Exam in „Fundamentals of Analysis", 06.02.2019

- Duration: 3 hours
- One need to answer all 4 questions.
- The grade of every question is 25 points.
- One need to present a complete and detailed solution for all questions.
- Eeach student can Bring 4 pages (2 lists from both sides) of formulas of a standard size A4. There is no calculator.
- Instead of a solution every student can write "I don't know". Then this question will not be checked and the grade for this question / or part of the question will be 20% of its maximal value.

Question 1 (א) (10 points) Let $X \subset \mathbb{R} \setminus \{0\}$ be any non-empty set. Define $d(x,y) = \left|\frac{1}{x} - \frac{1}{y}\right|$ for every $x, y \in X$.
Prove that d defines a metric on X.

(ב) (10 points) Let $X = (0,1]$. Metric d is defined as above. Prove that (X,d) is a complete metric space.

(ג) (5 points) Let $X = [1,\infty)$. Prove that (X,d) is not complete.

Hint: One can use the fact that a set $Y \subset \mathbb{R}$ with the usual Euclidean metric is a complete metric space if and only if Y is a closed subset of $\mathbb{R}$ (with Euclidean metric)

Question 2 (25 points) Let $X \subset \mathbb{R}^n$ be any subset of an Euclidean space $\mathbb{R}^n$ with a standard metric. Assume that X satisfies to the following property:

If $f(x): X \to \mathbb{R}$ is any continuous function, then its image $f(X) = \{f(x) : x \in X\}$ is a bounded subset of $\mathbb{R}$.

Prove that X is a compact set.

Question 3 (25 points). Let $E \subset [a,b]$ be a measurable set in the sense of Lebesgue. Assume that for every open interval Δ the following property holds:

$\mu(E \cap \Delta) \leq \frac{1}{2}\mu(\Delta)$. Prove that $\mu(E)=0$.

Question 4 (1) (10 points). Assume that $f_n(x): [0,1] \to [0,1]$ is a Lebesgue measurable function for every natural $n \in \mathbb{N}$, and such that $\int_{[0,1]} f_n(x)\, d\mu = 1$. Is it possible that $\lim_{n\to\infty} f_n(x) = 0$ for every $x \in [0,1]$?

(2) (10 points). Assume that $f_n(x): [0,\infty) \to [0,1]$ is a Lebesgue measurable function $\forall n \in \mathbb{N}$, and such that $\int_{[0,\infty)} f_n(x)\, d\mu = 1$.

Is it possible that $\lim_{n\to\infty} f_n(x) = 0$ for every $x \in [0,\infty)$?

Solutions.

Question 1 (K). Denote by $Y \subseteq \mathbb{R}$ the following set: $Y = \{ \frac{1}{x} : x \in X \}$

We consider Y with a standard Euclidean metric d_R: $d_R(y_1, y_2) = |y_1 - y_2| \quad \forall y_1, y_2 \in Y$.

So, we define a mapping

$$\varphi(y) = \frac{1}{y} : (Y, d_R) \to (X, d)$$

This mapping is a bijective mapping which preserves the distances, i.e.

$$d(\varphi(y_1), \varphi(y_2)) = d_R(y_1, y_2)$$

Therefore d satisfies all axioms of a metric, since d_R is a metric in Y.

(1) If $X = (0, 1]$, then $Y = \{ \frac{1}{x} : x \in X \} = [1, \infty)$.

The set $[1, \infty)$ is closed is closed in $(\mathbb{R}, d_R)$, therefore (Y, d_R) is a complete metric space, so (X, d) also is a complete metric space.

(2) If $X = [1, \infty)$, then $Y = (0, 1]$.

$Y \subset (\mathbb{R}, d_R)$ is not closed, therefore (X, d) is not complete. Direct proof of the same claim: take a sequence $\{x_n = n\}_{n=1}^{\infty}$. $d(x_n, x_m) = |\frac{1}{n} - \frac{1}{m}|$, so $\{x_n\}_{n=1}^{\infty}$ is a Cauchy sequence in (X, d). But $\lim_{n \to \infty} x_n$ in (X, d) does not exist, since $\lim_{n \to \infty} x_n = 0$ in $\mathbb{R}$, however $0 \notin Y$.

Question 2 On the contrary, assume that $X \subset \mathbb{R}^n$ is not a compact set. Then X or not bounded in $\mathbb{R}^n$, or not closed in $\mathbb{R}^n$.

1) Let X is not bounded. Consider the following function $f: X \to R$, $f(x) = d(x, 0) = \|x\|$.
$f(x)$ is a continuous function, and the image $f(X) \subset \mathbb{R}$ is not a bounded set in $\mathbb{R}$.
This is a contradiction with the assumption.

2) Let X is not a closed subset of $\mathbb{R}^n$. Then there is a sequence $\{x_n\}_{n=1}^{\infty} \subset X$ such that $x_n \to x_0 \notin X$. Consider the following function

$$f: X \to \mathbb{R}, \quad f(x) = \frac{1}{d(x, x_0)}.$$

Since $d(x, x_0)$ is a continuous function, and $d(x, x_0) \neq 0$ $\forall x \in X$, we have that $f(x)$ is a continuous function.

$$d(x_n, x_0) \xrightarrow[n\to\infty]{} 0 \implies f(x_n, x_0) = \frac{1}{d(x_n, x_0)} \xrightarrow[n\to\infty]{} \infty.$$

Again, we conclude that the image $f(X) \subset R$ is not a bounded set in R.
This is a contradiction with the assumption.

So, if X is not a compact set, then there exists a continuous function $f(x): X \to \mathbb{R}$ such that the image $f(X) \subset \mathbb{R}$ is not bounded. This contradiction shows that X has to be compact.

Question 3 By definition,

$$\mu(E) = \inf\{\mu(U) : E \subset U, U \text{ is open}\}$$

Every open set U can be represented as

$$U = \biguplus_n \Delta_n$$

where each Δ_n is an open interval, and the union is finite or countable.

Let $E \subset U$ - open, then

$$\mu(E) = \sum_n \mu(E \cap \Delta_n) \leq \text{ by assumption } \leq$$

$$\sum_n \frac{1}{2}\mu(\Delta_n) = \frac{1}{2}\mu(U)$$

So, for every open set U such that $E \subset U$ we have that

$$\mu(E) \leq \frac{1}{2}\mu(U)$$

Then $\mu(E) \leq \frac{1}{2} \inf_{E \subset U \text{ open}} \{\mu(U)\} = \frac{1}{2}\mu(E)$

We obtain the following inequality

$$0 \leq \mu(E) \leq \frac{1}{2}\mu(E)$$

It is possible iff $\mu(E) = 0$ only.

Question 4 (1) According to the Lebesgue Dominated Convergence Theorem

if $\lim_{n \to \infty} f_n(x) = 0$ for every $x \in [0,1]$

and $|f_n(x)| \leq 1$ for every $x \in [0,1]$ $\forall n \in \mathbb{N}$

then $\lim_{n \to \infty} \int_{[0,1]} f_n(x) \, d\mu = \int_{[0,1]} 0 \, d\mu = 0$

therefore, $\int_{[0,1]} f_n(x) \, d\mu = 1 \quad \forall n \in \mathbb{N}$

is impossible

(1.2) Example: $f_n(x) : [0,\infty) \to [0,1]$.

$$f_n(x) = \begin{cases} 1, & n-1 \leq x < n \\ 0, & \text{otherwise} \end{cases}$$

Then $\int_{[0,\infty)} f_n(x) \, d\mu = 1 \quad \forall n \in \mathbb{N}$.

$\lim_{n \to \infty} f_n(x) = 0 \quad \forall x \in [0,\infty)$.

It's possible.