Section 6. Random Vectors. Joint CDF and Joint Density Function. Marginal and Conditional Distributions.

a. Random Vector and Its Joint CDF. Marginal CDF.Independence.

In many applications, it is necessary to consider the properties of two or more random variables simultaneously. Suppose, that random variables $X_1, X_2, ..., X_k$ are defined on the same sample space. Then the ordered sequence $\mathbf{Y} = (X_1, X_2, ..., X_k)$ is called *random vector*. In this section, we will study the probabilistic description of random vectors.

Definition 1. The joint CDF of $\mathbf{Y}$ is defined as

$$F_{\mathbf{Y}}(t_1, t_2, ..., t_k) = P(X_1 \leq t_1, ..., X_k \leq t_k). \tag{1}$$

$F_{\mathbf{Y}}$ is defined for $-\infty < t_i < \infty, i = 1, ..., k$.

Let us consider in more detail the case of $k = 2$, a two-dimensional r.v. The corresponding joint CDF $F_{\mathbf{Y}}(t_1, t_2)$ gives the value of the probabilistic mass in the area which is "south-west" from the point t_1, t_2 on the plane, see Fig.1,a.

It follows from the definition, that the joint CDF has the following properties:

(i) $F_{\mathbf{Y}}(\cdot, \cdot)$ is a nondecreasing continuous from the right function of each of its arguments;

(ii) $\lim_{t_1 \to \infty, t_2 \to \infty} F_{\mathbf{Y}} = 1$;

(iii) $\lim_{t_i \to -\infty} F_{\mathbf{Y}} = 0$ for each t_i.

Example 1. A point is randomly thrown on the triangle with vertices (0,0),(0,1) and (1,0), see Fig. 1, b. Let X_1 and X_2 be the random coordinates of this point. Find their joint CDF.

Solution. It is advisable to consider separately the following cases:

(i) $t = (t_1, t_2)$ lies not in the first quarter;

(ii) $t_1 + t_2 \leq 1$, and both t_i are nonnegative, i.e. the point t lies inside the triangle;

(iii) t lies in the upper triangle (1,0),(0,1),(1,1);

(iv) $t_1 \leq 1, t_2 \geq 1$;

(v) $t_1 \geq 1, t_2 \leq 1$;
(vi) Both coordinates of the random point are greater than 1.
Elementary calculations based on geometric arguments lead to the following:
For (i): $F_{\mathbf{Y}}(t_1, t_2) = 0$;
For (ii): $F_{\mathbf{Y}}(t_1, t_2) = 2t_1 t_2$;
For (iii): $F_{\mathbf{Y}}(t_1, t_2) = 1 - (1 - t_1)^2 - (1 - t_2)^2$;
For (iv): $F_{\mathbf{Y}}(t_1, t_2) = 1 - (1 - t_1)^2$;
For (v): $F_{\mathbf{Y}}(t_1, t_2) = 1 - (1 - t_2)^2$;
For (vi): $F_{\mathbf{Y}}(t_1, t_2) = 1$. $\square$

Suppose that we know the joint CDF of $\mathbf{Y} = (X_1, X_2)$. Then the CDF of just the single r.v. X_1 can be derived from the joint CDF as follows, for $-\infty < t_1 < \infty$:

$$F_{X_1}(t_1) = P(X_1 \leq t_1) = lim_{t_2 \to \infty} P(X_1 \leq t_1 \text{ and } X_2 \leq t_2) = F_{\mathbf{Y}}(t_1, \infty).$$

The CDF of X_1 derived from the joint CDF $F_{\mathbf{Y}}(\cdot, \cdot)$, $F_{X_1}(\cdot)$ is called the *marginal* CDF of X_1. Similarly, the marginal CDF of X_2 is derived from the joint CDF by letting $t_1 \to \infty$.

Example 1 continued. Verify that for $t_1 \in [0, 1]$, $F_{X_1}(t_1) = 1 - (1 - t_1)^2$. By symmetry, for $t_2 \in [0, 1]$, $F_{X_2}(t_2) = 1 - (1 - t_2)^2$.

Definition 2.We say that two r.v.'s X_1 and X_2 are *independent* if for any reals t_1 and t_2,

$$P(X_1 \leq t_1 \text{ and } X_2 \leq t_2) = P(X_1 \leq t_1) P(X_2 \leq t_2). \tag{2}$$

In terms of the joint and marginal CDF's, this definition is equivalent to the following: X_1 and X_2 are independent if the joint CDF of X_1, X_2 equals, for any pair t_1, t_2, to the product of marginal CDF's $F_{X_1}(t_1)$ and $F_{X_2}(t_2)$.

Question: in Example 1, are the coordinates of the random vector $\mathbf{Y}$ independent?

b. Continuous Random Vectors. Joint Density Function. Independence

In this subsection we will be dealing with a two-dimensional random vector. The generalization for $k > 2$ dimensions could be done by analogy.

Definition 3. The function $f_{X_1, X_2}(u, v)$ is called the *joint density function* of r.v.'s X_1 and X_2 if

$$F_{X_1, X_2}(t, s) = \int_{-\infty}^{t} \int_{-\infty}^{s} f_{X_1, X_2}(u, v) du dv. \tag{3}$$

If the joint CDF can be expressed as a Riemann integral of the joint density function similar to (3), then the corresponding vector r.v. is called *continuous*.

In the above Example 1, $f_{X_1, X_2}(u, v)$ equals 2 if $u + v \leq 1, u > 0, v > 0$, and equals zero, otherwise (see Fig 1.b).

The probabilistic meaning of the joint density function is the following:

$$f_{X_1, X_2}(u, v) \cdot \Delta u \cdot \Delta v = P(X_1 \in [u, u + \Delta u] \& X_2 \in [v, v + \Delta v]). \tag{4}$$

An important formula for computing the probability that $(X_1, X_2) \in S$, where S is a two-dimensional region in the $u, v-$ plane, is the following one :

$$P((X_1, X_2) \in S) = \int\int_{(u,v)\in S} f_{X_1,X_2}(u,v)dudv. \tag{5}$$

For a continuous random variable, there is a simple relationship between the joint CDF and the joint density function:

$$f_{X_1,X_2}(u,v) = \frac{\partial^2 F_{X_1,X_2}(u,v)}{\partial u \partial v}. \tag{6}$$

This relationship is valid for all points (u, v) at which the derivative exists.

It is possible to express the marginal CDF of r.v. X_1 through the joint density function, as follows.

$$F_{X_1}(t) = \int_{-\infty}^{t} (\int_{-\infty}^{\infty} f_{X_1,X_2}(u,v)dv)du. \tag{7}$$

On the other hand, the CDF of X_1 can be expressed through its "own" (marginal) density function $f_{X_1}(\cdot)$ as follows :

$$F_{X_1}(t) = \int_{-\infty}^{t} f_{X_1}(u)du. \tag{8}$$

Comparing the last two formulas, we conclude that

$$f_{X_1}(u) = \int_{-\infty}^{\infty} f_{X_1,X_2}(u,v)dv. \tag{9}$$

Definition 3. Continuous r.v.'s X_1 and X_2 are called *independent* if for all u, v,

$$f_{X_1,X_2}(u,v) = f_{X_1}(u) f_{X_2}(v). \tag{10}$$

Example 2. $f_{X_1,X_2}(u,v) = e^{-u}$ for $0 \le u < \infty$ & $v \in [0,1]$, and 0 othertwise. Are X_1 and X_2 independent ?

Solution. Integrating the joint density with respect to v, we obtain that $f_{X_1}(u) = e^{-u}$, if $u \in [0, \infty)$, and 0 otherwise. Similarly, integrating the joint density with respect to u, we obtain that $f_{X_2}(u) = 1$ if $u \in [0,1]$ and 0 otherwise. Now it remains to check that indeed, $f_{X_1,X_2}(u,v)$ factorizes as $f_{X_1}(u) \cdot f_{X_1}(v)$, for all u, v. It is extremely helpful to sketch the support of (X_1, X_2), see Fig. 2.□

c. Conditional CDF and Conditional Density Function.

Let $\mathbf{V} = (X, Y)$ be a two-dimensional random variable. Very often we have to calculate probabilities related to one component of $\mathbf{V}$, say X, given some condition on another component Y. These calculations are carried out through the so-called conditional CDF and the conditional density functions. All formulas below will be given for the case of continuous random variables. Let $f_{X,Y}(u,v)$ be the corresponding joint density fucntion.

Let us first derive the formula for the conditional probability $P(X \le t | a < Y \le b)$. Using the definition of the conditional probability and the formula for the marginal density function for Y, we obtain

$$P(X \le t | a < Y \le b) = \frac{P(X \le t,\ a < Y \le b)}{P(a < Y \le b)} = \frac{\int_{-\infty}^{t} (\int_a^b f_{X,Y}(u,v)dv)du}{\int_{-\infty}^{\infty} (\int_a^b f_{X,Y}(u,v)dv)du}$$

The conditional probability in the left-hand side of this formula is a nondecreasing function of t and has all other properies of a CDF. It is, in fact, the conditional CDF of r.v. X *given* $a < Y \le b$. We will denote it as $F_X(t|a < Y \le b)$. The corresponding *conditional density* will be obtained by differentiation with respect to t :

$$f_X(t|a < Y \le b) = \frac{\int_a^b f_{X,Y}(t,v)dv}{\int_{-\infty}^{\infty} \int_a^b f_{X,Y}(u,v)dvdu}. \tag{11}$$

Now suppose that b in the above formulas equals $a + da$.Then

$$P(X \le t | a < Y \le a + da) = \frac{P(X \le t,\ a < Y \le a + da)}{P(a < Y \le a + da)} = \frac{(\int_{-\infty}^{t} f_{X,Y}(u,a)du)da}{(\int_{-\infty}^{\infty} f_{X,Y}(u,a)du)da}.$$

Cancelling da, we arrive at the following *conditional* CDF of X given $Y = a$:

$$F_X(t|Y = a) = P(X \le t|Y = a) = \frac{\int_{-\infty}^{t} f_{X,Y}(u,a)du}{\int_{-\infty}^{\infty} f_{X,Y}(u,a)du}. \tag{12}$$

Differentiating this expression with respect to t we arrive at the *conditional density function* of X given $Y = a$:

$$f_X(t|Y = a) = P(X \le t|Y = a) = \frac{f_{X,Y}(t,a)}{\int_{-\infty}^{\infty} f_{X,Y}(u,a)du}. \tag{13}$$

Note that the denominator in the last formula equals to the marginal density of Y evaluated at $Y = a$. Thus the last formula takes the form :

$$f_X(t|Y = a) = \frac{f_{X,Y}(t,a)}{f_Y(a)}. \tag{14}$$

Example 1 completed . Let us compute the marginal density of r.v. X_2 and the conditional density of X_1 given $X_2 = a$.

Obviously, $f_{X_2}(a) = 2(1-a)$ for $a \in [0,1]$ and 0 otherwise. The joint density of (X_1, X_2) equals 2 on the support , i.e. in the triangle (0,0), (1,0),(0,1), and equals zero otherwise. Therefore, $f_{X_1}(t|X_2 = a) = 1/(1-a)$ for $0 \leq t, a\&\ t + a \leq 1$ and zero otherwise. An interesting observation is that if the joint density function is constant on its support then the conditional densities are also constant. In other words, conditionally on $X_2 = a$, the r.v. $X_1 \sim \mathcal{U}(0, 1-a)$. □

d. Mixed Moments. Covariance.

Let $f_{X,Y}(u,v)$ be the joint density for the random vector $\mathbf{V} = (X, Y)$. The *mixed moment* of X and Y is defined as

$$E(X \cdot Y) = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} uv f_{X,Y}(u,v) du dv. \tag{15}$$

The *covariance* of X and Y, $Cov[X,Y]$, is defined as

$$Cov[X,Y] = E[(X - \mu_X)(Y - \mu_Y)]. \tag{16}$$

It is a matter of a simple computation to derive the following formula:

$$Cov[X,Y] = E[XY] - \mu_X \mu_Y. \tag{17}$$

An important characteristic of the joint distribution of r.v.'s X and Y is their *coefficient of correlation* denoted by $\rho[X,Y]$ and defined as

$$\rho[X,Y] = \frac{Cov[X,Y]}{\sigma_X \sigma_Y}, \tag{18}$$

where σ_X and σ_Y are the standard deviations of r.v.'s X and Y, respectively.

Random variables X and Y are not correlated if $\rho[X,Y] = 0$.

Proposition $|\rho[X,Y]| = 1$ if and only if r.v. X and Y are linearly depended, that is $Y = aX + b$ for some constants a, b.