Rigid Dualizing Complexes and Perverse Coherent Sheaves

Amnon Yekutieli

Department of Mathematics
Ben Gurion University

Notes available at http://www.math.bgu.ac.il/∼amyekut/lectures

Written 30 Oct 2007

Here is the plan of my lecture:

Here is the plan of my lecture:

1. Background on Dualizing Complexes
2. Rigid Complexes over Rings
3. Rigid Dualizing Complexes over Rings
4. Rigid Dualizing Complexes over Schemes
5. Perverse Coherent Sheaves
6. Cohen-Macaulay Complexes

Here is the plan of my lecture:

1. Background on Dualizing Complexes
2. Rigid Complexes over Rings
3. Rigid Dualizing Complexes over Rings
4. Rigid Dualizing Complexes over Schemes
5. Perverse Coherent Sheaves
6. Cohen-Macaulay Complexes

This talk is about joint work with James Zhang (Seattle).

1. Background on Dualizing Complexes

1. Background on Dualizing Complexes

Dualizing complexes over schemes were introduced by Grothendieck in the 1960's (see [RD]), as a vast generalization of Serre duality.

1. Background on Dualizing Complexes

Dualizing complexes over schemes were introduced by Grothendieck in the 1960's (see [RD]), as a vast generalization of Serre duality.

Suppose X is a noetherian scheme.

1. Background on Dualizing Complexes

Dualizing complexes over schemes were introduced by Grothendieck in the 1960's (see [RD]), as a vast generalization of Serre duality.

Suppose X is a noetherian scheme.

We denote by $\mathsf{Mod}\,\mathcal{O}_X$ the category of sheaves of $\mathcal{O}_X$-modules, and by $\mathsf{D}(\mathsf{Mod}\,\mathcal{O}_X)$ its derived category.

1. Background on Dualizing Complexes

Dualizing complexes over schemes were introduced by Grothendieck in the 1960's (see [RD]), as a vast generalization of Serre duality.

Suppose X is a noetherian scheme.

We denote by $\mathsf{Mod}\,\mathcal{O}_X$ the category of sheaves of $\mathcal{O}_X$-modules, and by $\mathsf{D}(\mathsf{Mod}\,\mathcal{O}_X)$ its derived category.

The full subcategory of bounded complexes with coherent cohomologies is $\mathsf{D}^{\mathrm{b}}_{\mathrm{c}}(\mathsf{Mod}\,\mathcal{O}_X)$. It is equivalent to $\mathsf{D}^{\mathrm{b}}(\mathsf{Coh}\,\mathcal{O}_X)$.

Definition 1.1. (Grothendieck [RD]) A dualizing complex on X is a complex $\mathcal{R} \in \mathsf{D}^{\mathrm{b}}_{\mathrm{c}}(\mathsf{Mod}\,\mathcal{O}_X)$ satisfying the two conditions:

(i) $\mathcal{R}$ has finite injective dimension.

(ii) The canonical morphism $\mathcal{O}_X \to \mathrm{R}\mathcal{H}om_{\mathcal{O}_X}(\mathcal{R}, \mathcal{R})$ is an isomorphism.

Definition 1.1. (Grothendieck [RD]) A dualizing complex on X is a complex $\mathcal{R} \in \mathsf{D}^{\mathrm{b}}_{\mathrm{c}}(\mathsf{Mod}\,\mathcal{O}_X)$ satisfying the two conditions:

(i) $\mathcal{R}$ has finite injective dimension.

(ii) The canonical morphism $\mathcal{O}_X \to \mathrm{R}\mathcal{H}om_{\mathcal{O}_X}(\mathcal{R}, \mathcal{R})$ is an isomorphism.

It follows that the functor

$$\mathcal{M} \mapsto \mathrm{R}\mathcal{H}om_{\mathcal{O}_X}(\mathcal{M}, \mathcal{R})$$

is an auto-duality of $\mathsf{D}^{\mathrm{b}}_{\mathrm{c}}(\mathsf{Mod}\,\mathcal{O}_X)$.

Definition 1.1. (Grothendieck [RD]) A dualizing complex on X is a complex $\mathcal{R} \in \mathsf{D}^{\mathrm{b}}_{\mathrm{c}}(\mathsf{Mod}\, \mathcal{O}_X)$ satisfying the two conditions:

(i) $\mathcal{R}$ has finite injective dimension.

(ii) The canonical morphism $\mathcal{O}_X \to \mathrm{R}\mathcal{H}om_{\mathcal{O}_X}(\mathcal{R}, \mathcal{R})$ is an isomorphism.

It follows that the functor

$$\mathcal{M} \mapsto \mathrm{R}\mathcal{H}om_{\mathcal{O}_X}(\mathcal{M}, \mathcal{R})$$

is an auto-duality of $\mathsf{D}^{\mathrm{b}}_{\mathrm{c}}(\mathsf{Mod}\, \mathcal{O}_X)$.

When $X = \operatorname{Spec} A$ is affine, the complex $R := \mathrm{R}\Gamma(X, \mathcal{R}) \in \mathsf{D}^{\mathrm{b}}_{\mathrm{f}}(\mathsf{Mod}\, A)$ is called a dualizing complex over A, and

$$M \mapsto \mathrm{RHom}_A(M, R)$$

is an auto-duality of $\mathsf{D}^{\mathrm{b}}_{\mathrm{f}}(\mathsf{Mod}\, A)$.

Suppose $\mathbb{K}$ is a regular noetherian ring of finite Krull dimension, and X is a finite type $\mathbb{K}$-scheme, with structural morphism $\pi : X \to \operatorname{Spec} \mathbb{K}$.

Suppose $\mathbb{K}$ is a regular noetherian ring of finite Krull dimension, and X is a finite type $\mathbb{K}$-scheme, with structural morphism $\pi : X \to \operatorname{Spec} \mathbb{K}$.

Then, according to [RD], there is a special dualizing complex on X, namely the Grothendieck dualizing complex $\mathcal{R}_X := \pi^! \mathbb{K}$.

Suppose $\mathbb{K}$ is a regular noetherian ring of finite Krull dimension, and X is a finite type $\mathbb{K}$-scheme, with structural morphism $\pi : X \to \operatorname{Spec} \mathbb{K}$.

Then, according to [RD], there is a special dualizing complex on X, namely the Grothendieck dualizing complex $\mathcal{R}_X := \pi^{!} \mathbb{K}$.

The proof of existence of this complex, and its properties, is very difficult.

Suppose $\mathbb{K}$ is a regular noetherian ring of finite Krull dimension, and X is a finite type $\mathbb{K}$-scheme, with structural morphism $\pi : X \to \operatorname{Spec} \mathbb{K}$.

Then, according to [RD], there is a special dualizing complex on X, namely the Grothendieck dualizing complex $\mathcal{R}_X := \pi^! \mathbb{K}$.

The proof of existence of this complex, and its properties, is very difficult.

In this lecture I will explain an alternative approach to Grothendieck duality.

Suppose $\mathbb{K}$ is a regular noetherian ring of finite Krull dimension, and X is a finite type $\mathbb{K}$-scheme, with structural morphism $\pi : X \to \operatorname{Spec} \mathbb{K}$.

Then, according to [RD], there is a special dualizing complex on X, namely the Grothendieck dualizing complex $\mathcal{R}_X := \pi^! \mathbb{K}$.

The proof of existence of this complex, and its properties, is very difficult.

In this lecture I will explain an alternative approach to Grothendieck duality.

For other approaches see the papers in the references, mainly by Joseph Lipman and his coauthors.

2. Rigid Complexes over Rings

2. Rigid Complexes over Rings

Suppose A is a commutative ring, and B is a commutative A-algebra.

2. Rigid Complexes over Rings

Suppose A is a commutative ring, and B is a commutative A-algebra.

In [YZ4] we constructed a functor

$$\mathrm{Sq}_{B/A} : \mathsf{D}(\mathsf{Mod}\, B) \to \mathsf{D}(\mathsf{Mod}\, B),$$

called the squaring operation.

2. Rigid Complexes over Rings

Suppose A is a commutative ring, and B is a commutative A-algebra.

In [YZ4] we constructed a functor

$$\operatorname{Sq}_{B/A} : \mathsf{D}(\mathsf{Mod}\, B) \to \mathsf{D}(\mathsf{Mod}\, B),$$

called the squaring operation.

When B is flat over A one has

$$\operatorname{Sq}_{B/A} M = \operatorname{RHom}_{B \otimes_A B}(B, M \otimes^{\mathrm{L}}_A M)$$

for $M \in \mathsf{D}(\mathsf{Mod}\, B)$.

2. Rigid Complexes over Rings

Suppose A is a commutative ring, and B is a commutative A-algebra.

In [YZ4] we constructed a functor

$$\mathrm{Sq}_{B/A} : \mathsf{D}(\mathsf{Mod}\, B) \to \mathsf{D}(\mathsf{Mod}\, B),$$

called the squaring operation.

When B is flat over A one has

$$\mathrm{Sq}_{B/A}\, M = \mathrm{RHom}_{B \otimes_A B}(B, M \otimes^{\mathrm{L}}_{A} M)$$

for $M \in \mathsf{D}(\mathsf{Mod}\, B)$.

But in general one has to use DG algebras to define $\mathrm{Sq}_{B/A}\, M$.

The functor $\operatorname{Sq}_{B/A}$ is quadratic, in the following sense. Given a morphism $\phi : M \to N$ in $\mathsf{D}(\mathsf{Mod}\, B)$, and an element $b \in B$, one has

$$\operatorname{Sq}_{B/A}(b\phi) = b^2 \operatorname{Sq}_{B/A}(\phi)$$

in

$$\operatorname{Hom}_{\mathsf{D}(\mathsf{Mod}\, B)}(\operatorname{Sq}_{B/A} M, \operatorname{Sq}_{B/A} N).$$

The functor $\operatorname{Sq}_{B/A}$ is quadratic, in the following sense. Given a morphism $\phi : M \to N$ in $\mathsf{D}(\mathsf{Mod}\, B)$, and an element $b \in B$, one has

$$\operatorname{Sq}_{B/A}(b\phi) = b^2 \operatorname{Sq}_{B/A}(\phi)$$

in

$$\operatorname{Hom}_{\mathsf{D}(\mathsf{Mod}\, B)}(\operatorname{Sq}_{B/A} M, \operatorname{Sq}_{B/A} N).$$

Definition 2.1. Let B be a noetherian A-algebra, and let M be a complex in $\mathsf{D}^{\mathrm{b}}_{\mathrm{f}}(\mathsf{Mod}\, B)$ that has finite flat dimension over A. Assume

$$\rho : M \xrightarrow{\simeq} \operatorname{Sq}_{B/A} M$$

is an isomorphism in $\mathsf{D}(\mathsf{Mod}\, B)$. Then the pair (M, ρ) is called a rigid complex over B relative to A.

Definition 2.2. Say (M, ρ) and (N, σ) are rigid complexes over B relative to A. A morphism $\phi : M \to N$ in $\mathsf{D}(\mathsf{Mod}\, B)$ is called a rigid morphism relative to A if the diagram

$$\begin{array}{ccc} M & \xrightarrow{\rho} & \operatorname{Sq}_{B/A} M \\ {\scriptstyle \phi}\downarrow & & \downarrow{\scriptstyle \operatorname{Sq}_{B/A}(\phi)} \\ N & \xrightarrow{\sigma} & \operatorname{Sq}_{B/A} N \end{array}$$

is commutative.

3. Rigid Dualizing Complexes over Rings

3. Rigid Dualizing Complexes over Rings

From now on $\mathbb{K}$ denotes a fixed noetherian regular ring of finite Krull dimension (e.g. a field or the ring of integers).

3. Rigid Dualizing Complexes over Rings

From now on $\mathbb{K}$ denotes a fixed noetherian regular ring of finite Krull dimension (e.g. a field or the ring of integers).

Let A be a noetherian $\mathbb{K}$-algebra. The next definition is due to Michel Van den Bergh [VdB].

3. Rigid Dualizing Complexes over Rings

From now on $\mathbb{K}$ denotes a fixed noetherian regular ring of finite Krull dimension (e.g. a field or the ring of integers).

Let A be a noetherian $\mathbb{K}$-algebra. The next definition is due to Michel Van den Bergh [VdB].

Definition 3.1. A rigid dualizing complex over A relative to $\mathbb{K}$ is a rigid complex (R, ρ), such that R is a dualizing complex.

3. Rigid Dualizing Complexes over Rings

From now on $\mathbb{K}$ denotes a fixed noetherian regular ring of finite Krull dimension (e.g. a field or the ring of integers).

Let A be a noetherian $\mathbb{K}$-algebra. The next definition is due to Michel Van den Bergh [VdB].

Definition 3.1. A rigid dualizing complex over A relative to $\mathbb{K}$ is a rigid complex (R, ρ), such that R is a dualizing complex.

Note that the only rigid automorphism of a rigid dualizing complex (R, ρ) is the identity $\mathbf{1}_R$. Indeed, any automorphism ϕ of R has to be of the form $\phi = a\mathbf{1}_R$ for some invertible element $a \in A$. If ϕ is rigid then $a^2 = a$, and hence $a = 1$.

Recall that an A-algebra B is called essentially finite type if it is a localization of some finitely generated A-algebra.

Recall that an A-algebra B is called essentially finite type if it is a localization of some finitely generated A-algebra.

The next theorems are taken from [YZ5]

Recall that an A-algebra B is called essentially finite type if it is a localization of some finitely generated A-algebra.

The next theorems are taken from [YZ5]

Theorem 3.2. *Let A be an essentially finite type $\mathbb{K}$-algebra. Then A has a rigid dualizing complex (R_A, ρ_A), which is unique up to a unique rigid isomorphism.*

Recall that an A-algebra B is called essentially finite type if it is a localization of some finitely generated A-algebra.

The next theorems are taken from [YZ5]

Theorem 3.2. *Let A be an essentially finite type $\mathbb{K}$-algebra. Then A has a rigid dualizing complex (R_A, ρ_A), which is unique up to a unique rigid isomorphism.*

Recall that a ring homomorphism $f^* : A \to B$ is called finite if B is a finitely generated A-module.

Recall that an A-algebra B is called essentially finite type if it is a localization of some finitely generated A-algebra.

The next theorems are taken from [YZ5]

Theorem 3.2. *Let A be an essentially finite type $\mathbb{K}$-algebra. Then A has a rigid dualizing complex (R_A, ρ_A), which is unique up to a unique rigid isomorphism.*

Recall that a ring homomorphism $f^* : A \to B$ is called finite if B is a finitely generated A-module.

Theorem 3.3. *Let A and B be essentially finite type $\mathbb{K}$-algebras, and let $f^* : A \to B$ be a finite homomorphism. Then the complex $\mathrm{RHom}_A(B, R_A)$ has an induced rigidifying isomorphism, and there is a unique rigid isomorphism*

$$\mathrm{RHom}_A(B, R_A) \cong R_B.$$

We say that B is essentially smooth of relative dimension n over A if it is essentially finite type, formally smooth, and the rank of the projective B-module $\Omega^1_{B/A}$ is n. When $n = 0$ we say B is essentially étale.

We say that B is essentially smooth of relative dimension n over A if it is essentially finite type, formally smooth, and the rank of the projective B-module $\Omega^1_{B/A}$ is n. When $n = 0$ we say B is essentially étale.

Example 3.4. If A' is a localization of A then $A \to A'$ is essentially étale. If $B = A[t_1, \ldots, t_n]$ is a polynomial algebra then $A \to B$ is essentially smooth of relative dimension n.

We say that B is essentially smooth of relative dimension n over A if it is essentially finite type, formally smooth, and the rank of the projective B-module $\Omega^1_{B/A}$ is n. When $n = 0$ we say B is essentially étale.

Example 3.4. If A' is a localization of A then $A \to A'$ is essentially étale. If $B = A[t_1, \ldots, t_n]$ is a polynomial algebra then $A \to B$ is essentially smooth of relative dimension n.

Theorem 3.5. *Let A and B be essentially finite type $\mathbb{K}$-algebras, and let $f^* : A \to B$ be an essentially smooth homomorphism of relative dimension n. Then the complex $\Omega^n_{B/A}[n] \otimes_A R_A$ has an induced rigidifying isomorphism, and there is a unique rigid isomorphism*

$$\Omega^n_{B/A}[n] \otimes_A R_A \cong R_B.$$

Taking $n = 0$ we get an important corollary:

Taking $n = 0$ we get an important corollary:

Corollary 3.6. *Given an essentially étale homomorphism $f^* : A \to B$, there is a unique rigid isomorphism*

$$B \otimes_A R_A \cong R_B.$$

4. Rigid Dualizing Complexes over Schemes

4. Rigid Dualizing Complexes over Schemes

Definition 4.1. Let X be a finite type separated $\mathbb{K}$-scheme. A rigid dualizing complex over X (relative to $\mathbb{K}$) is a the data $(\mathcal{R}, \boldsymbol{\rho})$, where:

4. Rigid Dualizing Complexes over Schemes

Definition 4.1. Let X be a finite type separated $\mathbb{K}$-scheme. A rigid dualizing complex over X (relative to $\mathbb{K}$) is a the data $(\mathcal{R}, \boldsymbol{\rho})$, where:

1. $\mathcal{R} \in \mathsf{D}^{\mathrm{b}}_{\mathrm{c}}(\mathsf{Mod}\, \mathcal{O}_X)$ is a dualizing complex on X.
2. $\boldsymbol{\rho} = \{\rho_U\}$ is a collection of rigidifying isomorphisms, indexed by the affine open sets of X. Namely, for any affine open set U, ρ_U is a rigidifying isomorphism for the dualizing complex $\mathrm{R}\Gamma(U, \mathcal{R})$ over the $\mathbb{K}$-algebra $A := \Gamma(U, \mathcal{O}_X)$.

4. Rigid Dualizing Complexes over Schemes

Definition 4.1. Let X be a finite type separated $\mathbb{K}$-scheme. A rigid dualizing complex over X (relative to $\mathbb{K}$) is a the data $(\mathcal{R}, \boldsymbol{\rho})$, where:

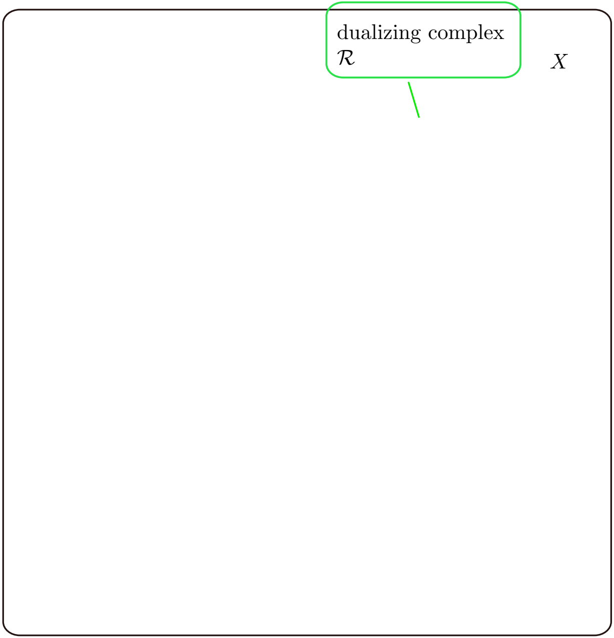
1. $\mathcal{R} \in \mathsf{D}^{\mathrm{b}}_{\mathrm{c}}(\mathsf{Mod}\, \mathcal{O}_X)$ is a dualizing complex on X.
2. $\boldsymbol{\rho} = \{\rho_U\}$ is a collection of rigidifying isomorphisms, indexed by the affine open sets of X. Namely, for any affine open set U, ρ_U is a rigidifying isomorphism for the dualizing complex $\mathrm{R}\Gamma(U, \mathcal{R})$ over the $\mathbb{K}$-algebra $A := \Gamma(U, \mathcal{O}_X)$.

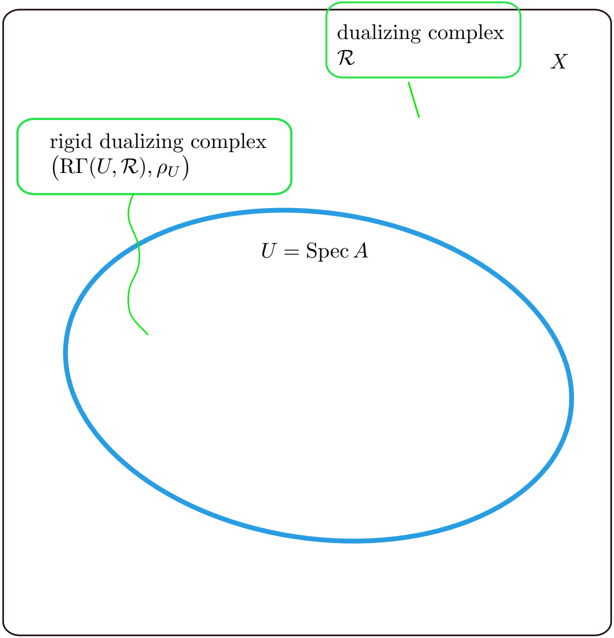
The condition is:

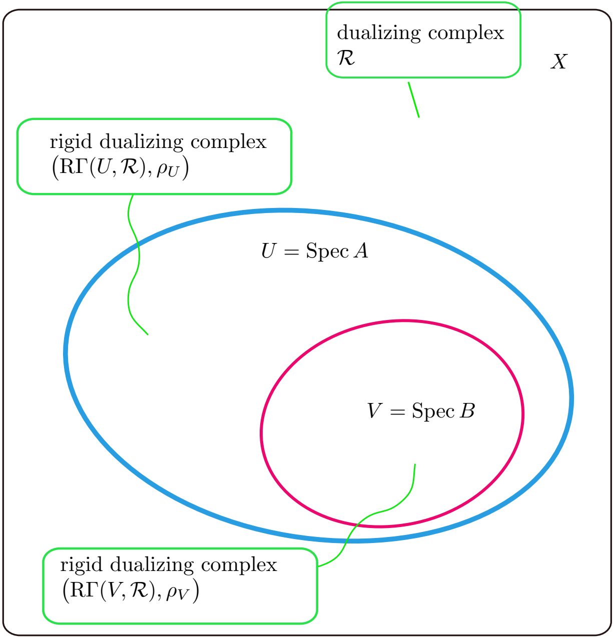
(†) For any inclusion $V \subset U$ of affine open sets, with $A := \Gamma(U, \mathcal{O}_X)$ and $B := \Gamma(V, \mathcal{O}_X)$, the canonical isomorphism

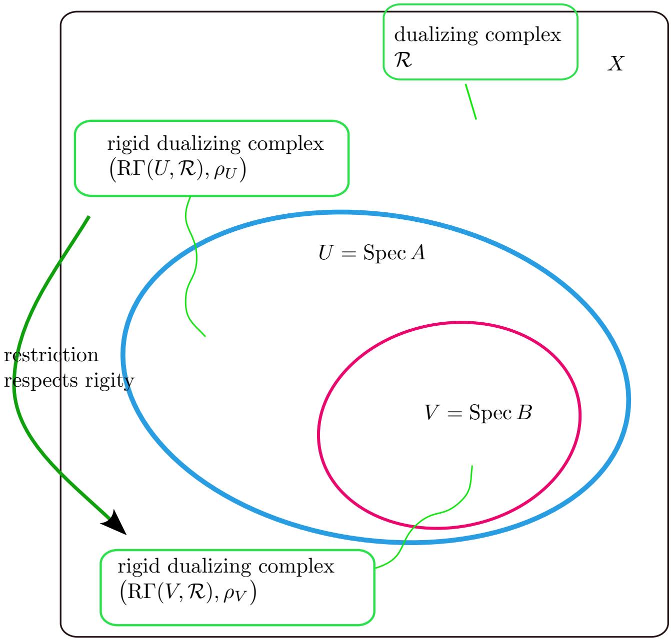
$$B \otimes_A \mathrm{R}\Gamma(U, \mathcal{R}) \cong \mathrm{R}\Gamma(V, \mathcal{R})$$

is rigid, with respect to the rigidifying isomorphisms ρ_U and ρ_V.


dualizing complex
$\mathcal{R}$
X


dualizing complex
$\mathcal{R}$
X
rigid dualizing complex
$\big(\mathrm{R}\Gamma(U, \mathcal{R}), \rho_U\big)$
$U = \operatorname{Spec} A$


dualizing complex
$\mathcal{R}$
X
rigid dualizing complex
$(\mathrm{R}\Gamma(U, \mathcal{R}), \rho_U)$
$U = \operatorname{Spec} A$
$V = \operatorname{Spec} B$
rigid dualizing complex
$(\mathrm{R}\Gamma(V, \mathcal{R}), \rho_V)$


dualizing complex $\mathcal{R}$
X
rigid dualizing complex $\big(\mathrm{R}\Gamma(U, \mathcal{R}), \rho_U\big)$
$U = \operatorname{Spec} A$
restriction
respects rigity
$V = \operatorname{Spec} B$
rigid dualizing complex $\big(\mathrm{R}\Gamma(V, \mathcal{R}), \rho_V\big)$

We would like to prove existence and uniqueness of a rigid dualizing complex on X.

We would like to prove existence and uniqueness of a rigid dualizing complex on X.

Consider an affine open set $U \subset X$, and let $A := \Gamma(U, \mathcal{O}_X)$. According to Theorem 3.2 there exists a rigid dualizing complex (R_A, ρ_A) over A. Let us denote by $\mathcal{R}_U$ the corresponding complex of sheaves on U, which is of course a dualizing complex over U. And let's write $\rho_U := \rho_A$.

We would like to prove existence and uniqueness of a rigid dualizing complex on X.

Consider an affine open set $U \subset X$, and let $A := \Gamma(U, \mathcal{O}_X)$. According to Theorem 3.2 there exists a rigid dualizing complex (R_A, ρ_A) over A. Let us denote by $\mathcal{R}_U$ the corresponding complex of sheaves on U, which is of course a dualizing complex over U. And let's write $\rho_U := \rho_A$.

Now suppose $V \subset U$ is another affine open set, with $B := \Gamma(V, \mathcal{O}_X)$ and rigid dualizing complex (R_B, ρ_B). According to Corollary 3.6 there is a unique isomorphism

$$\phi_{V/U} : \mathcal{R}_U|_V \xrightarrow{\simeq} \mathcal{R}_V \tag{4.2}$$

in $\mathsf{D}(\mathsf{Mod}\, \mathcal{O}_V)$ which respects rigidity.

We would like to prove existence and uniqueness of a rigid dualizing complex on X.

Consider an affine open set $U \subset X$, and let $A := \Gamma(U, \mathcal{O}_X)$. According to Theorem 3.2 there exists a rigid dualizing complex (R_A, ρ_A) over A. Let us denote by $\mathcal{R}_U$ the corresponding complex of sheaves on U, which is of course a dualizing complex over U. And let's write $\rho_U := \rho_A$.

Now suppose $V \subset U$ is another affine open set, with $B := \Gamma(V, \mathcal{O}_X)$ and rigid dualizing complex (R_B, ρ_B). According to Corollary 3.6 there is a unique isomorphism

$$\phi_{V/U} : \mathcal{R}_U|_V \xrightarrow{\simeq} \mathcal{R}_V \tag{4.2}$$

in $\mathsf{D}(\mathsf{Mod}\, \mathcal{O}_V)$ which respects rigidity.

Therefore given an affine open set $W \subset V$, these isomorphisms satisfy

$$\phi_{W/V} \circ \phi_{V/U} = \phi_{W/U}.$$

The next step would be to try to glue the affine dualizing complexes $\mathcal{R}_U$ to a global complex $\mathcal{R}_X \in \mathsf{D}^{\mathrm{b}}_{\mathrm{c}}(\mathsf{Mod}\,\mathcal{O}_X)$.

The next step would be to try to glue the affine dualizing complexes $\mathcal{R}_U$ to a global complex $\mathcal{R}_X \in \mathsf{D}^{\mathrm{b}}_{\mathrm{c}}(\mathsf{Mod}\, \mathcal{O}_X)$.

But here we encounter a genuine problem: usually objects in derived categories cannot be glued!

The next step would be to try to glue the affine dualizing complexes $\mathcal{R}_U$ to a global complex $\mathcal{R}_X \in \mathsf{D}^{\mathrm{b}}_{\mathrm{c}}(\mathsf{Mod}\,\mathcal{O}_X)$.

But here we encounter a genuine problem: usually objects in derived categories cannot be glued!

Grothendieck's solution in the commutative case, in [RD], was to use Cousin complexes. This solution can be used in our setup too, but it has a disadvantage: we are forced to leave the derived category, and then return to it.

The next step would be to try to glue the affine dualizing complexes $\mathcal{R}_U$ to a global complex $\mathcal{R}_X \in \mathsf{D}^{\mathrm{b}}_{\mathrm{c}}(\mathsf{Mod}\, \mathcal{O}_X)$.

But here we encounter a genuine problem: usually objects in derived categories cannot be glued!

Grothendieck's solution in the commutative case, in [RD], was to use Cousin complexes. This solution can be used in our setup too, but it has a disadvantage: we are forced to leave the derived category, and then return to it.

We propose an alternative solution: perverse coherent sheaves.

The next step would be to try to glue the affine dualizing complexes $\mathcal{R}_U$ to a global complex $\mathcal{R}_X \in \mathsf{D}^{\mathrm{b}}_{\mathrm{c}}(\mathsf{Mod}\, \mathcal{O}_X)$.

But here we encounter a genuine problem: usually objects in derived categories cannot be glued!

Grothendieck's solution in the commutative case, in [RD], was to use Cousin complexes. This solution can be used in our setup too, but it has a disadvantage: we are forced to leave the derived category, and then return to it.

We propose an alternative solution: perverse coherent sheaves.

Remark 4.3. For noncommutative ringed schemes one is forced to use perverse coherent sheaves, since Cousin complexes are ill-behaved. See [YZ3].

5. Perverse Coherent Sheaves

5. Perverse Coherent Sheaves

The notion of t-structures and perverse sheaves were introduced by Beilinson, Bernstein and Deligne [BBD] around 1980. This was in the context of intersection cohomology on singular spaces. For such a space X they were interested in t-structures on subcategories of $\mathsf{D}(\mathsf{Mod}\,\mathbb{K}_X)$, where $\mathbb{K}_X$ is a constant sheaf of rings on X.

5. Perverse Coherent Sheaves

The notion of t-structures and perverse sheaves were introduced by Beilinson, Bernstein and Deligne [BBD] around 1980. This was in the context of intersection cohomology on singular spaces. For such a space X they were interested in t-structures on subcategories of $\mathsf{D}(\mathsf{Mod}\,\mathbb{K}_X)$, where $\mathbb{K}_X$ is a constant sheaf of rings on X.

Perverse coherent sheaves came into the scene only very recently, independently in the work of Bezrukavnikov (after Deligne) [Bz], Bridgeland [Br], Kashiwara [Ka] and our paper [YZ3].

Let me recall what is a t-structure on a triangulated category D. It consists of the datum of two full subcategories $\mathsf{D}^{\leq 0}$ and $\mathsf{D}^{\geq 0}$ satisfying the axioms below, where $\mathsf{D}^{\leq n} := \mathsf{D}^{\leq 0}[-n]$ and $\mathsf{D}^{\geq n} := \mathsf{D}^{\geq 0}[-n]$.

Let me recall what is a t-structure on a triangulated category D. It consists of the datum of two full subcategories $\mathsf{D}^{\leq 0}$ and $\mathsf{D}^{\geq 0}$ satisfying the axioms below, where $\mathsf{D}^{\leq n} := \mathsf{D}^{\leq 0}[-n]$ and $\mathsf{D}^{\geq n} := \mathsf{D}^{\geq 0}[-n]$.

(i) $\mathsf{D}^{\leq -1} \subset \mathsf{D}^{\leq 0}$ and $\mathsf{D}^{\geq 1} \subset \mathsf{D}^{\geq 0}$.

(ii) $\mathrm{Hom}_{\mathsf{D}}(M, N) = 0$ for $M \in \mathsf{D}^{\leq 0}$ and $N \in \mathsf{D}^{\geq 1}$.

(iii) For any $M \in \mathsf{D}$ there is a distinguished triangle

$$M' \to M \to M'' \to M'[1]$$

in D with $M' \in \mathsf{D}^{\leq 0}$ and $M'' \in \mathsf{D}^{\geq 1}$.

Let me recall what is a t-structure on a triangulated category D. It consists of the datum of two full subcategories $\mathsf{D}^{\leq 0}$ and $\mathsf{D}^{\geq 0}$ satisfying the axioms below, where $\mathsf{D}^{\leq n} := \mathsf{D}^{\leq 0}[-n]$ and $\mathsf{D}^{\geq n} := \mathsf{D}^{\geq 0}[-n]$.

(i) $\mathsf{D}^{\leq -1} \subset \mathsf{D}^{\leq 0}$ and $\mathsf{D}^{\geq 1} \subset \mathsf{D}^{\geq 0}$.

(ii) $\mathrm{Hom}_{\mathsf{D}}(M, N) = 0$ for $M \in \mathsf{D}^{\leq 0}$ and $N \in \mathsf{D}^{\geq 1}$.

(iii) For any $M \in \mathsf{D}$ there is a distinguished triangle

$$M' \to M \to M'' \to M'[1]$$

in D with $M' \in \mathsf{D}^{\leq 0}$ and $M'' \in \mathsf{D}^{\geq 1}$.

When these conditions are satisfied one defines the heart of D to be the full subcategory $\mathsf{D}^{0} := \mathsf{D}^{\leq 0} \cap \mathsf{D}^{\geq 0}$. This is an abelian category.

Given a scheme X, the derived category $\mathsf{D}^{\mathrm{b}}_{\mathrm{c}}(\mathsf{Mod}\,\mathcal{O}_X)$ has the standard t-structure, in which

$$\begin{aligned}
&\mathsf{D}^{\mathrm{b}}_{\mathrm{c}}(\mathsf{Mod}\,\mathcal{O}_X)^{\leq 0} := \\
&\quad \{\mathcal{M} \in \mathsf{D}^{\mathrm{b}}_{\mathrm{c}}(\mathsf{Mod}\,\mathcal{O}_X) \mid \mathrm{H}^i \mathcal{M} = 0 \text{ for all } i > 0\}, \\
&\mathsf{D}^{\mathrm{b}}_{\mathrm{c}}(\mathsf{Mod}\,\mathcal{O}_X)^{\geq 0} := \\
&\quad \{\mathcal{M} \in \mathsf{D}^{\mathrm{b}}_{\mathrm{c}}(\mathsf{Mod}\,\mathcal{O}_X) \mid \mathrm{H}^i \mathcal{M} = 0 \text{ for all } i < 0\}.
\end{aligned}$$

Given a scheme X, the derived category $\mathsf{D}^{\mathrm{b}}_{\mathrm{c}}(\mathsf{Mod}\,\mathcal{O}_X)$ has the standard t-structure, in which

$$\begin{aligned}
&\mathsf{D}^{\mathrm{b}}_{\mathrm{c}}(\mathsf{Mod}\,\mathcal{O}_X)^{\leq 0} := \\
&\quad \{\mathcal{M} \in \mathsf{D}^{\mathrm{b}}_{\mathrm{c}}(\mathsf{Mod}\,\mathcal{O}_X) \mid \mathrm{H}^i \mathcal{M} = 0 \text{ for all } i > 0\}, \\
&\mathsf{D}^{\mathrm{b}}_{\mathrm{c}}(\mathsf{Mod}\,\mathcal{O}_X)^{\geq 0} := \\
&\quad \{\mathcal{M} \in \mathsf{D}^{\mathrm{b}}_{\mathrm{c}}(\mathsf{Mod}\,\mathcal{O}_X) \mid \mathrm{H}^i \mathcal{M} = 0 \text{ for all } i < 0\}.
\end{aligned}$$

The heart $\mathsf{D}^{\mathrm{b}}_{\mathrm{c}}(\mathsf{Mod}\,\mathcal{O}_X)^{0}$ is equivalent to the category $\mathsf{Coh}\,\mathcal{O}_X$ of coherent sheaves.

Given a scheme X, the derived category $\mathsf{D}^{\mathrm{b}}_{\mathrm{c}}(\mathsf{Mod}\,\mathcal{O}_X)$ has the standard t-structure, in which

$$\begin{aligned}
&\mathsf{D}^{\mathrm{b}}_{\mathrm{c}}(\mathsf{Mod}\,\mathcal{O}_X)^{\leq 0} := \\
&\quad \{\mathcal{M} \in \mathsf{D}^{\mathrm{b}}_{\mathrm{c}}(\mathsf{Mod}\,\mathcal{O}_X) \mid \mathrm{H}^i \mathcal{M} = 0 \text{ for all } i > 0\}, \\
&\mathsf{D}^{\mathrm{b}}_{\mathrm{c}}(\mathsf{Mod}\,\mathcal{O}_X)^{\geq 0} := \\
&\quad \{\mathcal{M} \in \mathsf{D}^{\mathrm{b}}_{\mathrm{c}}(\mathsf{Mod}\,\mathcal{O}_X) \mid \mathrm{H}^i \mathcal{M} = 0 \text{ for all } i < 0\}.
\end{aligned}$$

The heart $\mathsf{D}^{\mathrm{b}}_{\mathrm{c}}(\mathsf{Mod}\,\mathcal{O}_X)^0$ is equivalent to the category $\mathsf{Coh}\,\mathcal{O}_X$ of coherent sheaves.

Other t-structures will be referred to as perverse t-structures.

Here is an observation. Suppose A is an essentially finite type $\mathbb{K}$-algebra, where as before $\mathbb{K}$ is a finite dimensional regular noetherian ring. Let R_A be the rigid dualizing complex of A.

Here is an observation. Suppose A is an essentially finite type $\mathbb{K}$-algebra, where as before $\mathbb{K}$ is a finite dimensional regular noetherian ring. Let R_A be the rigid dualizing complex of A.

Then the duality $\mathrm{D} := \mathrm{RHom}_A(-, R_A)$ gives rise to a perverse t-structure

$$\begin{aligned} &{}^{\mathrm{p}}\mathsf{D}^{\mathrm{b}}_{\mathrm{f}}(\mathsf{Mod}\, A)^{\leq 0} := \\ &\qquad \{M \mid \mathrm{H}^i \mathrm{D} M = 0 \text{ for all } i < 0\}, \\ &{}^{\mathrm{p}}\mathsf{D}^{\mathrm{b}}_{\mathrm{f}}(\mathsf{Mod}\, A)^{\geq 0} := \\ &\qquad \{M \mid \mathrm{H}^i \mathrm{D} M = 0 \text{ for all } i > 0\}. \end{aligned}$$

on $\mathsf{D}^{\mathrm{b}}_{\mathrm{f}}(\mathsf{Mod}\, A)$.

Here is an observation. Suppose A is an essentially finite type $\mathbb{K}$-algebra, where as before $\mathbb{K}$ is a finite dimensional regular noetherian ring. Let R_A be the rigid dualizing complex of A.

Then the duality $\mathrm{D} := \mathrm{RHom}_A(-, R_A)$ gives rise to a perverse t-structure

$$\begin{aligned} &{}^{\mathrm{p}}\mathsf{D}^{\mathrm{b}}_{\mathrm{f}}(\mathsf{Mod}\, A)^{\leq 0} := \\ &\qquad \{M \mid \mathrm{H}^i \mathrm{D} M = 0 \text{ for all } i < 0\}, \\ &{}^{\mathrm{p}}\mathsf{D}^{\mathrm{b}}_{\mathrm{f}}(\mathsf{Mod}\, A)^{\geq 0} := \\ &\qquad \{M \mid \mathrm{H}^i \mathrm{D} M = 0 \text{ for all } i > 0\}. \end{aligned}$$

on $\mathsf{D}^{\mathrm{b}}_{\mathrm{f}}(\mathsf{Mod}\, A)$.

We call it the rigid perverse t-structure. The heart is denoted by ${}^{\mathrm{p}}\mathsf{D}^{\mathrm{b}}_{\mathrm{f}}(\mathsf{Mod}\, A)^{0}$.

Here is an observation. Suppose A is an essentially finite type $\mathbb{K}$-algebra, where as before $\mathbb{K}$ is a finite dimensional regular noetherian ring. Let R_A be the rigid dualizing complex of A.

Then the duality $\mathrm{D} := \mathrm{RHom}_A(-, R_A)$ gives rise to a perverse t-structure

$$\begin{aligned} &{}^{\mathrm{p}}\mathsf{D}^{\mathrm{b}}_{\mathrm{f}}(\mathsf{Mod}\, A)^{\leq 0} := \\ &\qquad \{M \mid \mathrm{H}^i \mathrm{D} M = 0 \text{ for all } i < 0\}, \\ &{}^{\mathrm{p}}\mathsf{D}^{\mathrm{b}}_{\mathrm{f}}(\mathsf{Mod}\, A)^{\geq 0} := \\ &\qquad \{M \mid \mathrm{H}^i \mathrm{D} M = 0 \text{ for all } i > 0\}. \end{aligned}$$

on $\mathsf{D}^{\mathrm{b}}_{\mathrm{f}}(\mathsf{Mod}\, A)$.

We call it the rigid perverse t-structure. The heart is denoted by ${}^{\mathrm{p}}\mathsf{D}^{\mathrm{b}}_{\mathrm{f}}(\mathsf{Mod}\, A)^{0}$.

The next theorem was proved in [YZ3].

Theorem 5.1. *Let X be a finite type $\mathbb{K}$-scheme. Let $\star$ denote either ≤ 0, ≥ 0 or 0.*

Theorem 5.1. *Let X be a finite type $\mathbb{K}$-scheme. Let $\star$ denote either ≤ 0, ≥ 0 or 0.*

Define ${}^{\mathrm{p}}\mathsf{D}^{\mathrm{b}}_{\mathrm{c}}(\mathsf{Mod}\, \mathcal{O}_X)^{\star}$ to be the class of complexes $\mathcal{M} \in \mathsf{D}^{\mathrm{b}}_{\mathrm{c}}(\mathsf{Mod}\, \mathcal{O}_X)$ such that

$$\mathrm{R}\Gamma(U, \mathcal{M}) \in {}^{\mathrm{p}}\mathsf{D}^{\mathrm{b}}_{\mathrm{f}}\big(\mathsf{Mod}\, A\big)^{\star}$$

for any affine open set U, with $A := \Gamma(U, \mathcal{O}_X)$.

Theorem 5.1. *Let X be a finite type $\mathbb{K}$-scheme. Let $\star$ denote either ≤ 0, ≥ 0 or 0.*

Define ${}^{\mathrm{p}}\mathsf{D}^{\mathrm{b}}_{\mathrm{c}}(\mathsf{Mod}\,\mathcal{O}_X)^{\star}$ to be the class of complexes $\mathcal{M} \in \mathsf{D}^{\mathrm{b}}_{\mathrm{c}}(\mathsf{Mod}\,\mathcal{O}_X)$ such that

$$\mathrm{R}\Gamma(U, \mathcal{M}) \in {}^{\mathrm{p}}\mathsf{D}^{\mathrm{b}}_{\mathrm{f}}\big(\mathsf{Mod}\,A\big)^{\star}$$

for any affine open set U, with $A := \Gamma(U, \mathcal{O}_X)$.

Then:

(1) *The pair*

$$\big({}^{\mathrm{p}}\mathsf{D}^{\mathrm{b}}_{\mathrm{c}}(\mathsf{Mod}\,\mathcal{O}_X)^{\leq 0}, {}^{\mathrm{p}}\mathsf{D}^{\mathrm{b}}_{\mathrm{c}}(\mathsf{Mod}\,\mathcal{O}_X)^{\geq 0}\big)$$

is a t-structure on $\mathsf{D}^{\mathrm{b}}_{\mathrm{c}}(\mathsf{Mod}\,\mathcal{O}_X)$.

Theorem 5.1. *Let X be a finite type $\mathbb{K}$-scheme. Let $\star$ denote either ≤ 0, ≥ 0 or 0.*

Define ${}^{\mathrm{p}}\mathsf{D}^{\mathrm{b}}_{\mathrm{c}}(\mathsf{Mod}\,\mathcal{O}_X)^{\star}$ *to be the class of complexes* $\mathcal{M} \in \mathsf{D}^{\mathrm{b}}_{\mathrm{c}}(\mathsf{Mod}\,\mathcal{O}_X)$ *such that*

$$\mathrm{R}\Gamma(U, \mathcal{M}) \in {}^{\mathrm{p}}\mathsf{D}^{\mathrm{b}}_{\mathrm{f}}\big(\mathsf{Mod}\,A\big)^{\star}$$

for any affine open set U, with $A := \Gamma(U, \mathcal{O}_X)$.

Then:

(1) *The pair*

$$\big({}^{\mathrm{p}}\mathsf{D}^{\mathrm{b}}_{\mathrm{c}}(\mathsf{Mod}\,\mathcal{O}_X)^{\leq 0}, {}^{\mathrm{p}}\mathsf{D}^{\mathrm{b}}_{\mathrm{c}}(\mathsf{Mod}\,\mathcal{O}_X)^{\geq 0}\big)$$

is a t-structure on $\mathsf{D}^{\mathrm{b}}_{\mathrm{c}}(\mathsf{Mod}\,\mathcal{O}_X)$.

(2) *The assignment* $V \mapsto {}^{\mathrm{p}}\mathsf{D}^{\mathrm{b}}_{\mathrm{c}}(\mathsf{Mod}\,\mathcal{O}_V)^{0}$, *for $V \subset X$ open, is a stack of abelian categories on X.*

Part (2) says that the objects of ${}^{\mathrm{p}}\mathsf{D}^{\mathrm{b}}_{\mathrm{c}}(\mathsf{Mod}\,\mathcal{O}_X)^0$, which we call perverse coherent sheaves, can be glued. They behave like sheaves, and hence the name.

Part (2) says that the objects of ${}^{\mathrm{p}}\mathsf{D}^{\mathrm{b}}_{\mathrm{c}}(\mathsf{Mod}\,\mathcal{O}_X)^0$, which we call perverse coherent sheaves, can be glued. They behave like sheaves, and hence the name.

For any affine open set $U = \operatorname{Spec} A$, the dualizing complex $\mathcal{R}_U$ (the sheafification of the rigid dualizing complex R_A) is clearly a perverse coherent sheaf on U.

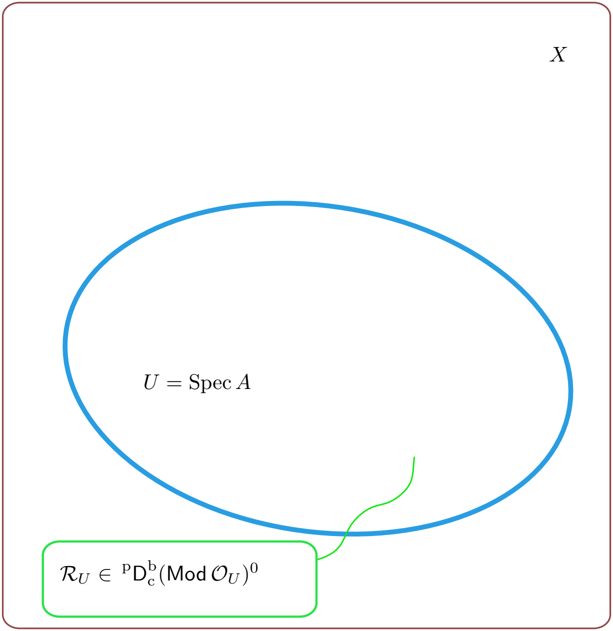
Part (2) says that the objects of ${}^{\mathrm{p}}\mathsf{D}^{\mathrm{b}}_{\mathrm{c}}(\mathsf{Mod}\,\mathcal{O}_X)^0$, which we call perverse coherent sheaves, can be glued. They behave like sheaves, and hence the name.

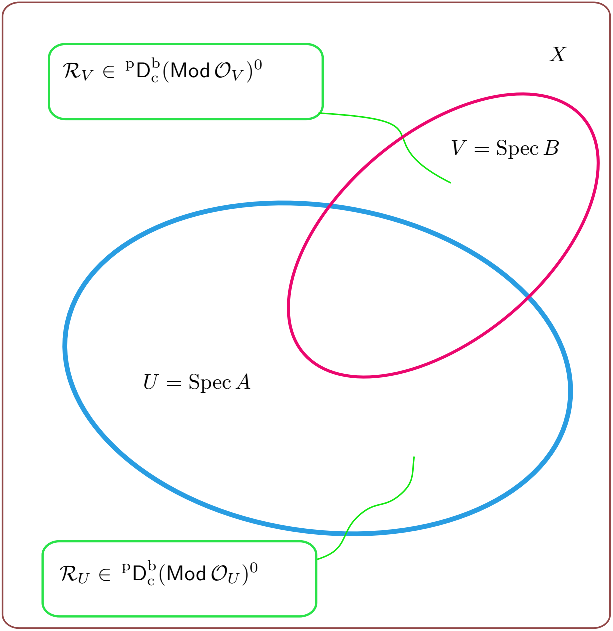
For any affine open set $U = \operatorname{Spec} A$, the dualizing complex $\mathcal{R}_U$ (the sheafification of the rigid dualizing complex R_A) is clearly a perverse coherent sheaf on U.

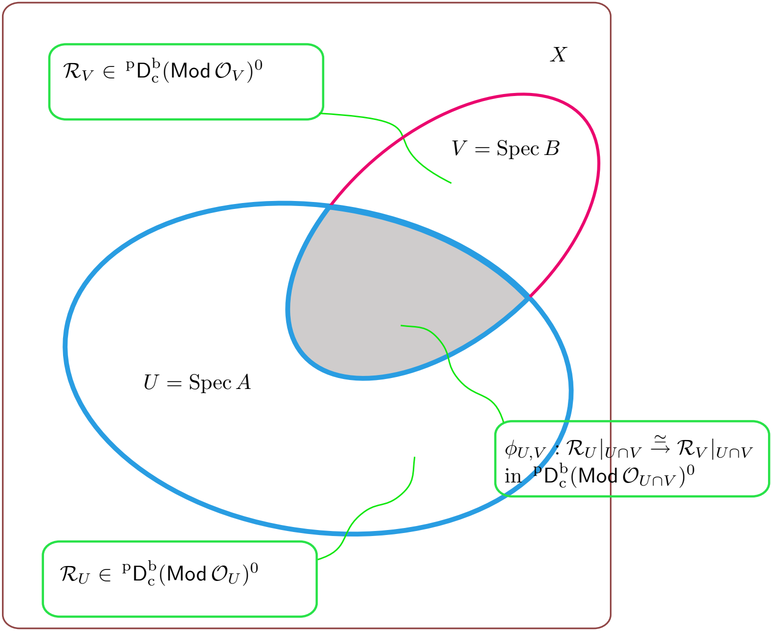
Thus we can use the isomorphisms

$$\phi_{U,V} : \mathcal{R}_U|_{U\cap V} \xrightarrow{\simeq} \mathcal{R}_V|_{U\cap V},$$

deduced from equation (4.2), to glue the affine dualizing complexes.


X
$U = \operatorname{Spec} A$
$\mathcal{R}_U \in {}^{\mathrm{p}}\mathsf{D}^{\mathrm{b}}_{\mathrm{c}}(\mathsf{Mod}\,\mathcal{O}_U)^0$


$\mathcal{R}_V \in {}^{\mathrm{p}}\mathsf{D}^{\mathrm{b}}_{\mathrm{c}}(\mathsf{Mod}\,\mathcal{O}_V)^0$
X
$V = \operatorname{Spec} B$
$U = \operatorname{Spec} A$
$\mathcal{R}_U \in {}^{\mathrm{p}}\mathsf{D}^{\mathrm{b}}_{\mathrm{c}}(\mathsf{Mod}\,\mathcal{O}_U)^0$


X
$\mathcal{R}_V \in {}^{\mathrm{p}}\mathsf{D}^{\mathrm{b}}_{\mathrm{c}}(\mathsf{Mod}\,\mathcal{O}_V)^0$
$V = \operatorname{Spec} B$
$U = \operatorname{Spec} A$
$\phi_{U,V} : \mathcal{R}_U|_{U \cap V} \xrightarrow{\simeq} \mathcal{R}_V|_{U \cap V}$
in ${}^{\mathrm{p}}\mathsf{D}^{\mathrm{b}}_{\mathrm{c}}(\mathsf{Mod}\,\mathcal{O}_{U \cap V})^0$
$\mathcal{R}_U \in {}^{\mathrm{p}}\mathsf{D}^{\mathrm{b}}_{\mathrm{c}}(\mathsf{Mod}\,\mathcal{O}_U)^0$

In this way we obtain:

In this way we obtain:

Theorem 5.2. *Let X be a finite type $\mathbb{K}$-scheme. There exists a rigid dualizing complex $(\mathcal{R}_X, \boldsymbol{\rho}_X)$ over X relative to $\mathbb{K}$, and it is unique up to a unique rigid isomorphism.*

Along the same lines, using Theorems 3.3 and 3.5, we can also prove:

Along the same lines, using Theorems 3.3 and 3.5, we can also prove:

Theorem 5.3. *Let X and Y be finite type $\mathbb{K}$-schemes, and let $f : X \to Y$ be a morphism.*

Along the same lines, using Theorems 3.3 and 3.5, we can also prove:

Theorem 5.3. *Let X and Y be finite type $\mathbb{K}$-schemes, and let $f : X \to Y$ be a morphism.*

1. *If f is finite, then there is a unique isomorphism*

$$\mathrm{R}f_* \mathcal{R}_X \cong \mathrm{R}\mathcal{H}om_{\mathcal{O}_Y}(f_* \mathcal{O}_X, \mathcal{R}_Y)$$

   which respects rigidity.

Along the same lines, using Theorems 3.3 and 3.5, we can also prove:

Theorem 5.3. *Let X and Y be finite type $\mathbb{K}$-schemes, and let $f : X \to Y$ be a morphism.*

1. *If f is finite, then there is a unique isomorphism*

$$\mathrm{R}f_* \mathcal{R}_X \cong \mathrm{R}\mathcal{H}om_{\mathcal{O}_Y}(f_* \mathcal{O}_X, \mathcal{R}_Y)$$

   which respects rigidity.

2. *If f is smooth of relative dimension n, then there is a unique isomorphism*

$$\Omega^n_{X/Y}[n] \otimes_{\mathcal{O}_X} f^* \mathcal{R}_Y \cong \mathcal{R}_X$$

   which respects rigidity.

Proper morphisms and residues are treated in [Ye6].

Proper morphisms and residues are treated in [Ye6].

Remark 5.4. Recently I discovered a totally new proof of the duality theorem for proper morphisms, which uses perverse sheaves only, avoiding residue calculations. If correct, the new proof will make the paper [Ye6] much shorter.

Proper morphisms and residues are treated in [Ye6].

Remark 5.4. Recently I discovered a totally new proof of the duality theorem for proper morphisms, which uses perverse sheaves only, avoiding residue calculations. If correct, the new proof will make the paper [Ye6] much shorter.

Remark 5.5. I think all the results here work also for essentially finite type $\mathbb{K}$-schemes.

6. Cohen-Macaulay Complexes

6. Cohen-Macaulay Complexes

As before X is a finite type $\mathbb{K}$-scheme. Let $\mathcal{R}_X$ be the rigid dualizing complex of X.

6. Cohen-Macaulay Complexes

As before X is a finite type $\mathbb{K}$-scheme. Let $\mathcal{R}_X$ be the rigid dualizing complex of X.

Given a point $x \in X$ let $\boldsymbol{k}(x)$ be its residue field. We denote by $\dim_{\mathbb{K}}(x)$ the unique integer i such that

$$\operatorname{Ext}^{-i}_{\mathcal{O}_{X,x}}\big(\boldsymbol{k}(x), \mathcal{R}_{X,x}\big) \neq 0.$$

6. Cohen-Macaulay Complexes

As before X is a finite type $\mathbb{K}$-scheme. Let $\mathcal{R}_X$ be the rigid dualizing complex of X.

Given a point $x \in X$ let $\boldsymbol{k}(x)$ be its residue field. We denote by $\dim_{\mathbb{K}}(x)$ the unique integer i such that

$$\operatorname{Ext}^{-i}_{\mathcal{O}_{X,x}}\big(\boldsymbol{k}(x), \mathcal{R}_{X,x}\big) \neq 0.$$

Then the function

$$\dim_{\mathbb{K}} : X \to \mathbb{Z}$$

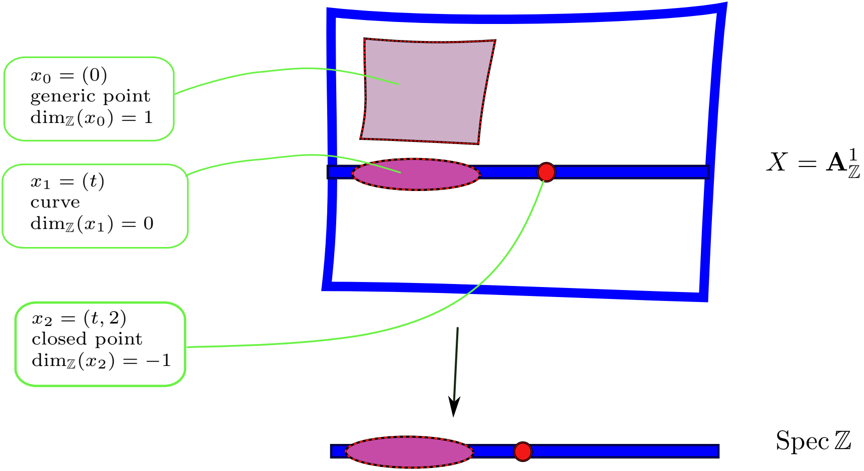
is a dimension function, i.e.

$$\dim_{\mathbb{K}}(y) = \dim_{\mathbb{K}}(x) - 1$$

when y is an immediate specialization of x.

Example 6.1. Take $\mathbb{K} := \mathbb{Z}$, the ring of integers, and $X := \mathbf{A}^1_{\mathbb{K}} = \operatorname{Spec} \mathbb{K}[t]$, the affine line. Consider the following points in X: x_0 is the generic point; x_1 is the prime ideal (t); and x_2 is the maximal ideal $(t, 2)$. Then

$$\dim_{\mathbb{K}}(x_i) = 1 - i.$$


$x_0 = (0)$
generic point
$\dim_{\mathbb{Z}}(x_0) = 1$
$x_1 = (t)$
curve
$\dim_{\mathbb{Z}}(x_1) = 0$
$x_2 = (t, 2)$
closed point
$\dim_{\mathbb{Z}}(x_2) = -1$
$X = \mathbf{A}^1_{\mathbb{Z}}$
$\operatorname{Spec} \mathbb{Z}$

Recall from [RD] that a complex $\mathcal{M} \in \mathsf{D}^{\mathrm{b}}_{\mathrm{c}}(\mathsf{Mod}\,\mathcal{O}_X)$ is called Cohen-Macaulay if for every point x the local cohomologies $\mathrm{H}^i_x \mathcal{M}$ all vanish except for $i = -\dim_{\mathbb{K}}(x)$.

Recall from [RD] that a complex $\mathcal{M} \in \mathsf{D}^{\mathrm{b}}_{\mathrm{c}}(\mathsf{Mod}\, \mathcal{O}_X)$ is called Cohen-Macaulay if for every point x the local cohomologies $\mathrm{H}^i_x \mathcal{M}$ all vanish except for $i = -\dim_{\mathbb{K}}(x)$.

Here is another result from [YZ3].

Theorem 6.2. *Let X be a finite type scheme over $\mathbb{K}$, let $\mathcal{R}_X$ be the rigid dualizing complex of X, and let D be the duality functor $\mathrm{R}\mathcal{H}om_{\mathcal{O}_X}(-, \mathcal{R}_X)$.*

Theorem 6.2. *Let X be a finite type scheme over $\mathbb{K}$, let $\mathcal{R}_X$ be the rigid dualizing complex of X, and let D be the duality functor $\mathrm{R}\mathcal{H}om_{\mathcal{O}_X}(-, \mathcal{R}_X)$.*

Then the following conditions are equivalent for $\mathcal{M} \in \mathsf{D}^{\mathrm{b}}_{\mathrm{c}}(\mathsf{Mod}\, \mathcal{O}_X)$.

Theorem 6.2. *Let X be a finite type scheme over $\mathbb{K}$, let $\mathcal{R}_X$ be the rigid dualizing complex of X, and let D be the duality functor $\mathrm{R}\mathcal{H}om_{\mathcal{O}_X}(-, \mathcal{R}_X)$.*

Then the following conditions are equivalent for $\mathcal{M} \in \mathsf{D}^{\mathrm{b}}_{\mathrm{c}}(\mathsf{Mod}\,\mathcal{O}_X)$.

(i) *$\mathcal{M}$ is a perverse coherent sheaf (for the rigid perverse t-structure).*

Theorem 6.2. *Let X be a finite type scheme over $\mathbb{K}$, let $\mathcal{R}_X$ be the rigid dualizing complex of X, and let D be the duality functor $\mathrm{R}\mathcal{H}om_{\mathcal{O}_X}(-, \mathcal{R}_X)$.*

Then the following conditions are equivalent for $\mathcal{M} \in \mathsf{D}^{\mathrm{b}}_{\mathrm{c}}(\mathsf{Mod}\, \mathcal{O}_X)$.

(i) *$\mathcal{M}$ is a perverse coherent sheaf (for the rigid perverse t-structure).*

(ii) *$\mathrm{D}\mathcal{M}$ is a coherent sheaf, i.e. $\mathrm{H}^i \mathrm{D}\mathcal{M} = 0$ for all $i \neq 0$.*

Theorem 6.2. *Let X be a finite type scheme over $\mathbb{K}$, let $\mathcal{R}_X$ be the rigid dualizing complex of X, and let D be the duality functor $\mathrm{R}\mathcal{H}om_{\mathcal{O}_X}(-, \mathcal{R}_X)$.*

Then the following conditions are equivalent for $\mathcal{M} \in \mathsf{D}^{\mathrm{b}}_{\mathrm{c}}(\mathsf{Mod}\, \mathcal{O}_X)$.

(i) *$\mathcal{M}$ is a perverse coherent sheaf (for the rigid perverse t-structure).*

(ii) *$\mathrm{D}\mathcal{M}$ is a coherent sheaf, i.e. $\mathrm{H}^i \mathrm{D}\mathcal{M} = 0$ for all $i \neq 0$.*

(iii) *$\mathcal{M}$ is a Cohen-Macaulay complex.*

Theorem 6.2. *Let X be a finite type scheme over $\mathbb{K}$, let $\mathcal{R}_X$ be the rigid dualizing complex of X, and let D be the duality functor $\mathrm{R}\mathcal{H}om_{\mathcal{O}_X}(-, \mathcal{R}_X)$.*

Then the following conditions are equivalent for $\mathcal{M} \in \mathsf{D}^{\mathrm{b}}_{\mathrm{c}}(\mathsf{Mod}\, \mathcal{O}_X)$.

(i) *$\mathcal{M}$ is a perverse coherent sheaf (for the rigid perverse t-structure).*

(ii) *$\mathrm{D}\mathcal{M}$ is a coherent sheaf, i.e. $\mathrm{H}^i \mathrm{D}\mathcal{M} = 0$ for all $i \neq 0$.*

(iii) *$\mathcal{M}$ is a Cohen-Macaulay complex.*

In particular this implies the Cohen-Macaulay complexes form an abelian subcategory of $\mathsf{D}^{\mathrm{b}}_{\mathrm{c}}(\mathsf{Mod}\, \mathcal{O}_X)$, a fact that seems to have eluded Grothendieck.

Theorem 6.2. *Let X be a finite type scheme over $\mathbb{K}$, let $\mathcal{R}_X$ be the rigid dualizing complex of X, and let D be the duality functor $\mathrm{R}\mathcal{H}om_{\mathcal{O}_X}(-, \mathcal{R}_X)$.*

Then the following conditions are equivalent for $\mathcal{M} \in \mathsf{D}^{\mathrm{b}}_{\mathrm{c}}(\mathsf{Mod}\,\mathcal{O}_X)$.

(i) *$\mathcal{M}$ is a perverse coherent sheaf (for the rigid perverse t-structure).*

(ii) *$\mathrm{D}\mathcal{M}$ is a coherent sheaf, i.e. $\mathrm{H}^i \mathrm{D}\mathcal{M} = 0$ for all $i \neq 0$.*

(iii) *$\mathcal{M}$ is a Cohen-Macaulay complex.*

In particular this implies the Cohen-Macaulay complexes form an abelian subcategory of $\mathsf{D}^{\mathrm{b}}_{\mathrm{c}}(\mathsf{Mod}\,\mathcal{O}_X)$, a fact that seems to have eluded Grothendieck.

- END -

- L. Alonso, A. Jeremías and J. Lipman, “Studies in Duality on Noetherian Formal Schemes and Non-Noetherian Ordinary Schemes,” Contemp. Math. **244**, Amer. Math. Soc., 1999.
- A. Altman and S. Kleiman, “Introduction to Grothendieck Duality,” Lecture Notes in Math. **20**, Springer, 1970.
- A.A. Beilinson, J. Bernstein and P. Deligne, Faisceaux perverse, in “Analyse et Topologie sur les Espaces Singulieres,” Astérisque **100**, Soc. Math. France, 1982, 5-171.
- T. Bridgeland, Flops and derived categories, Invent. Math. **147** (2002), no. 3, 613-632.
- R. Bezrukavnikov, Perverse coherent sheaves (after Deligne), eprint math.AG/0005152 at http://arXiv.org.
- B. Conrad, “Grothendieck Duality and Base Change,” Lecture Notes in Math. **1750**, Springer, 2000.

- R. Hübl and P. Sastry, Regular differential forms and relative duality, Amer. J. Math. **115** (1993), no. 4, 749-787.
- R. Hübl and E. Kunz, Regular differential forms and duality for projective morphisms, J. Reine Angew. Math. **410** (1990), 84-108.
- M. Kashiwara, T-structures on the derived categories of holonomic D-modules and coherent O-modules, Moscow Mathematical Journal, **4** (2004), no. 4, 847-868.
- M. Kashiwara and P. Schapira, “Sheaves on Manifolds,” Springer, 1990.
- J. Lipman, Dualizing sheaves, differentials and residues on algebraic varieties, Astérisque **117** (1984).
- J. Lipman, S. Nayak and P. Sastry, “Variance and Duality for Cousin Complexes on Formal Schemes”, Contemp. Math. **375**, AMS 2005.
- A. Neeman, The Grothendieck duality theorem via Bousfield’s techniques and Brown representability, Jour. Amer. Math. Soc. **9** (1996), 205-236.

- R. Hartshorne, “Residues and Duality,” Lecture Notes in Math. **20**, Springer-Verlag, Berlin, 1966.
- M. Van den Bergh, Existence theorems for dualizing complexes over non-commutative graded and filtered rings, J. Algebra **195** (1997), no. 2, 662-679.
- A. Yekutieli, Dualizing complexes over noncommutative graded algebras, J. Algebra **153** (1992), 41-84.
- A. Yekutieli, “An Explicit Construction of the Grothendieck Residue Complex” (with an appendix by P. Sastry), Astérisque **208** (1992).
- A. Yekutieli, Smooth formal embeddings and the residue complex, Canadian J. Math. **50** (1998), 863-896.
- A. Yekutieli, Residues and differential operators on schemes, Duke Math. J. **95** (1998), 305-341.

- A. Yekutieli, Dualizing complexes, Morita equivalence and the derived Picard group of a ring, J. London Math. Soc. (2) **60** (1999), no. 3, 723-746.
- A. Yekutieli, Rigidity, Residues, and Grothendieck Duality for Schemes, in preparation.
- A. Yekutieli and J.J. Zhang, Rings with Auslander dualizing complexes, J. Algebra **213** (1999), no. 1, 1-51.
- A. Yekutieli and J.J. Zhang, Rigid Dualizing Complexes and Perverse Modules over Differential Algebras, Compositio Math. **141** (2005), 620-654.
- A. Yekutieli and J.J. Zhang, Dualizing Complexes and Perverse Sheaves on Noncommutative Ringed Schemes, Selecta Math. **12** (2006), 137-177.
- A. Yekutieli and J.J. Zhang, Rigid Complexes via DG Algebras, to appear in Trans. Amer. Math. Soc. Eprint math.AC/0603733 at http://arXiv.org.

A. Yekutieli and J.J. Zhang, Rigid Dualizing Complexes over Commutative Rings, to appear in Algebr. Represent. Theory, eprint math.AG/0601654 http://arXiv.org.