אינטגרציה של פונקציות טריגונומטריות. הצבה $t=tg\frac{x}{2}$

2018 $\displaystyle\int \frac{dx}{2\sin x-\cos x+5} = \rightarrow$

$t = tg\frac{x}{2}$ $\quad x = 2\,\mathrm{arctg}\, t$ $\quad \sin x = \frac{2t}{1+t^2}$

$(-\pi < x < \pi)$ $\quad dx = \frac{2\,dt}{1+t^2}$ $\quad \cos x = \frac{1-t^2}{1+t^2}$

$$2\sin x - \cos x + 5 = \frac{4t}{1+t^2} - \frac{1-t^2}{1+t^2} + 5 =$$

$$= \frac{4t-1+t^2+5+5t^2}{1+t^2} = \frac{6t^2+4t+4}{1+t^2} = \frac{2(3t^2+2t+2)}{1+t^2}$$

$$\rightarrow = \int \frac{2\,dt\cdot(1+t^2)}{(1+t^2)\,2\,(3t^2+2t+2)} = \int \frac{dt}{3t^2+2t+2} = \rightarrow$$

$$3t^2+2t+2 = 3\left(t^2+\frac{2}{3}t+\frac{2}{3}\right) = 3\left(t^2+2\cdot\frac{1}{3}t+\frac{1}{9}-\frac{1}{9}+\frac{2}{3}\right) =$$

$$= 3\left[\left(t+\frac{1}{3}\right)^2+\frac{5}{9}\right]$$

$$\rightarrow = \frac{1}{3}\int \frac{dt}{\left(t+\frac{1}{3}\right)^2+\frac{5}{9}} = \frac{1}{3}\cdot\frac{3}{\sqrt5}\,\mathrm{arctg}\,\frac{\left(t+\frac{1}{3}\right)3}{\sqrt5} + c =$$

$$= \frac{1}{\sqrt5}\,\mathrm{arctg}\,\frac{3t+1}{\sqrt5} + c = \frac{1}{\sqrt5}\,\mathrm{arctg}\,\frac{3tg\frac{x}{2}+1}{\sqrt5} + c$$

2025 $\displaystyle\int \frac{\sin x\cdot\cos x}{\sin x+\cos x}\,dx = \frac{1}{2}\int \frac{(\sin x+\cos x)^2-1}{\sin x+\cos x}\,dx =$

$$\sin x\cdot\cos x = \frac{1}{2}\left[(\sin x+\cos x)^2-1\right]$$

$$= \frac{1}{2}\int(\sin x+\cos x)\,dx - \frac{1}{2}\int\frac{dx}{\sin x+\cos x} = \frac{1}{2}(-\cos x+\sin x) -$$

$$-\frac{1}{2}\int\frac{dx}{\sqrt2\,\sin\left(x+\frac{\pi}{4}\right)} = \frac{1}{2}(\sin x-\cos x) - \frac{1}{2\sqrt2}\ln\left|tg\left(\frac{x}{2}+\frac{\pi}{8}\right)\right| + c$$

סניה וקמן

1983 $\int \cos^5 x\,dx = \int (1-\sin^2 x)^2 \cos x\,dx =$

$\sin x = t \quad \cos x \cdot dx = dt$

$= \int (1-t^2)^2 dt = \int (1-2t^2+t^4)dt = t - 2\frac{t^3}{3} + \frac{t^5}{5} + c =$

$= \sin x - \frac{2}{3}\sin^2 x + \frac{1}{5}\sin^5 x + c.$

1984 $\int \sin^6 x\,dx = \int \left(\frac{1-\cos 2x}{2}\right)^3 dx =$

$= \frac{1}{8}\int (1 - 3\cos 2x + 3\cos^2 2x - \cos^3 2x)\,dx =$

$= \frac{x}{8} - \frac{3}{2\cdot 8}\sin 2x + \frac{3}{8}\int \frac{1+\cos 4x}{2}dx - \frac{1}{8}\int (1-\sin^2 2x)\cos 2x\,dx$

$= \frac{x}{8} - \frac{3}{16}\sin 2x + \frac{3}{16}x + \frac{3}{16}\cdot\frac{\sin 4x}{4} - \frac{1}{8}\cdot\frac{\sin 2x}{2} +$

$+ \frac{1}{8}\int \sin^2 2x \cos 2x\,dx = \frac{5x}{16} - \frac{1}{4}\sin 2x + \frac{3}{64}\sin 4x +$

$\sin 2x = t \quad 2\cos 2x\,dx = dt \quad \cos 2x\,dx = \frac{1}{2}dt$

$+ \frac{1}{16}\int t^2 dt = \frac{5}{16}x - \frac{1}{4}\sin 2x + \frac{3}{64}\sin 4x + \frac{1}{48}\sin^3 2x + c.$

1986 $\int \sin^2 x \cdot \cos^4 x\,dx = \int \frac{1-\cos 2x}{2}\cdot\left(\frac{1+\cos 2x}{2}\right)^2 dx =$

$= \frac{1}{8}\int (1-\cos^2 2x)(1+\cos 2x)\,dx = \frac{1}{8}\int (1+\cos 2x - \cos^2 2x - \cos^3 2x)d$

$= \frac{1}{8}x + \frac{1}{16}\sin 2x - \frac{1}{8}\int \frac{1+\cos 4x}{2}dx - \frac{1}{8}\int (1-\sin^2 2x)\cos 2x\,dx =$

$= \frac{1}{8}x + \frac{1}{16}\sin 2x - \frac{1}{16}x - \frac{1}{16}\cdot\frac{\sin 4x}{4} - \frac{1}{16}\sin 2x +$

$+ \frac{1}{8}\int \sin^2 2x \cos 2x\,dx =$

$\sin 2x = t \quad 2\cos 2x\,dx = dt \quad \cos 2x\,dx = \frac{1}{2}dt$

$= \frac{1}{16}x - \frac{1}{64}\sin 4x + \frac{1}{16}\int t^2 dt = \frac{1}{16}x - \frac{1}{64}\sin 4x + \frac{\sin^3 2x}{48} + c$

בס"ד

טניה וקסמן

1747 $\int \frac{dx}{\sin^2 x \cdot \cos x} = \int \frac{\sin^2 x + \cos^2 x}{\sin^2 x \cdot \cos x}\, dx =$

$= \int \frac{dx}{\cos x} + \int \frac{\cos x\, dx}{\sin^2 x} \underset{\sin x = t}{=} \int \frac{dx}{\cos x} + \underset{\cos x\, dx = dt}{\int \frac{dt}{t^2}} =$

$1694.\ \int \frac{dx}{\sin x} = \ln\left|tg \frac{x}{2}\right| \qquad 1695.\ \int \frac{dx}{\cos x} = \ln\left|tg\left(\frac{x}{2} + \frac{\pi}{4}\right)\right|$

$= \ln\left|tg\left(\frac{x}{2} + \frac{\pi}{4}\right)\right| - \frac{1}{t} + c = \ln\left|tg\left(\frac{x}{2} + \frac{\pi}{4}\right)\right| - \frac{1}{\sin x} + c.$

1748 $\int \frac{dx}{\sin x \cdot \cos^3 x} = \int \frac{\sin^2 x + \cos^2 x}{\sin x \cdot \cos^3 x}\, dx =$

$= \int \frac{\sin x\, dx}{\cos^3 x} + \int \frac{dx}{\sin x \cdot \cos x} = -\int \frac{dt}{t^3} + 2\int \frac{dx}{\sin 2x} =$

$2x = t_1 \quad 2\,dx = dt_1$

$\cos x = t \quad -\sin x\, dx = dt$

$= -\frac{t^{-3+1}}{-3+1} + \int \frac{dt_1}{\sin t_1} = \frac{-1}{2t^2} + \ln\left|tg \frac{t_1}{2}\right| + c =$

$= -\frac{1}{2\cos^2 x} + \ln|tg\, x| + c$

1749 $\int \frac{\cos^3 x}{\sin x}\, dx = \int \frac{(1 - \sin^2 x)\cos x}{\sin x}\, dx = \int \frac{(1 - t^2)\,dt}{t} =$

$\sin x = t \quad \cos x\, dx = dt$

$= \ln|t| - \frac{t^2}{2} + c = \ln|\sin x| - \frac{\sin^2 x}{2} + c$

1750 $\int \frac{dx}{\cos^4 x} = \int \frac{1}{\cos^2 x} \cdot \frac{dx}{\cos^2 x} = \int (1 + tg^2 x)\frac{dx}{\cos^2 x} =$

$\frac{1}{\cos^2 x} = 1 + tg^2 x$

$= tg\, x + \frac{tg^3 x}{3} + c.$

טעות וזמן

(1741.) $\int \sin^4 x\,dx = \int \left(\frac{1-\cos 2x}{2}\right)^2 dx =$

$= \frac{1}{4}\int \left(1 - 2\cos 2x + \cos^2 2x\right)dx = \frac{1}{4}x - \frac{2}{4}\cdot\frac{1}{2}\sin 2x + \frac{1}{4}\int \frac{1+\cos 4x}{2}d$

$= \frac{x}{4} - \frac{\sin 2x}{4} + \frac{x}{8} + \frac{\sin 4x}{4\cdot 2\cdot 4} + c = \frac{3x}{8} - \frac{\sin 2x}{4} + \frac{\sin 4x}{32} + c$

(1742) $\int \cos^4 x\,dx = \int \frac{(1+\cos 2x)^2}{4}dx = \frac{1}{4}\int\left(1 + 2\cos 2x + \cos^2 2x\right)dx =$

$= \frac{x}{4} + \frac{\sin 2x}{4} + \frac{1}{4}\int \frac{1+\cos 4x}{2}dx = \frac{x}{4} + \frac{\sin 2x}{4} + \frac{x}{8} + \frac{\sin 4x}{32} + c =$

$= \frac{3x}{8} + \frac{\sin 2x}{4} + \frac{\sin 4x}{32} + c.$

(1743) $\int \operatorname{ctg}^2 x\,dx = \int\left(\frac{1}{\sin^2 x} - 1\right)dx = -\operatorname{ctg} x - x + c$

$1 + \operatorname{ctg}^2\alpha = \frac{1}{\sin^2\alpha}$

(1744) $\int \operatorname{tg}^3 x\,dx = \int \operatorname{tg} x\cdot \operatorname{tg}^2 x\,dx = \int \operatorname{tg} x\left(\frac{1}{\cos^2 x} - 1\right)dx =$

$1 + \operatorname{tg}^2\alpha = \frac{1}{\cos^2\alpha}$

$= \int \operatorname{tg} x\cdot\frac{dx}{\cos^2 x} - \int \operatorname{tg} x\,dx = \int t\,dt - \int \frac{\sin x}{\cos x}dx =$

$\operatorname{tg} x = t \quad \frac{1}{\cos^2 x}\cdot dx = dt$

$\cos x = t_1$

$-\sin x\,dx = dt_1$

$= \frac{\operatorname{tg}^2 x}{2} + \ln|\cos x| + C.$

(1746) $\int \frac{dx}{\sin^2 x\cdot\cos^2 x} = \int \frac{\sin^2 x + \cos^2 x}{\sin^2 x\cdot\cos^2 x}dx = \int\frac{dx}{\cos^2 x} + \int\frac{dx}{\sin^2 x} =$

$= \operatorname{tg} x - \operatorname{ctg} x + c$

בס"ד

טניה וינמן

אינטגרציה של פונקציות טריגונומטריות

1733, 1734, 1738–1744, 1746–1750, 1983, 1984, 1986

2018, 2025.

(1733) $\int \sin^2 x\,dx = \int \frac{1-\cos 2x}{2}\,dx = \frac{1}{2}\int dx - \frac{1}{2}\int \cos 2x\,dx$

$1-\cos\alpha = 2\sin^2\frac{\alpha}{2}$

$\sin^2 x = \frac{1}{2}(1-\cos 2x)$

$= \frac{1}{2}x - \frac{1}{2}\cdot\frac{1}{2}\sin 2x + c = \frac{x}{2} - \frac{\sin 2x}{4} + c$

(1734) $\int \cos^2 x\,dx = \int \frac{1+\cos 2x}{2}\,dx = \frac{x}{2} + \frac{\sin 2x}{4} + c$

(1738) $\int \sin^3 x\,dx = \int \sin^2 x\cdot\sin x\,dx = \int (1-\cos^2 x)\sin x\,dx$

$= \int \sin x\,dx - \int \cos^2 x\cdot\sin x\,dx = -\cos x + \int t^2\,dt =$

$\cos x = t \quad -\sin x\,dx = dt$

$= -\cos x + \frac{t^3}{3} + c = -\cos x + \frac{\cos^3 x}{3} + c$

(1739) $\int \sin(\overbrace{2x-\tfrac{\pi}{6}}^{\alpha})\cdot\cos(\overbrace{3x+\tfrac{\pi}{4}}^{\beta})\,dx = \frac{1}{2}\int[\sin(\alpha+\beta)+\sin(\alpha-\beta)]\,dx =$

$2\sin\alpha\cdot\sin\beta = \sin(\alpha+\beta)+\sin(\alpha-\beta)$

$\alpha+\beta = 2x-\frac{\pi}{6}+3x+\frac{\pi}{4} = 5x+\frac{\pi}{12} \qquad \alpha-\beta = 2x-\frac{\pi}{6}-3x-\frac{\pi}{4} = -x-\frac{5\pi}{12}$

$= \frac{1}{2}\left[\int \sin\left(5x+\frac{\pi}{12}\right)dx - \int \sin\left(x+\frac{5\pi}{12}\right)dx\right] =$

$= -\frac{1}{10}\cos\left(5x+\frac{\pi}{12}\right) + \frac{1}{2}\cos\left(x+\frac{5\pi}{12}\right) + C$

(1740) $\int \cos^3 x\,dx = \int \cos^2 x\cdot\cos x\,dx = \int (1-\sin^2 x)\cos x\,dx =$

$= \int \cos x\,dx - \int \sin^2 x\cdot\cos x\,dx = \sin x - \frac{\sin^3 x}{3} + c$

$\sin x = t \quad \cos x\,dx = dt$

טניה וקסמן

1872 המשך

$$= \ln|x| - \ln|1+x| - \int \frac{dx}{1+x+x^2} =$$

$$1+x+x^2 = x^2 + 2\cdot\frac{1}{2}\cdot x + \frac{1}{4} - \frac{1}{4} + 1 = \left(x+\frac{1}{2}\right)^2 + \frac{3}{4}$$

$$= \ln\left|\frac{x}{1+x}\right| - \int \frac{dx}{\left(x+\frac{1}{2}\right)^2 + \frac{3}{4}} = \ln\left|\frac{x}{1+x}\right| - \int \frac{d\left(x+\frac{1}{2}\right)}{\left(x+\frac{1}{2}\right)^2 + \left(\frac{\sqrt{3}}{2}\right)^2} =$$

$$= \ln\left|\frac{x}{1+x}\right| - \frac{2}{\sqrt{3}} \operatorname{arctg} \frac{\left(x+\frac{1}{2}\right)\cdot 2}{\sqrt{3}} + C =$$

$$= \ln\left|\frac{x}{1+x}\right| - \frac{2}{\sqrt{3}} \operatorname{arctg} \frac{2x+1}{\sqrt{3}} + C.$$

1873 $\displaystyle \int \frac{dx}{x^3+1} = \int \frac{dx}{(x+1)(x^2-x+1)} = \rightarrow$

$$\frac{1}{(x+1)(x^2-x+1)} = \frac{A}{x+1} + \frac{Bx+C}{x^2-x+1}$$

$$1 \equiv A(x^2-x+1) + (Bx+C)(x+1)$$

$x=-1$	$1 = A\cdot 3$	$A = \frac{1}{3}$	$A = \frac{1}{3}$
$x=0$	$1 = A + C$	$C = 1 - A = \frac{2}{3}$	$B = -\frac{1}{3}$
$x=1$	$1 = A + (B+C)\cdot 2$	$2B = 1 - A - 2C = -\frac{2}{3}$	$C = \frac{2}{3}$

$$\rightarrow = \frac{1}{3}\int\frac{dx}{x+1} - \frac{1}{3}\int\frac{(x-2)\,dx}{x^2-x+1} = \frac{1}{3}\ln|x+1| - \frac{1}{3\cdot 2}\int\frac{2x-4}{x^2-x+1}\,dx =$$

$$d(x^2-x+1) = (2x-1)\,dx$$

$$= \frac{1}{3}\ln|x+1| - \frac{1}{6}\int\left(\frac{2x-1}{x^2-x+1} - \frac{3}{x^2-x+1}\right)dx = \frac{1}{3}\ln|x+1| -$$

$$-\frac{1}{6}\ln(x^2-x+1) + \frac{1}{2}\int\frac{dx}{x^2-x+1} = \frac{1}{3}\ln|x+1| - \frac{1}{6}\ln(x^2-x+1) + \frac{1}{2}\int\frac{d\left(x-\frac{1}{2}\right)}{\left(x-\frac{1}{2}\right)^2+\frac{3}{4}}$$

$$x^2 - x + 1 = x^2 - 2\cdot\frac{1}{2}x + \frac{1}{4} - \frac{1}{4} + 1 = \left(x-\frac{1}{2}\right)^2 + \frac{3}{4}$$

$+C$

$$= \frac{1}{6}\ln\frac{(x+1)^2}{x^2-x+1} + \frac{1}{2}\cdot\frac{2}{\sqrt{3}}\operatorname{arctg}\frac{\left(x-\frac{1}{2}\right)2}{\sqrt{3}} + C = \frac{1}{6}\ln\frac{(x+1)^2}{x^2-x+1} + \frac{1}{\sqrt{3}}\operatorname{arctg}\frac{2x-1}{\sqrt{3}}$$

טנוה וקסאן

בס״ד

1864. $\int \frac{x^2+1}{(x+1)^2(x-1)}dx = \frac{1}{2}\int\frac{dx}{x+1} - \int\frac{dx}{(x+1)^2} + \frac{1}{2}\int\frac{dx}{x-1} =$

$$\frac{x^2+1}{(x+1)^2(x-1)} = \frac{A}{x+1} + \frac{B}{(x+1)^2} + \frac{C}{x-1}$$

$$x^2+1 = A(x+1)(x-1) + B(x-1) + C(x+1)^2$$

$$\begin{array}{l|lll} x=1 & 2 = 4C & C = \frac{1}{2} & A = 1 - C = \frac{1}{2} \\ x=-1 & 2 = -2B & B = -1 & A = \frac{1}{2} \\ x^2 & 1 = A + C. & & \end{array}$$

$= \frac{1}{2}\ln|x+1| - \frac{1}{x+1} + \frac{1}{2}\ln|x-1| + C = \frac{1}{2}\ln|x^2-1| - \frac{1}{x+1} + C$

1869 $\int\frac{dx}{(x+1)(x^2+1)} = \frac{1}{2}\int\frac{dx}{x+1} - \frac{1}{2}\int\frac{x-1}{x^2+1}dx =$

$$\frac{1}{(x+1)(x^2+1)} = \frac{A}{x+1} + \frac{Bx+C}{x^2+1}$$

$$1 \equiv A(x^2+1) + (Bx+C)(x+1)$$

$$\begin{array}{l|lll} x=-1 & 1 = A\cdot 2 & A = \frac{1}{2} & A = \frac{1}{2} \\ x=0 & 1 = A + C & C = 1 - A = \frac{1}{2} & B = -\frac{1}{2} \\ x^2 & 0 = A + B & B = -A & C = \frac{1}{2} \end{array}$$

$= \frac{1}{2}\ln|x+1| - \frac{1}{2}\int\frac{x\,dx}{x^2+1} + \frac{1}{2}\int\frac{dx}{x^2+1} = \frac{1}{2}\ln|x+1| - \frac{1}{2}\int\frac{t\,dt}{t^2} +$

$x^2+1 = t^2$

$x\,dx = t\,dt$

$+ \frac{1}{2}\operatorname{arctg} x = \frac{1}{2}\ln|x+1| - \frac{1}{2}\ln\sqrt{x^2+1} + \frac{1}{2}\operatorname{arctg} x + C =$

$= \frac{1}{2}\ln|x+1| - \frac{1}{4}\ln(x^2+1) + \frac{1}{2}\operatorname{arctg} x + C.$

1879 $\int\frac{dx}{x(1+x)(1+x+x^2)} = \int\frac{dx}{x} - \int\frac{dx}{1+x} - \int\frac{dx}{1+x+x^2} =$

$$\frac{1}{x(1+x)(1+x+x^2)} = \frac{A}{x} + \frac{B}{1+x} + \frac{Cx+D}{1+x+x^2}$$

$$1 = A(1+x)(1+x+x^2) + Bx(1+x+x^2) + (Cx+D)x(1+x)$$

$C = -A - B = -1 + 1 = 0$

$$\begin{array}{l|l} x=0 & 1 = A \\ x=-1 & 1 = -B\cdot 1 \\ x^3 & 0 = A + B + C \end{array} \quad \begin{array}{l} x=1 \Rightarrow 1 = A\cdot 2\cdot 3 + B\cdot 3 + D\cdot 2 \\ 2D = 1 - 6A - 3B = 1 - 6 + 3 = -2 \end{array} \quad \begin{array}{l} A = 1 \\ B = -1 \\ C = 0 \\ D = -1 \end{array}$$

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טנה וקסמן

פסיז

אינטגרציה של פונקציות רציונליות. 1858, 1859, 1863, 1864, 1869, 1872, 1873:

1858 $\int \frac{(2x+3)dx}{(x-2)(x+5)} = \int \frac{dx}{x-2} + \int \frac{dx}{x+5} = \ln|x-2| + \ln|x+5| + C$

$$\frac{2x+3}{(x-2)(x+5)} = \frac{A}{x-2} + \frac{B}{x+5}$$

$$2x+3 = A(x+5) + B(x-2)$$

$x=-5$ | $-10+3 = B\cdot(-7)$ $\qquad B=1$

$x=2$ | $4+3 = A\cdot 7$ $\qquad A=1$

1859 $\int \frac{x\,dx}{(x+1)(x+2)(x+3)} = \rightarrow$

$$\frac{x}{(x+1)(x+2)(x+3)} = \frac{A}{x+1} + \frac{B}{x+2} + \frac{C}{x+3}$$

$$x \equiv A(x+2)(x+3) + B(x+1)(x+3) + C(x+1)(x+2)$$

$x=-1$ | $-1 = A\cdot(+1)\cdot 2$ $\qquad 3A=-1$ $\qquad A=-\frac{1}{3}$

$x=-2$ | $-2 = B\cdot(-1)$ $\qquad B=2$ $\qquad B=2$

$x=-3$ | $-3 = C\cdot(-2)(-1)$ $\qquad 2C=-3$ $\qquad C=-\frac{3}{2}$

$$\rightarrow = -\frac{1}{3}\int \frac{dx}{x+1} + 2\int \frac{dx}{x+2} - \frac{3}{2}\int \frac{dx}{x+3} =$$

$$= -\frac{1}{3}\ln|x+1| + 2\ln|x+2| - \frac{3}{2}\ln|x+3| + C.$$

1863 $\int \frac{x\,dx}{x^3-3x+2} = \int \frac{x\,dx}{(x-1)^2(x+2)} = -\frac{1}{9}\int \frac{dx}{x-1} + \frac{1}{3}\int \frac{dx}{(x-1)^2} - \frac{2}{9}\int \frac{dx}{x+2} =$

עושה חילוק פולינום במונה

$x=1$ שורש $\Rightarrow x^3-3x+2 = (x-1)(x^2+x-2) = (x-1)^2(x+2)$

$= (x-1)(x-1)(x+2)$

$$\begin{array}{l|l} x^3-3x+2 & x-1 \\ \underline{x^3-x^2} & \overline{x^2+x-2} \\ x^2-3x+2 & \\ \underline{x^2-x} & \\ -2x+2 & \end{array}$$

$$\frac{x}{(x-1)^2(x+2)} = \frac{A}{x-1} + \frac{B}{(x-1)^2} + \frac{C}{x+2}$$

$$x \equiv A(x-1)(x+2) + B(x+2) + C(x-1)^2$$

$A = -B-C = -\frac{1}{3} + \frac{2}{9} = \frac{-1}{9}$

$x=1$ | $1 = B\cdot 3$ $\qquad B=\frac{1}{3}$

$x=-2$ | $-2 = C\cdot 9$ $\qquad C=-\frac{2}{9}$

x^2 | $0 = A+B+C$ $\qquad A=-\frac{1}{9}$

$\rightarrow = -\frac{1}{9}\ln|x-1| + \frac{1}{3(x-1)} - \frac{2}{9}\ln|x+2| + C.$

בוריס

טניה וקסמן

1820. $I = \int e^{ax} \sin bx\, dx = \frac{1}{a} e^{ax} \cdot \sin bx - \frac{b}{a} \int e^{ax} \cos bx\, dx =$

$\sin bx = u \quad \Big| \quad du = b \cos bx\, dx$

$e^{ax} dx = dV \quad \Big| \quad V = \frac{e^{ax}}{a}$

$\cos bx = u \quad \Big| \quad du = -b \sin bx$

$e^{ax} dx = dV \quad \Big| \quad V = \frac{e^{ax}}{a}$

$$= \frac{1}{a} e^{ax} \sin bx - \frac{b}{a} \left(\frac{e^{ax} \cos bx}{a} + \frac{b}{a} \underbrace{\int e^{ax} \sin bx\, dx}_{I} \right).$$

$$I = \frac{1}{a} e^{ax} \sin bx - \frac{b}{a^2} e^{ax} \cos bx - \frac{b^2}{a^2} I.$$

$$I \left(1 + \frac{b^2}{a^2}\right) = \frac{1}{a^2} e^{ax} (a \sin bx - b \cos bx)$$

$$I = \frac{e^{ax} (a \sin bx - b \cos bx)}{a^2 + b^2}$$

1821. $\int e^{2x} \sin^2 x\, dx = \int e^{2x} \frac{1 - \cos 2x}{2} dx =$

$= \frac{1}{2} \int e^{2x} dx - \frac{1}{2} \int e^{2x} \cos 2x\, dx = \frac{1}{4} e^{2x} - \frac{1}{4} \int e^t \cos t\, dt$

$2x = t \quad 2\,dx = dt \quad dx = \frac{1}{2} dt$

$I = \int e^t \cos t\, dt = e^t \cdot \cos t + \int e^t \sin t\, dt =$

$\cos t = u \quad \Big| \quad du = -\sin t\, dt$

$e^t dt = dV \quad \Big| \quad V = e^t$

$\sin t = u \quad \Big| \quad du = \cos t\, dt$

$e^t dt = dV \quad \Big| \quad V = e^t$

$= e^t \cdot \cos t + e^t \sin t - \underbrace{\int e^t \cos t\, dt}_{I}. \quad I = e^t (\cos t + \sin t) - I.$

$2I = e^t (\sin t + \cos t) \qquad I = \frac{1}{2} e^t (\cos t + \sin t)$

$\int e^{2x} \sin^2 x\, dx = \frac{1}{4} e^{2x} - \frac{1}{4} \cdot \frac{1}{2} e^{2x} (\sin 2x + \cos 2x) + C =$

$$= \frac{e^{2x} (2 - \sin 2x - \cos 2x)}{8} + C.$$

1825. $\int \frac{x\, dx}{\cos^2 x} = x \operatorname{tg} x - \int \operatorname{tg} x\, dx = x \operatorname{tg} x - \int \frac{\sin x\, dx}{\cos x} =$

$\cos x = t$

$-\sin x\, dx = dt$

$x = u \quad \Big| \quad du = dx$

$\frac{dx}{\cos^2 x} = dV \quad \Big| \quad V = \operatorname{tg} x$

$= x \operatorname{tg} x + \int \frac{dt}{t} = x \operatorname{tg} x + \ln|\cos x| + C$

טנהוקסאן

(1794) $\int \arcsin x\,dx = x\arcsin x - \int \frac{x\,dx}{\sqrt{1-x^2}} =$

$\arcsin x = u \;\Big|\; du = \frac{dx}{\sqrt{1-x^2}}$ $\qquad \sqrt{1-x^2} = t \quad 1-x^2 = t^2$

$dx = dV \;\Big|\; V = x$ $\qquad -2x\,dx = 2t\,dt \quad x\,dx = -t\,dt$

$= x\arcsin x + \int \frac{t\,dt}{t} = x\arcsin x + \sqrt{1-x^2} + c$

(1795) $\int x \,\mathrm{arctg}\, x\,dx = \frac{x^2}{2}\mathrm{arctg}\, x - \frac{1}{2}\int \frac{x^{2\,+1-1}dx}{1+x^2} =$

$\mathrm{arctg}\, x = u \;\Big|\; du = \frac{dx}{1+x^2}$

$x\,dx = dV \;\Big|\; V = \frac{x^2}{2}$

$= \frac{x^2}{2}\mathrm{arctg}\, x - \frac{1}{2}\int\left(1 - \frac{1}{1+x^2}\right)dx = \frac{x^2}{2}\mathrm{arctg}\, x - \frac{x}{2} + \mathrm{arctg}\, x + c$

(1818) $\int \cos(\ln x)\,dx = \int e^t \cos t\,dt = I =$

$\ln x = t \quad x = e^t \quad dx = e^t dt$ $\qquad \cos t = u \;\Big|\; du = -\sin t\,dt$

$\qquad e^t dt = dV \;\Big|\; V = e^t$

$= e^t\cos t + \int e^t \sin t\,dt = e^t\cos t + e^t \sin t - \int e^t \cos t\,dt.$

$\sin t = u \;\Big|\; du = \cos t\,dt$

$e^t dt = dV \;\Big|\; V = e^t$

קיבלנו שוויון:

$I = \int e^t \cos t\,dt = e^t\cos t + e^t \sin t - \overline{\int e^t \cos t\,dt}^{\,I}.$

$2I = e^t(\cos t + \sin t) \qquad I = \frac{e^t}{2}(\sin t + \cos t) + c$

$\int \cos(\ln x)\,dx = \frac{e^{\ln x}}{2}(\sin \ln x + \cos \ln x) + c = \frac{x}{2}(\sin \ln x + \cos \ln x).$

(1819) $\int e^{ax}\cos bx\,dx = \frac{e^{ax}}{a}\cos bx + \frac{b}{a}\int e^{ax}\sin bx\,dx =$

$\cos bx = u \;\Big|\; du = -b\sin bx\,dx$ $\qquad \sin bx = u \;\Big|\; du = b\cos bx\,dx$

$e^{ax}dx = dV \;\Big|\; V = \frac{1}{a}e^{ax}$ $\qquad e^{ax}dx = dV \;\Big|\; V = \frac{e^{ax}}{a}$

$= \frac{1}{a}e^{ax}\cos bx + \frac{b}{a}\left(\frac{1}{a}e^{ax}\sin bx - \frac{b}{a}\underbrace{\int e^{ax}\cos bx\,dx}_{I}\right).$

$I = \frac{1}{a}e^{ax}\cos bx + \frac{b}{a^2}e^{ax}\sin bx - \frac{b^2}{a^2}I.$

$I\left(1 + \frac{b^2}{a^2}\right) = \frac{1}{a^2}e^{ax}(a\cos bx + b\sin bx).$ $\qquad I = \frac{e^{ax}(a\cos bx + b\sin bx)}{a^2+b^2} + c$

בס"ד

סניה ויקסמן

אינטגרציה לפי חלקים

1782, 1784, 1790, 1793–1795

1818–1821, 1825. $\int u\,dv = uv - \int v\,du$ $\quad u=u(x)\quad v=v(x)$

(1782) $\int \ln x\,dx = x\ln x - \int dx = x\ln x - x + C$

$\ln x = u \quad dx = dv \;\Big|\; du = \frac{dx}{x} \quad v = x$

(1784) $\int \left(\frac{\ln x}{x}\right)^2 dx = \int \frac{\ln^2 x}{x^2}\,dx = -\frac{\ln^2 x}{x} + 2\int \frac{\ln x}{x^2}\,dx =$

$\ln^2 x = u \quad \frac{dx}{x^2} = dv \;\Big|\; du = \frac{2\ln x}{x}\,dx \quad v = -\frac{1}{x}$ $\Big\}$ $\ln x = u \quad \frac{dx}{x^2} = dv \;\Big|\; du = \frac{dx}{x} \quad v = -\frac{1}{x}$

$= -\frac{\ln^2 x}{x} + 2\left(-\frac{\ln x}{x} + \int \frac{dx}{x^2}\right) = -\frac{\ln^2 x}{x} - \frac{2\ln x}{x} - \frac{2}{x} + C$

(1790) $\int x^2 \sin 2x\,dx = -\frac{x^2\cos 2x}{2} + \int x\cos 2x\,dx =$

$x^2 = u \quad \sin 2x\,dx = dv \;\Big|\; du = 2x\,dx \quad v = -\frac{\cos 2x}{2}$

$x = u \quad \cos 2x\,dx = dv \;\Big|\; du = dx \quad v = \frac{\sin 2x}{2}$

$= -\frac{x^2\cos 2x}{2} + \frac{x\sin 2x}{2} - \frac{1}{2}\int \sin 2x\,dx = -\frac{x^2\cos 2x}{2} + \frac{x\sin 2x}{2} +$

$+ \frac{\cos 2x}{4} + C.$

(1793) $\int \operatorname{arctg} x\,dx = x\operatorname{arctg} x - \int \frac{x\,dx}{1+x^2} = x\operatorname{arctg} x - \int \frac{t\,dt}{t^2} =$

$\operatorname{arctg} x = u \quad dx = dv \;\Big|\; du = \frac{dx}{1+x^2} \quad v = x$

$1+x^2 = t^2 \quad 2x\,dx = 2t\,dt$

$x\,dx = t\,dt$

$= x\operatorname{arctg} x - \ln(1+x^2) + C$

בס"ד

טניה ויסמן

1695 $\int \frac{dx}{\cos x} = \int \frac{d(x+\frac{\pi}{2})}{\sin(x+\frac{\pi}{2})} = \ln\left|tg\left(\frac{x}{2}+\frac{\pi}{4}\right)\right| + c$

ראה 1694

1703 $\int \frac{x^2+1}{x^4+1}\,dx = \int \frac{x^2\left(1+\frac{1}{x^2}\right)}{x^2\left(x^2+\frac{1}{x^2}\right)}\,dx =$

$x-\frac{1}{x}=t \qquad \left(1+\frac{1}{x^2}\right)dx = dt \qquad \left(x-\frac{1}{x}\right)^2 = x^2-2+\frac{1}{x^2}$

$x^2+\frac{1}{x^2} = \left(x-\frac{1}{x}\right)^2+2 = t^2+2$

$= \int \frac{\left(1+\frac{1}{x^2}\right)dx}{x^2+\frac{1}{x^2}} = \int \frac{dt}{t^2+2} = \frac{1}{\sqrt{2}}\,arctg\,\frac{t}{\sqrt{2}} + c =$

$= \frac{1}{\sqrt{2}}\,arctg\,\frac{x-\frac{1}{x}}{\sqrt{2}} + c = \frac{1}{\sqrt{2}}\,arctg\,\frac{x^2-1}{\sqrt{2}\,x} + c.$

1751 $\int \frac{dx}{1+e^x} = \int \frac{e^x\,dx}{e^x(1+e^x)} = \int \frac{dt}{t(1+t)} =$

$e^x = t \qquad e^x\,dx = dt$

$= \int \left(\frac{1}{t} - \frac{1}{t+1}\right)dt = \ln|t| - \ln|t+1| + c =$

$= \ln\left|\frac{t}{t+1}\right| + c = \ln\frac{e^x}{e^x+1} + c$

1639 $\int tg^2x\,dx = \int\left(\frac{1}{\cos^2x}-1\right)dx = tg\,x - x + c$

$tg^2\alpha+1 = \frac{1}{\cos^2\alpha} \Leftarrow \sin^2\alpha+\cos^2\alpha = 1$

3"од

(1677) $\int x e^{-x^2}dx = -\frac{1}{2}\int e^{t}dt = -\frac{1}{2}e^{t}+c =$

$-x^2 = t \quad -2x\,dx = dt \quad x\,dx = -\frac{1}{2}dt$

$= -\frac{1}{2}e^{-x^2}+c.$

(1678) $\int \frac{dx}{\sqrt{1+e^{2x}}} = \int \frac{dx}{e^{x}\sqrt{\frac{1}{e^{2x}}+1}} = \int \frac{e^{-x}dx}{\sqrt{e^{-2x}+1}} =$

$e^{-x} = t \quad -e^{-x}dx = dt \quad e^{-x}dx = -dt$

$= -\int \frac{dt}{\sqrt{t^2+1}} = -\ln\left|t+\sqrt{t^2+1}\right|+c = -\ln\left(e^{-x}+\sqrt{e^{-2x}+1}\right)+c$

(1681) $\int \frac{\ln^2 x}{x}dx = \int t^2dt = \frac{t^3}{3}+c = \frac{\ln^3 x}{3}+c$

$\ln x = t \quad \frac{1}{x}dx = dt$

(1685) $\int tg\,x\,dx = \int \frac{\sin x}{\cos x}dx = -\int\frac{dt}{t} = -\ln|t|+c =$

$\cos x = t \quad -\sin x\,dx = dt \quad \sin x\,dx = -dt$

$= -\ln|\cos x|+c$

(1686) $\int ctg\,x\,dx = \int \frac{\cos x}{\sin x}dx = \int\frac{dt}{t} = \ln|\sin x|+c$

$\sin x = t \quad \cos x\,dx = dt$

(1694) $\int\frac{dx}{\sin x} = \int \frac{\sin^2\frac{x}{2}+\cos^2\frac{x}{2}}{2\sin\frac{x}{2}\cdot\cos\frac{x}{2}}dx = \int\frac{\sin\frac{x}{2}}{2\cos\frac{x}{2}}dx + \int\frac{\cos\frac{x}{2}}{2\sin\frac{x}{2}}dx =$

$\cos\frac{x}{2} = t_1 \quad -\frac{1}{2}\sin\frac{x}{2}dx = dt_1 \quad \| \quad \sin\frac{x}{2} = t_2 \quad \frac{1}{2}\cos\frac{x}{2}dx = dt_2$

$= -\int\frac{dt_1}{t_1} + \int\frac{dt_2}{t_2} = -\ln\left|\cos\frac{x}{2}\right| + \ln\left|\sin\frac{x}{2}\right|+c =$

$= \ln\left|tg\,\frac{x}{2}\right|+c$

טניה וקסמן

חדו"א א-1

בס"ד

(1649) $\int \frac{\sqrt[5]{1-2x+x^2}}{1-x}dx = \int \frac{\sqrt[5]{(1-x)^2}}{1-x}dx = \int (1-x)^{-\frac{3}{5}}dx =$

$1-x=t \quad -dx = dt \quad dx = -dt$

$= -\int t^{-\frac{3}{5}}dt = -\frac{t^{-\frac{3}{5}+1}}{-\frac{3}{5}+1}+c = -\frac{5}{2}t^{\frac{2}{5}}+c = -\frac{5}{2}\sqrt[5]{(1-x)^2}+c$

החלפת משתנים

1663, 1664, 1666, 1667, 1668, 1670, 1677, 1678, 1681, 1685, 1686, 1694, 1695, 1703, 1751. 1639.

(1663) $\int x^2\sqrt[3]{1+x^3}\,dx = \int t^3 dt = \frac{t^4}{4}+c = \sqrt[3]{(1+x^3)^4}+c$

$1+x^3 = t^3 \quad 3x^2dx = 3t^2dt \quad x^2dx = t^2dt$

(1664) $\int \frac{x\,dx}{3-2x^2} = -\frac{1}{4}\int\frac{dt}{t} = -\frac{1}{4}\ln|t|+c = -\frac{1}{4}\ln|3-2x^2|+c$

$3-2x^2 = t \quad -4x\,dx = dt \quad x\,dx = -\frac{1}{4}dt$

(1666) $\int \frac{x\,dx}{4+x^4} = \frac{1}{2}\int\frac{dt}{2^2+t^2} = \frac{1}{2}\cdot\frac{1}{2}\operatorname{arctg}\frac{t}{2}+c =$

$x^2 = t \quad 2x\,dx = dt \quad x\,dx = \frac{1}{2}dt \quad \{\} = \frac{1}{4}\operatorname{arctg}\frac{x^2}{2}+c$

(1667) $\int \frac{x^3dx}{x^8-2} = \frac{1}{4}\int\frac{dt}{t^2-2} = \frac{1}{4}\cdot\frac{1}{2}\ln\left|\frac{t-\sqrt{2}}{t+\sqrt{2}}\right|+c$

$x^4 = t \quad 4x^3dx = dt \quad x^3dx = \frac{1}{4}dt \quad \{\} = \frac{1}{8}\ln\left|\frac{x^4-\sqrt{2}}{x^4+\sqrt{2}}\right|+c$

(1668) $\int\frac{dx}{(1+x)\sqrt{x}} = 2\int\frac{dt}{1+t^2} = 2\operatorname{arctg}t+c = 2\operatorname{arctg}\sqrt{x}+c$

$\sqrt{x} = t \quad \frac{dx}{2\sqrt{x}} = dt \quad \frac{dx}{\sqrt{x}} = 2dt$

(1670) $\int\frac{dx}{x\sqrt{x^2+1}} = \int\frac{x\,dx}{x^2\cdot\sqrt{x^2+1}} = \int\frac{t\,dt}{(t^2-1)\cdot t} = \frac{1}{2}\ln\left|\frac{t-1}{t+1}\right|+c =$

$\sqrt{x^2+1} = t \quad x^2+1 = t^2 \quad 2x\,dx = 2t\,dt \quad x\,dx = t\,dt \quad x^2 = t^2-1$

$= \frac{1}{2}\ln\left|\frac{\sqrt{x^2+1}-1}{\sqrt{x^2+1}+1}\right|+c$

בס"ד

טניה ויקסמן

המשך 1627 $= 2x - 2\sqrt{2}\sqrt[3]{3}\ \frac{x^{-\frac{1}{6}+1}}{-\frac{1}{6}+1} + \sqrt[3]{9}\cdot\frac{x^{-\frac{1}{3}+1}}{-\frac{1}{3}+1} + c =$

$$= 2x - 12\sqrt{2}\cdot\sqrt[3]{3}\cdot\frac{1}{5}\,x^{\frac{5}{6}} + \frac{3}{2}\sqrt[3]{9}\,x^{\frac{2}{3}} + c$$

1628 $\int \frac{\sqrt{x^4+x^{-4}+2}}{x^3}\,dx = \int \frac{\sqrt{(x^2+x^{-2})^2}}{x^3}\,dx = \int \frac{x^2+x^{-2}}{x^3}\,dx =$

$$= \int\left(\frac{1}{x} + x^{-5}\right)dx = \ln|x| + \frac{x^{-5+1}}{-5+1} + c = \ln|x| - \frac{1}{4}x^{-4} + c$$

1629 $\int \frac{x^2\,dx}{1-x^2} = \int \frac{x^2-1+1}{1-x^2}\,dx = \int\left(\frac{dx}{1-x^2} - 1\right)dx =$

$$= \frac{1}{2}\ln\left|\frac{1+x}{1-x}\right| - x + c$$

1630 $\int \frac{x^2+3}{x^2-1}\,dx = \int \frac{x^2+4-1}{x^2-1}\,dx = \int\left(1 + \frac{4}{x^2-1}\right)dx =$

$$= x + 2\ln\left|\frac{x-1}{x+1}\right| + c$$

1646 − 1649 מיידים אינטגרליים

1646 $\int \sqrt[3]{1-3x}\,dx = -\frac{1}{3}\int t^{\frac{1}{3}}\,dt = -\frac{1}{3}\,\frac{t^{\frac{1}{3}+1}}{\frac{1}{3}+1} + c =$

$1-3x = t \qquad -3\,dx = dt \qquad dx = -\frac{1}{3}\,dt$

$$= -4\,t^{\frac{4}{3}} + c = -4\sqrt[3]{(1-3x)^4} + c$$

1647 $\int \frac{dx}{\sqrt{2-5x}} = -\frac{1}{5}\int t^{-\frac{1}{2}}\,dt = -\frac{1}{5}\,\frac{t^{-\frac{1}{2}+1}}{-\frac{1}{2}+1} + c =$

$2-5x = t \qquad -5\,dx = dt \qquad dx = -\frac{1}{5}\,dt$

$$= -\frac{2}{5}\,t^{\frac{1}{2}} + c = -\frac{2}{5}\sqrt{2-5x} + c$$

1648 $\int \frac{dx}{(5x-2)^{\frac{5}{2}}} = \frac{1}{5}\int t^{-\frac{5}{2}}\,dt = \frac{1}{5}\,\frac{t^{-\frac{5}{2}+1}}{-\frac{5}{2}+1} + c = -\frac{2}{15}\,t^{-\frac{3}{2}} + c = -\frac{2}{15}(5x-2)^{-\frac{3}{2}} + c$

$5x-2 = t \qquad 5\,dx = dt \qquad dx = \frac{1}{5}\,dt$

(1621.) $\int \left(\frac{1-x}{x}\right)^2 dx = \int \frac{1-2x+x^2}{x^2} dx = \int \left(x^{-2} - \frac{2}{x} + 1\right) dx =$

$= \frac{x^{-2+1}}{-2+1} - 2\ln|x| + x + c = -\frac{1}{x} - 2\ln|x| + x + c$

(1622.) $\int \left(\frac{a}{x} + \frac{a^2}{x^2} + \frac{a^3}{x^3}\right) dx = \int \left(\frac{a}{x} + a^2 x^{-2} + a^3 x^{-3}\right) dx =$

$= a\ln|x| + a^2 \frac{x^{-2+1}}{-2+1} + a^3 \frac{x^{-3+1}}{-3+1} + c = a\ln|x| - \frac{a^2}{x} - \frac{a^3}{2x^2} + c$

(1623) $\int \frac{x+1}{\sqrt{x}} dx = \int \frac{x+1}{x^{\frac{1}{2}}} dx = \int \left(x^{\frac{1}{2}} + x^{-\frac{1}{2}}\right) dx =$

$= \frac{x^{\frac{1}{2}+1}}{\frac{1}{2}+1} + \frac{x^{-\frac{1}{2}+1}}{-\frac{1}{2}+1} + c = \frac{2}{3}\sqrt{x^3} + 2\sqrt{x} + c$

(1624) $\int \frac{\sqrt{x} - 2\sqrt[3]{x^2} + 1}{\sqrt[4]{x}} dx = \int \left(\frac{x^{\frac{1}{2}}}{x^{\frac{1}{4}}} - 2\frac{x^{\frac{2}{3}}}{x^{\frac{1}{4}}} + \frac{1}{x^{\frac{1}{4}}}\right) dx =$

$= \int \left(x^{\frac{1}{4}} - 2x^{\frac{5}{12}} + x^{-\frac{1}{4}}\right) dx = \frac{x^{\frac{1}{4}+1}}{\frac{1}{4}+1} - 2\frac{x^{\frac{5}{12}+1}}{\frac{5}{12}+1} + \frac{x^{-\frac{1}{4}+1}}{-\frac{1}{4}+1} + c =$

$= \frac{4x^{\frac{5}{4}}}{5} - 2\cdot\frac{12}{17} x^{\frac{17}{12}} + \frac{4}{3} x^{\frac{3}{4}} + c$

(1625) $\int \frac{(1-x)^3}{x\sqrt[3]{x}} dx = \int \frac{1-3x+3x^2-x^3}{x^{\frac{4}{3}}} dx = \int \left(x^{-\frac{4}{3}} + 3x^{\frac{2}{3}} -\right.$

$\left. - 3x^{-\frac{1}{3}} - x^{\frac{5}{3}}\right) dx = \frac{x^{-\frac{4}{3}+1}}{-\frac{4}{3}+1} + 3\frac{x^{\frac{2}{3}+1}}{\frac{2}{3}+1} - 3\frac{x^{-\frac{1}{3}+1}}{-\frac{1}{3}+1} - \frac{x^{\frac{5}{3}+1}}{\frac{5}{3}+1} + c =$

$= -3x^{-\frac{1}{3}} + 3\cdot\frac{3}{5} x^{\frac{5}{3}} - 3\cdot\frac{3}{2} x^{\frac{2}{3}} - \frac{3}{8} x^{\frac{8}{3}} + c$

(1626) $\int \left(1 - \frac{1}{x^2}\right)\sqrt{x\sqrt{x}}\, dx = \int \left(1 - \frac{1}{x^2}\right) x^{\frac{1}{2}} x^{\frac{1}{4}} dx =$

$= \int \left(x^{\frac{3}{4}} - x^{-\frac{5}{4}}\right) dx = \frac{x^{\frac{3}{4}+1}}{\frac{3}{4}+1} - \frac{x^{-\frac{5}{4}+1}}{-\frac{5}{4}+1} + c = \frac{4}{7} x^{\frac{7}{4}} + 4x^{-\frac{1}{4}} + c$

(1627) $\int \frac{(\sqrt{2x} - \sqrt[3]{3x})^2}{x} dx = \int \frac{2x - 2\sqrt{2x}\cdot\sqrt[3]{3x} + \sqrt[3]{9}\cdot x^{\frac{2}{3}}}{x} dx =$

$= \int \left(2 - 2\sqrt{2}\cdot x^{-\frac{1}{2}}\cdot\sqrt[3]{3}\cdot x^{\frac{1}{3}} + \sqrt[3]{9}\, x^{-\frac{1}{3}}\right) dx = \int \left(2 - 2\sqrt{2}\cdot\sqrt[3]{3}\, x^{-\frac{1}{6}} + \sqrt[3]{9}\, x^{-\frac{1}{3}}\right) dx$