid Dualizing Complexes over Rings
4. Rigid Dualizing Complexes over Schemes
5. Perverse Coherent Sheaves
6. Cohen-Macaulay Complexes

This talk is about joint work with James Zhang (Seattle).

1. Background on Dualizing Complexes

Dualizing complexes over schemes were introduced by Grothendieck in the 1960's (see [RD]), as a vast generalization of Serre duality.

Suppose X is a noetherian scheme.

We denote by $\mathsf{Mod}\, \mathcal{O}_X$ the category of sheaves of $\mathcal{O}_X$-modules, and by $\mathsf{D}(\mathsf{Mod}\, \mathcal{O}_X)$ its derived category.

The full subcategory of bounded complexes with coherent cohomologies is $\mathsf{D}^{\mathrm{b}}_{\mathrm{c}}(\mathsf{Mod}\, \mathcal{O}_X)$. It is equivalent to $\mathsf{D}^{\mathrm{b}}(\mathsf{Coh}\, \mathcal{O}_X)$.

Definition 1.1. (Grothendieck [RD]) A *dualizing complex* on X is a complex $\mathcal{R} \in \mathsf{D}^{\mathrm{b}}_{\mathrm{c}}(\mathsf{Mod}\, \mathcal{O}_X)$ satisfying the two conditions:

(i) $\mathcal{R}$ has finite injective dimension.
(ii) The canonical morphism $\mathcal{O}_X \to \mathrm{R}\mathcal{H}om_{\mathcal{O}_X}(\mathcal{R}, \mathcal{R})$ is an isomorphism.

It follows that the functor

$$\mathcal{M} \mapsto \mathrm{R}\mathcal{H}om_{\mathcal{O}_X}(\mathcal{M}, \mathcal{R})$$

is an auto-duality of $\mathsf{D}^{\mathrm{b}}_{\mathrm{c}}(\mathsf{Mod}\, \mathcal{O}_X)$.

When $X = \operatorname{Spec} A$ is affine, the complex $R := \mathrm{R}\Gamma(X, \mathcal{R}) \in \mathsf{D}^{\mathrm{b}}_{\mathrm{f}}(\mathsf{Mod}\, A)$ is called a dualizing complex over A, and

$$M \mapsto \mathrm{RHom}_A(M, R)$$

is an auto-duality of $\mathsf{D}^{\mathrm{b}}_{\mathrm{f}}(\mathsf{Mod}\, A)$.

Suppose $\mathbb{K}$ is a regular noetherian ring of finite Krull dimension, and X is a finite type $\mathbb{K}$-scheme, with structural morphism $\pi : X \to \operatorname{Spec} \mathbb{K}$.

Then, according to [RD], there is a special dualizing complex on X, namely the *Grothendieck dualizing complex* $\mathcal{R}_X := \pi^! \mathbb{K}$.

The proof of existence of this complex, and its properties, is very difficult.

In this lecture I will explain an alternative approach to Grothendieck duality.

For other approaches see the papers in the references, mainly by Joseph Lipman and his coauthors.

2. Rigid Complexes over Rings

Suppose A is a commutative ring, and B is a commutative A-algebra.

In [YZ4] we constructed a functor

$$\mathrm{Sq}_{B/A} : \mathsf{D}(\mathsf{Mod}\, B) \to \mathsf{D}(\mathsf{Mod}\, B),$$

called the *squaring operation*.

When B is flat over A one has

$$\mathrm{Sq}_{B/A} M = \mathrm{RHom}_{B \otimes_A B}(B, M \otimes^{\mathrm{L}}_A M)$$

for $M \in \mathsf{D}(\mathsf{Mod}\, B)$.

But in general one has to use DG algebras to define $\mathrm{Sq}_{B/A} M$.

The functor $\mathrm{Sq}_{B/A}$ is quadratic, in the following sense. Given a morphism $\phi : M \to N$ in $\mathsf{D}(\mathsf{Mod}\, B)$, and an element $b \in B$, one has

$$\mathrm{Sq}_{B/A}(b\phi) = b^2 \, \mathrm{Sq}_{B/A}(\phi)$$

in

$$\mathrm{Hom}_{\mathsf{D}(\mathsf{Mod}\, B)}(\mathrm{Sq}_{B/A} M, \mathrm{Sq}_{B/A} N).$$

Definition 2.1. Let B be a noetherian A-algebra, and let M be a complex in $\mathsf{D}^{\mathrm{b}}_{\mathrm{f}}(\mathsf{Mod}\, B)$ that has finite flat dimension over A. Assume

$$\rho : M \xrightarrow{\simeq} \mathrm{Sq}_{B/A} M$$

is an isomorphism in $\mathsf{D}(\mathsf{Mod}\, B)$. Then the pair (M, ρ) is called a *rigid complex over B relative to A.*

Definition 2.2. Say (M, ρ) and (N, σ) are rigid complexes over B relative to A. A morphism $\phi : M \to N$ in $\mathsf{D}(\mathsf{Mod}\, B)$ is called a *rigid morphism relative to A* if the diagram

$$\begin{array}{ccc} M & \xrightarrow{\rho} & \operatorname{Sq}_{B/A} M \\ \phi \downarrow & & \downarrow \operatorname{Sq}_{B/A}(\phi) \\ N & \xrightarrow{\sigma} & \operatorname{Sq}_{B/A} N \end{array}$$

is commutative.

3. Rigid Dualizing Complexes over Rings

From now on $\mathbb{K}$ denotes a fixed noetherian regular ring of finite Krull dimension (e.g. a field or the ring of integers).

Let A be a noetherian $\mathbb{K}$-algebra. The next definition is due to Michel Van den Bergh [VdB].

Definition 3.1. A *rigid dualizing complex* over A relative to $\mathbb{K}$ is a rigid complex (R, ρ), such that R is a dualizing complex.

Note that the only rigid automorphism of a rigid dualizing complex (R, ρ) is the identity $\mathbf{1}_R$. Indeed, any automorphism ϕ of R has to be of the form $\phi = a\mathbf{1}_R$ for some invertible element $a \in A$. If ϕ is rigid then $a^2 = a$, and hence $a = 1$.

Recall that an A-algebra B is called *essentially finite type* if it is a localization of some finitely generated A-algebra.

The next theorems are taken from [YZ5]

Theorem 3.2. *Let A be an essentially finite type $\mathbb{K}$-algebra. Then A has a rigid dualizing complex (R_A, ρ_A), which is unique up to a unique rigid isomorphism.*

Recall that a ring homomorphism $f^* : A \to B$ is called *finite* if B is a finitely generated A-module.

Theorem 3.3. *Let A and B be essentially finite type $\mathbb{K}$-algebras, and let $f^* : A \to B$ be a finite homomorphism. Then the complex $\operatorname{RHom}_A(B, R_A)$ has an induced rigidifying isomorphism, and there is a unique rigid isomorphism*

$$\operatorname{RHom}_A(B, R_A) \cong R_B.$$

We say that B is *essentially smooth* of relative dimension n over A if it is essentially finite type, formally smooth, and the rank of the projective B-module $\Omega^1_{B/A}$ is n. When $n = 0$ we say B is *essentially étale.*

Example 3.4. If A' is a localization of A then $A \to A'$ is essentially étale. If $B = A[t_1, \ldots, t_n]$ is a polynomial algebra then $A \to B$ is essentially smooth of relative dimension n.

Theorem 3.5. *Let A and B be essentially finite type $\mathbb{K}$-algebras, and let $f^* : A \to B$ be an essentially smooth homomorphism of relative dimension n. Then* *the complex* $\Omega^n_{B/A}[n] \otimes_A R_A$ *has an induced rigidifying isomorphism, and there is a unique rigid isomorphism*

$$\Omega^n_{B/A}[n] \otimes_A R_A \cong R_B.$$

Taking $n = 0$ we get an important corollary:

Corollary 3.6. *Given an essentially étale homomorphism* $f^* : A \to B$*, there is a unique rigid isomorphism*

$$B \otimes_A R_A \cong R_B.$$

4. Rigid Dualizing Complexes over Schemes

Definition 4.1. Let X be a finite type separated $\mathbb{K}$-scheme. A *rigid dualizing complex over* X (relative to $\mathbb{K}$) is a the data $(\mathcal{R}, \boldsymbol{\rho})$, where:

(1) $\mathcal{R} \in \mathsf{D}^{\mathrm{b}}_{\mathrm{c}}(\mathsf{Mod}\,\mathcal{O}_X)$ is a dualizing complex on X.
(2) $\boldsymbol{\rho} = \{\rho_U\}$ is a collection of rigidifying isomorphisms, indexed by the affine open sets of X. Namely, for any affine open set U, ρ_U is a rigidifying isomorphism for the dualizing complex $\mathrm{R}\Gamma(U, \mathcal{R})$ over the $\mathbb{K}$-algebra $A := \Gamma(U, \mathcal{O}_X)$.

The condition is:

(†) For any inclusion $V \subset U$ of affine open sets, with $A := \Gamma(U, \mathcal{O}_X)$ and $B := \Gamma(V, \mathcal{O}_X)$, the canonical isomorphism

$$B \otimes_A \mathrm{R}\Gamma(U, \mathcal{R}) \cong \mathrm{R}\Gamma(V, \mathcal{R})$$

is rigid, with respect to the rigidifying isomorphisms ρ_U and ρ_V.

We would like to prove *existence and uniqueness of a rigid dualizing complex on* X.

Consider an affine open set $U \subset X$, and let $A := \Gamma(U, \mathcal{O}_X)$. According to Theorem 3.2 there exists a rigid dualizing complex (R_A, ρ_A) over A. Let us denote by $\mathcal{R}_U$ the corresponding complex of sheaves on U, which is of course a dualizing complex over U. And let's write $\rho_U := \rho_A$.

Now suppose $V \subset U$ is another affine open set, with $B := \Gamma(V, \mathcal{O}_X)$ and rigid dualizing complex (R_B, ρ_B). According to Corollary 3.6 there is a unique isomorphism

$$\phi_{V/U} : \mathcal{R}_U|_V \xrightarrow{\simeq} \mathcal{R}_V \tag{4.2}$$

in $\mathsf{D}(\mathsf{Mod}\,\mathcal{O}_V)$ which respects rigidity.

Therefore given an affine open set $W \subset V$, these isomorphisms satisfy

$$\phi_{W/V} \circ \phi_{V/U} = \phi_{W/U}.$$

The next step would be to try to glue the affine dualizing complexes $\mathcal{R}_U$ to a global complex $\mathcal{R}_X \in \mathsf{D}^{\mathrm{b}}_{\mathrm{c}}(\mathsf{Mod}\,\mathcal{O}_X)$.

But here we encounter a genuine problem: *usually objects in derived categories cannot be glued!*

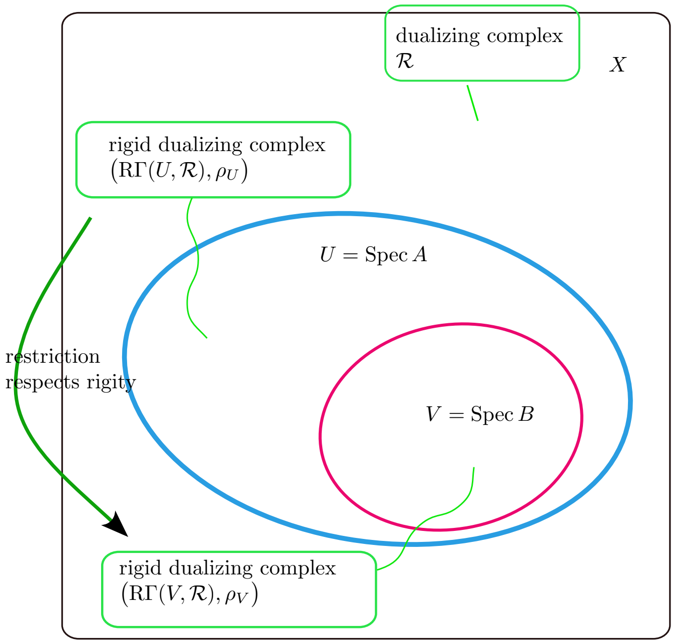

FIGURE 1

Grothendieck's solution in the commutative case, in [RD], was to use Cousin complexes. This solution can be used in our setup too, but it has a disadvantage: we are forced to leave the derived category, and then return to it.

We propose an alternative solution: *perverse coherent sheaves.*

Remark 4.3. For noncommutative ringed schemes one is forced to use perverse coherent sheaves, since Cousin complexes are ill-behaved. See [YZ3].

5. Perverse Coherent Sheaves

The notion of t-structures and perverse sheaves were introduced by Beilinson, Bernstein and Deligne [BBD] around 1980. This was in the context of intersection cohomology on singular spaces. For such a space X they were interested in t-structures on subcategories of $\mathsf{D}(\mathsf{Mod}\, \mathbb{K}_X)$, where $\mathbb{K}_X$ is a constant sheaf of rings on X.

Perverse coherent sheaves came into the scene only very recently, independently in the work of Bezrukavnikov (after Deligne) [Bz], Bridgeland [Br], Kashiwara [Ka] and our paper [YZ3].

Let me recall what is a t-structure on a triangulated category D. It consists of the datum of two full subcategories $\mathsf{D}^{\leq 0}$ and $\mathsf{D}^{\geq 0}$ satisfying the axioms below, where $\mathsf{D}^{\leq n} := \mathsf{D}^{\leq 0}[-n]$ and $\mathsf{D}^{\geq n} := \mathsf{D}^{\geq 0}[-n]$.

(i) $\mathsf{D}^{\leq -1} \subset \mathsf{D}^{\leq 0}$ and $\mathsf{D}^{\geq 1} \subset \mathsf{D}^{\geq 0}$.
(ii) $\operatorname{Hom}_{\mathsf{D}}(M, N) = 0$ for $M \in \mathsf{D}^{\leq 0}$ and $N \in \mathsf{D}^{\geq 1}$.

(iii) For any $M \in \mathsf{D}$ there is a distinguished triangle

$$M' \to M \to M'' \to M'[1]$$

in D with $M' \in \mathsf{D}^{\leq 0}$ and $M'' \in \mathsf{D}^{\geq 1}$.

When these conditions are satisfied one defines the *heart* of D to be the full subcategory $\mathsf{D}^0 := \mathsf{D}^{\leq 0} \cap \mathsf{D}^{\geq 0}$. This is an abelian category.

Given a scheme X, the derived category $\mathsf{D}^{\mathrm{b}}_{\mathrm{c}}(\mathsf{Mod}\,\mathcal{O}_X)$ has the *standard t-structure*, in which

$$\begin{aligned} &\mathsf{D}^{\mathrm{b}}_{\mathrm{c}}(\mathsf{Mod}\,\mathcal{O}_X)^{\leq 0} := \\ &\quad \{\mathcal{M} \in \mathsf{D}^{\mathrm{b}}_{\mathrm{c}}(\mathsf{Mod}\,\mathcal{O}_X) \mid \mathrm{H}^i\mathcal{M} = 0 \text{ for all } i > 0\}, \\ &\mathsf{D}^{\mathrm{b}}_{\mathrm{c}}(\mathsf{Mod}\,\mathcal{O}_X)^{\geq 0} := \\ &\quad \{\mathcal{M} \in \mathsf{D}^{\mathrm{b}}_{\mathrm{c}}(\mathsf{Mod}\,\mathcal{O}_X) \mid \mathrm{H}^i\mathcal{M} = 0 \text{ for all } i < 0\}. \end{aligned}$$

The heart $\mathsf{D}^{\mathrm{b}}_{\mathrm{c}}(\mathsf{Mod}\,\mathcal{O}_X)^0$ is equivalent to the category $\mathsf{Coh}\,\mathcal{O}_X$ of coherent sheaves.

Other t-structures will be referred to as *perverse t-structures.*

Here is an observation. Suppose A is an essentially finite type $\mathbb{K}$-algebra, where as before $\mathbb{K}$ is a finite dimensional regular noetherian ring. Let R_A be the rigid dualizing complex of A.

Then the duality $\mathrm{D} := \mathrm{RHom}_A(-, R_A)$ gives rise to a perverse t-structure

$$\begin{aligned} &{}^{\mathrm{p}}\mathsf{D}^{\mathrm{b}}_{\mathrm{f}}(\mathsf{Mod}\,A)^{\leq 0} := \\ &\quad \{M \mid \mathrm{H}^i\mathrm{D}M = 0 \text{ for all } i < 0\}, \\ &{}^{\mathrm{p}}\mathsf{D}^{\mathrm{b}}_{\mathrm{f}}(\mathsf{Mod}\,A)^{\geq 0} := \\ &\quad \{M \mid \mathrm{H}^i\mathrm{D}M = 0 \text{ for all } i > 0\}. \end{aligned}$$

on $\mathsf{D}^{\mathrm{b}}_{\mathrm{f}}(\mathsf{Mod}\,A)$.

We call it the *rigid perverse t-structure.* The heart is denoted by ${}^{\mathrm{p}}\mathsf{D}^{\mathrm{b}}_{\mathrm{f}}(\mathsf{Mod}\,A)^0$.

The next theorem was proved in [YZ3].

Theorem 5.1. *Let X be a finite type $\mathbb{K}$-scheme. Let $\star$ denote either ≤ 0, ≥ 0 or* 0.

Define ${}^{\mathrm{p}}\mathsf{D}^{\mathrm{b}}_{\mathrm{c}}(\mathsf{Mod}\,\mathcal{O}_X)^\star$ to be the class of complexes $\mathcal{M} \in \mathsf{D}^{\mathrm{b}}_{\mathrm{c}}(\mathsf{Mod}\,\mathcal{O}_X)$ such that

$$\mathrm{R}\Gamma(U, \mathcal{M}) \in {}^{\mathrm{p}}\mathsf{D}^{\mathrm{b}}_{\mathrm{f}}\big(\mathsf{Mod}\,A\big)^\star$$

for any affine open set U, with $A := \Gamma(U, \mathcal{O}_X)$.

Then:

(1) *The pair*

$$\big({}^{\mathrm{p}}\mathsf{D}^{\mathrm{b}}_{\mathrm{c}}(\mathsf{Mod}\,\mathcal{O}_X)^{\leq 0}, {}^{\mathrm{p}}\mathsf{D}^{\mathrm{b}}_{\mathrm{c}}(\mathsf{Mod}\,\mathcal{O}_X)^{\geq 0}\big)$$

is a t-structure on $\mathsf{D}^{\mathrm{b}}_{\mathrm{c}}(\mathsf{Mod}\,\mathcal{O}_X)$.

(2) *The assignment $V \mapsto {}^{\mathrm{p}}\mathsf{D}^{\mathrm{b}}_{\mathrm{c}}(\mathsf{Mod}\,\mathcal{O}_V)^0$, for $V \subset X$ open, is a stack of abelian categories on X.*

Part (2) says that the objects of ${}^{\mathrm{p}}\mathsf{D}^{\mathrm{b}}_{\mathrm{c}}(\mathsf{Mod}\,\mathcal{O}_X)^0$, which we call *perverse coherent sheaves*, can be glued. They behave like sheaves, and hence the name.

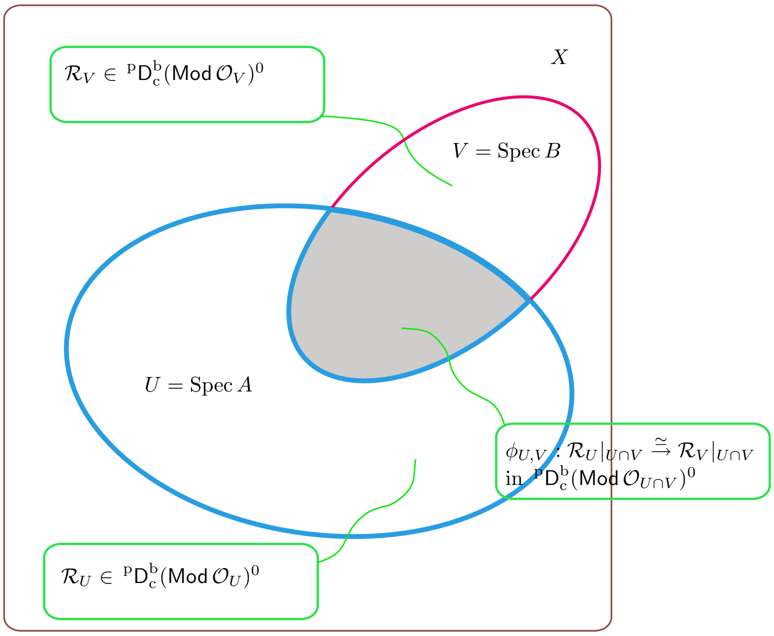

FIGURE 2

For any affine open set $U = \operatorname{Spec} A$, the dualizing complex $\mathcal{R}_U$ (the sheafification of the rigid dualizing complex R_A) is clearly a perverse coherent sheaf on U.

Thus we can use the isomorphisms

$$\phi_{U,V} : \mathcal{R}_U|_{U\cap V} \xrightarrow{\simeq} \mathcal{R}_V|_{U\cap V},$$

deduced from equation (4.2), to glue the affine dualizing complexes.

In this way we obtain:

Theorem 5.2. *Let X be a finite type $\mathbb{K}$-scheme. There exists a rigid dualizing complex $(\mathcal{R}_X, \boldsymbol{\rho}_X)$ over X relative to $\mathbb{K}$, and it is unique up to a unique rigid isomorphism.*

Along the same lines, using Theorems 3.3 and 3.5, we can also prove:

Theorem 5.3. *Let X and Y be finite type $\mathbb{K}$-schemes, and let $f : X \to Y$ be a morphism.*

(1) *If f is finite, then there is a unique isomorphism*

$$\mathrm{R}f_* \mathcal{R}_X \cong \mathrm{R}\mathcal{H}om_{\mathcal{O}_Y}(f_* \mathcal{O}_X, \mathcal{R}_Y)$$

which respects rigidity.

(2) *If f is smooth of relative dimension n, then there is a unique isomorphism*

$$\Omega^n_{X/Y}[n] \otimes_{\mathcal{O}_X} f^* \mathcal{R}_Y \cong \mathcal{R}_X$$

which respects rigidity.

Proper morphisms and residues are treated in [Ye6].

Remark 5.4. Recently I discovered a totally new proof of the duality theorem for proper morphisms, which uses perverse sheaves only, avoiding residue calculations. If correct, the new proof will make the paper [Ye6] much shorter.

Remark 5.5. I think all the results here work also for essentially finite type $\mathbb{K}$-schemes.

6. Cohen-Macaulay Complexes

As before X is a finite type $\mathbb{K}$-scheme. Let $\mathcal{R}_X$ be the rigid dualizing complex of X.

Given a point $x \in X$ let $\boldsymbol{k}(x)$ be its residue field. We denote by $\dim_{\mathbb{K}}(x)$ the unique integer i such that

$$\operatorname{Ext}^{-i}_{\mathcal{O}_{X,x}}\big(\boldsymbol{k}(x), \mathcal{R}_{X,x}\big) \neq 0.$$

Then the function

$$\dim_{\mathbb{K}} : X \to \mathbb{Z}$$

is a dimension function, i.e.

$$\dim_{\mathbb{K}}(y) = \dim_{\mathbb{K}}(x) - 1$$

when y is an immediate specialization of x.

Example 6.1. Take $\mathbb{K} := \mathbb{Z}$, the ring of integers, and $X := \mathbf{A}^1_{\mathbb{K}} = \operatorname{Spec} \mathbb{K}[t]$, the affine line. Consider the following points in X: x_0 is the generic point; x_1 is the prime ideal (t); and x_2 is the maximal ideal $(t, 2)$. Then

$$\dim_{\mathbb{K}}(x_i) = 1 - i.$$

Recall from [RD] that a complex $\mathcal{M} \in \mathsf{D}^{\mathrm{b}}_{\mathrm{c}}(\mathsf{Mod}\, \mathcal{O}_X)$ is called *Cohen-Macaulay* if for every point x the local cohomologies $\mathrm{H}^i_x \mathcal{M}$ all vanish except for $i = -\dim_{\mathbb{K}}(x)$.

Here is another result from [YZ3].

Theorem 6.2. *Let X be a finite type scheme over $\mathbb{K}$, let $\mathcal{R}_X$ be the rigid dualizing complex of X, and let D be the duality functor $\mathrm{R}\mathcal{H}om_{\mathcal{O}_X}(-, \mathcal{R}_X)$.*

Then the following conditions are equivalent for $\mathcal{M} \in \mathsf{D}^{\mathrm{b}}_{\mathrm{c}}(\mathsf{Mod}\, \mathcal{O}_X)$.

(i) *$\mathcal{M}$ is a perverse coherent sheaf (for the rigid perverse t-structure).*
(ii) *$\mathrm{D}\mathcal{M}$ is a coherent sheaf, i.e. $\mathrm{H}^i \mathrm{D}\mathcal{M} = 0$ for all $i \neq 0$.*
(iii) *$\mathcal{M}$ is a Cohen-Macaulay complex.*

In particular this implies the Cohen-Macaulay complexes form an abelian subcategory of $\mathsf{D}^{\mathrm{b}}_{\mathrm{c}}(\mathsf{Mod}\, \mathcal{O}_X)$, a fact that seems to have eluded Grothendieck.

- END -

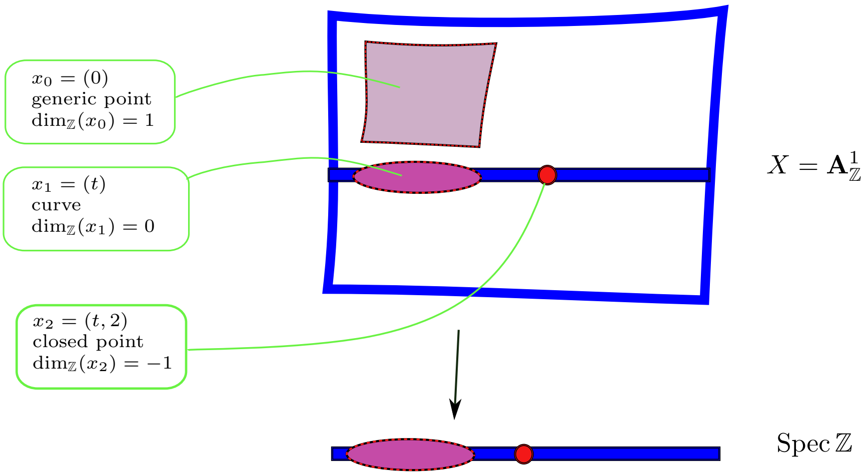

FIGURE 3

REFERENCES

[AJL] L. Alonso, A. Jeremías and J. Lipman, "Studies in Duality on Noetherian Formal Schemes and Non-Noetherian Ordinary Schemes," Contemp. Math. **244**, Amer. Math. Soc., 1999.

[AK] A. Altman and S. Kleiman, "Introduction to Grothendieck Duality," Lecture Notes in Math. **20**, Springer, 1970.

[BBD] A.A. Beilinson, J. Bernstein and P. Deligne, Faisceaux perverse, in "Analyse et Topologie sur les Espaces Singulieres," Astérisque **100**, Soc. Math. France, 1982, 5-171.

[Br] T. Bridgeland, Flops and derived categories, Invent. Math. **147** (2002), no. 3, 613-632.

[Bz] R. Bezrukavnikov, Perverse coherent sheaves (after Deligne), eprint math.AG/0005152 at http://arXiv.org.

[Co] B. Conrad, "Grothendieck Duality and Base Change," Lecture Notes in Math. **1750**, Springer, 2000.

[HS] R. Hübl and P. Sastry, Regular differential forms and relative duality, Amer. J. Math. **115** (1993), no. 4, 749-787.

[HK] R. Hübl and E. Kunz, Regular differential forms and duality for projective morphisms, J. Reine Angew. Math. **410** (1990), 84-108.

[Ka] M. Kashiwara, T-structures on the derived categories of holonomic D-modules and coherent O-modules, Moscow Mathematical Journal, **4** (2004), no. 4, 847-868.

[KS] M. Kashiwara and P. Schapira, "Sheaves on Manifolds," Springer, 1990.

[Li] J. Lipman, Dualizing sheaves, differentials and residues on algebraic varieties, Astérisque **117** (1984).

[LNS] J. Lipman, S. Nayak and P. Sastry, "Variance and Duality for Cousin Complexes on Formal Schemes", Contemp. Math. **375**, AMS 2005.

[Ne] A. Neeman, The Grothendieck duality theorem via Bousfield's techniques and Brown representability, Jour. Amer. Math. Soc. **9** (1996), 205-236.

[RD] R. Hartshorne, "Residues and Duality," Lecture Notes in Math. **20**, Springer-Verlag, Berlin, 1966.

[VdB] M. Van den Bergh, Existence theorems for dualizing complexes over non-commutative graded and filtered rings, J. Algebra **195** (1997), no. 2, 662-679.

[Ye1] A. Yekutieli, Dualizing complexes over noncommutative graded algebras, J. Algebra **153** (1992), 41-84.

[Ye2] A. Yekutieli, "An Explicit Construction of the Grothendieck Residue Complex" (with an appendix by P. Sastry), Astérisque **208** (1992).

[Ye3] A. Yekutieli, Smooth formal embeddings and the residue complex, Canadian J. Math. **50** (1998), 863-896.

[Ye4] A. Yekutieli, Residues and differential operators on schemes, Duke Math. J. **95** (1998), 305-341.

[Ye5] A. Yekutieli, Dualizing complexes, Morita equivalence and the derived Picard group of a ring, J. London Math. Soc. (2) **60** (1999), no. 3, 723-746.

[Ye6] A. Yekutieli, Rigidity, Residues, and Grothendieck Duality for Schemes, in preparation.

[YZ1] A. Yekutieli and J.J. Zhang, Rings with Auslander dualizing complexes, J. Algebra **213** (1999), no. 1, 1-51.

[YZ2] A. Yekutieli and J.J. Zhang, Rigid Dualizing Complexes and Perverse Modules over Differential Algebras, Compositio Math. **141** (2005), 620-654.

[YZ3] A. Yekutieli and J.J. Zhang, Dualizing Complexes and Perverse Sheaves on Noncommutative Ringed Schemes, Selecta Math. **12** (2006), 137-177.

[YZ4] A. Yekutieli and J.J. Zhang, Rigid Complexes via DG Algebras, to appear in Trans. Amer. Math. Soc. Eprint math.AC/0603733 at http://arXiv.org.

[YZ5] A. Yekutieli and J.J. Zhang, Rigid Dualizing Complexes over Commutative Rings, to appear in Algebr. Represent. Theory, eprint math.AG/0601654 http://arXiv.org.