Final #2

Mark the correct answer in each of the following questions.

1. In the space $L^2_{\mathrm{PC}}[1,\infty)$, let v_α be the function defined by $v_\alpha(x) = 1/x^\alpha$, where α is an arbitrary fixed real number $> 1/2$. Let $W = \mathrm{span}\{w\}$, where $w(x) = 1/x$. Denote by w^*_α a vector in W that is closest to v_α within W. Define the function $g : (1/2,\infty) \to \mathbf{R}$ by:

$$g(\alpha) = \|w^*_\alpha - v_\alpha\|_2, \qquad 1/2 < \alpha < \infty.$$

   (a) The function g is increasing over $(1/2,\infty)$.

   (b) The function g is decreasing over $(1/2,\infty)$.

   (c) The function g is increasing over the interval $(1/2,\alpha_0)$ for some $\alpha_0 > 1/2$, but not over the whole of $(1/2,\infty)$.

   (d) The function g is decreasing over the interval $(1/2,\alpha_0)$ for some $\alpha_0 > 1/2$, but not over the whole of $(1/2,\infty)$.

   (e) None of the above.

2. Let V be a vector space with an inner product $\langle\cdot,\cdot\rangle$ and induced norm $\|\cdot\|$, and $\{u_n\}_{n=1}^\infty$ a set of vectors in V. Define another sequence of vectors in V by:

$$v_n = \frac{1}{1!}u_1 + \frac{1}{2!}u_2 + \frac{1}{3!}u_3 + \ldots + \frac{1}{n!}u_n, \qquad n = 1,2,\ldots.$$

   (a) If $\{u_n\}_{n=1}^\infty$ is an orthonormal system then $(v_n)_{n=1}^\infty$ is a Cauchy sequence, but in general it does not have to be the case.

   (b) The sequence $(v_n)_{n=1}^\infty$ is a Cauchy sequence if and only if $u_n \underset{n\to\infty}{\longrightarrow} 0$.

(c) If $\|u_n\| \leq 2^n$ for each n, then $(v_n)_{n=1}^{\infty}$ is a Cauchy sequence. On the other hand, if $\|u_n\| \geq (n-1)!$ for each n, then it is not a Cauchy sequence.

(d) If $\{u_n\}_{n=1}^{\infty}$ is an orthonormal system and $v_n \underset{n\to\infty}{\longrightarrow} v$ for some $v \in V$, then $\|v\| = e - 1$.

(e) None of the above.

3. The function $f : [-\pi, \pi] \to \mathbf{C}$ is differentiable of every order on the whole interval (and $f^{(k)}(-\pi) = f^{(k)}(\pi)$ for $k = 0, 1, 2, \ldots$). All the numbers $c_n,\ n = \ldots, -3, -2, -1, 1, 2, 3, \ldots$ are given, but c_0 is not.

(a) The number c_0 is uniquely determined by the other c_n's.

(b) Whether or not c_0 is uniquely determined by the other c_n's depends on the other c_n's. Namely, for some choices of $c_n,\ n = \ldots, -3, -2, -1, 1, 2, 3, \ldots$, the number c_0 is uniquely determined, while for others it is not.

(c) For every choice of $c_n,\ n = \ldots, -3, -2, -1, 1, 2, 3, \ldots$, the number c_0 is not uniquely determined, but there always exists a constant $r > 0$ (depending on the other c_n's) such that $|c_0| < r$.

(d) For every choice of $c_n,\ n = \ldots, -3, -2, -1, 1, 2, 3, \ldots$, the number c_0 may be any complex number.

(e) None of the above.

4. The function f is known to belong to $L^2_{\mathrm{PC}}[-\pi, \pi]$ and is piecewise continuously differentiable. Denote:

$$l = f(0-), \qquad m = f(0), \qquad r = f(0+).$$

(a) If $l \neq m \neq r \neq l$, then the Fourier series of f cannot possibly converge pointwise in $[-\pi, \pi]$.

(b) If not all three numbers l, m, r are equal, then the Fourier series of f cannot possibly converge uniformly in $[-\pi, \pi]$.

(c) Even if $l \neq r$, it is still possible that the Fourier series of f converges uniformly in $[-\pi, \pi]$.

(d) It is possible that $l \neq m$ and $m \neq r$, yet the Fourier series of f converges uniformly in $[-\pi, \pi]$.

(e) None of the above.

5. We expand the function f, defined by

$$f(x) = \sin\left(e^{-2ix}\right), \qquad -\pi \leq x \leq \pi,$$

into a Fourier series

$$\sum_{n=-\infty}^{\infty} c_n e^{inx}.$$

(a) The series converges to the function pointwise and $c_{-10} = 0$.

(b) The series converges to the function pointwise and $c_{-10} = 1/5!$.

(c) The series does not converge to the function pointwise and $c_{-10} = 0$.

(d) The series does not converge to the function pointwise and $c_{-10} = 1/5!$.

(e) None of the above.

6. Let $(f_n)_{n=0}^{\infty}$ be the sequence of functions defined by

$$f_n(x) = e^{inx}, \qquad n = 0, 1, 2, \ldots, \ -\pi \leq x \leq \pi,$$

and $(g_n)_{n=0}^{\infty}$ the sequence defined by

$$g_n(x) = \frac{f_0(x) + f_1(x) + \ldots + f_n(x)}{n+1}, \qquad n = 0, 1, 2, \ldots, \ -\pi \leq x \leq \pi.$$

(a) Both sequences converge only at the point $x = 0$.

(b) The sequence (f_n) converges only at the point $x = 0$. The sequence (g_n) converges pointwise for every $x \in [-\pi, \pi]$. However, for no interval $[a, b]$ with $-\pi \le a < b \le \pi$ does it converge uniformly on $[a, b]$.

(c) The sequence (f_n) converges only at the point $x = 0$. The sequence (g_n) converges pointwise for every $x \in [-\pi, \pi]$. It also converges uniformly on $[-\pi, -\delta] \cup [\delta, \pi]$ for every $\delta > 0$, but not on $[-\pi, \pi]$.

(d) The sequence (f_n) converges only at the point $x = 0$. The sequence (g_n) converges uniformly on $[-\pi, \pi]$.

(e) None of the above.

7. Let f be the function defined by

$$f(x) = (x-1)^2, \qquad -\pi \le x < \pi,$$

and let $(S_n)_{n=1}^{\infty}$ be the sequence of partial sums of the Fourier series of f.

(a) For every $\varepsilon > 0$ there exists an $N = N(\varepsilon)$ such that for every $n > N$ we have

$$\left|S_n(-\pi) - \pi^2 - 1\right| < \varepsilon.$$

(b) For every $\varepsilon > 0$ there exists an $N = N(\varepsilon)$ such that for every $n > N$ we have

$$\left|S_n(-\pi) - (\pi + 1)^2\right| < \varepsilon.$$

(c) For every $\varepsilon > 0$ there exists an $N = N(\varepsilon)$ such that for every $n > N$ we have

$$|S_n(x) - f(x)| < \varepsilon, \qquad x \in (-\pi, -\pi + 0.1).$$

(d) For every $\varepsilon > 0$ and $x \in [-\pi, 0]$ there exists an $N = N(\varepsilon, x)$ such that for every $n > N$ we have

$$|S_n(x) - f(x)| < \varepsilon.$$

(e) None of the above.

8. Let $f : \mathbf{R} \to \mathbf{R}$ be defined by:

$$f(x) = \begin{cases} \frac{1}{2^{n-1}}, & n-1 \leq x < n, \ n = 1, 2, \ldots \\ 0, & x < 0. \end{cases}$$

(a) For $\omega \neq 0$ we have:

$$\hat{f}(\omega) = \frac{1}{2\pi i\omega} \cdot \frac{1 - e^{-i\omega}}{1 - e^{-i\omega}/2}.$$

Moreover, $\hat{f} \in L^2_{\mathrm{PC}}(-\infty, \infty)$.

(b) For $\omega \neq 0$ we have:

$$\hat{f}(\omega) = \frac{1}{2\pi i\omega} \cdot \frac{1 - e^{-i\omega}}{1 - e^{-i\omega}/2}.$$

However, $\hat{f} \notin L^2_{\mathrm{PC}}(-\infty, \infty)$.

(c) For $\omega \neq 0$ we have:

$$\hat{f}(\omega) = \frac{1}{2\pi i\omega} \cdot \frac{1 + e^{-i\omega}}{1 + e^{-i\omega}/2}.$$

Moreover, $\hat{f} \in L^2_{\mathrm{PC}}(-\infty, \infty)$.

(d) For $\omega \neq 0$ we have:

$$\hat{f}(\omega) = \frac{1}{2\pi i\omega} \cdot \frac{1 + e^{-i\omega}}{1 + e^{-i\omega}/2}.$$

However, $\hat{f} \notin L^2_{\mathrm{PC}}(-\infty, \infty)$.

(e) None of the above.

9. Let $f : \mathbf{R} \to \mathbf{C}$ be defined by:

$$f(x) = e^{-x-x^2/2}, \qquad x \in \mathbf{R}.$$

(a) We know that the function $g(x) = e^{-x^2/2}$ belongs to $L^1_{\text{PC}}(-\infty, \infty)$. Since the product of any function in the space $L^1_{\text{PC}}(-\infty, \infty)$ by e^{-x} still belongs to this space, the function f also belongs to this space. Moreover:

$$\hat{f}(\omega) = \sqrt{\frac{1}{2\pi}} e^{i\omega - \omega^2/2}, \qquad \omega \in \mathbf{R}.$$

(b) In general it is not true that the product of any function in the space $L^1_{\text{PC}}(-\infty, \infty)$ by e^{-x} still belongs to this space. However, in our case the function f indeed belongs to the space. Moreover:

$$\hat{f}(\omega) = \sqrt{\frac{1}{2\pi}} e^{i\omega - \omega^2/2}, \qquad \omega \in \mathbf{R}.$$

(c) We know that the function $g(x) = e^{-x^2/2}$ belongs to $L^1_{\text{PC}}(-\infty, \infty)$. Since the product of any function in the space $L^1_{\text{PC}}(-\infty, \infty)$ by e^{-x} still belongs to this space, the function f also belongs to this space. Moreover:

$$\hat{f}(\omega) = \sqrt{\frac{e}{2\pi}} e^{i\omega - \omega^2/2}, \qquad \omega \in \mathbf{R}.$$

(d) In general it is not true that the product of any function in the space $L^1_{\text{PC}}(-\infty, \infty)$ by e^{-x} still belongs to this space. However, in our case the function f indeed belongs to the space. Moreover:

$$\hat{f}(\omega) = \sqrt{\frac{e}{2\pi}} e^{i\omega - \omega^2/2}, \qquad \omega \in \mathbf{R}.$$

(e) None of the above.

Solutions

1. For $\alpha = 1$, the function v_1 belongs to W, so that $w_1^* = v_1$ and $g(1) = 0$. For $\alpha \neq 1$ we clearly have $g(\alpha) > 0$. Now:

$$g(\alpha) = \|v_\alpha - w_\alpha^*\| = \sqrt{\|v_\alpha\|^2 - \|w_\alpha^*\|^2} = \sqrt{\|v_\alpha\|^2 - \frac{|\langle v_\alpha, w\rangle|^2}{\|w\|^2}}.$$

We need to find the following inner products:

$$\|v_\alpha\|^2 = \int_1^\infty \frac{dx}{x^{2\alpha}} = \frac{1}{2\alpha - 1},$$

$$\langle v_\alpha, w\rangle = \int_1^\infty \frac{dx}{x^{\alpha+1}} = \frac{1}{\alpha},$$

$$\|w\|^2 = \int_1^\infty \frac{dx}{x^2} = 1.$$

Hence:

$$g^2(\alpha) = \frac{1}{2\alpha - 1} - \frac{1}{\alpha^2} = \frac{(\alpha - 1)^2}{\alpha^2(2\alpha - 1)}.$$

The numerator decreases in the interval $(\frac{1}{2}, 1)$, while each of the factors in the denominator increases there. Thus, g decreases on $(\frac{1}{2}, 1)$. Since g vanishes at 1 and is positive for $\alpha > 1$, it does not decrease on $[1, \infty)$.

Thus, (d) is true.

2. For $m > n \geq 1$ we have:

$$\|v_m - v_n\| = \left\| \sum_{k=n+1}^{m} \frac{1}{k!} u_k \right\| \leq \sum_{k=n+1}^{m} \frac{\|u_k\|}{k!} \leq \sum_{k=n+1}^{\infty} \frac{1}{k!} u_k.$$

If $\{u_n\}_{n=1}^\infty$ is an orthonormal system, then in particular $\|u_n\| = 1$ for all n, and hence the right-hand side of the last inequality is the tail of a convergent series, and thus converges to 0. The same holds if $\|u_n\| \leq 2^n$ for all n. If $\|u_n\| \geq (n-1)!$, then the sequence is not necessarily a Cauchy sequence, but it may be such. For example, if

$\{u_n\}_{n=1}^{\infty}$ is an orthogonal system and $\|u_n\| = (n-1)!$ for each n, then for $m > n \geq 1$:

$$\|v_m - v_n\| = \sqrt{\sum_{k=n+1}^{m} \frac{1}{k^2}} \xrightarrow[n\to\infty]{} 0.$$

Claim (d) fails because

$$\|v\| = \lim_{n\to\infty} \|v_n\| = \lim_{n\to\infty} \sqrt{\sum_{k=1}^{n} \frac{1}{k!^2}} \leq \lim_{n\to\infty} \sum_{k=1}^{n} \frac{1}{k!^2} < \lim_{n\to\infty} \sum_{k=1}^{n} \frac{1}{k!} = e-1.$$

Thus, (a) is true.

3. If f is such a function, then for all $c \in \mathbf{C}$, the function $\tilde{f}(x) = f(x) + c$ has the same properties. Since the Fourier coefficients of f and $\tilde{f}$ are the same except for c_0, and those differ by c, the coefficient c_0 may be any complex number, whatever all other coefficients may be.

   Thus, (d) is true.

4. The Fourier series of the function

$$f(x) = \begin{cases} 0, & x \neq 0, \\ 1, & x = 0, \end{cases}$$

   is identically 0, and in particular converges uniformly to 0, even though $l = r = 0$ while $m = 1$. Hence (b) is false and (d) is true.

   The function

$$g(x) = \begin{cases} 0, & -\pi < x < 0, \\ 1, & x = 0, \pm\pi, \\ 2, & 0 < x < \pi, \end{cases}$$

   has the property that, at each discontinuity point, its value coincides with the average of the two one-sided limits. Hence its Fourier series converges pointwise to g, so that (a) is also false.

If the Fourier series of f converges uniformly, then the limit is a continuous function. Now, in a sufficiently small neighborhood of 0 (not including 0), in which f is continuous, the Fourier series must converge pointwise to f. If $l \neq r$, then the limit of the Fourier series cannot be continuous, and the convergence cannot be uniform. Hence (c) is false.

Thus, (d) is true.

5. The Taylor series of the function sin converges uniformly to the function on bounded subsets of the plane. It follows that the series

$$\sum_{n=0}^{\infty} \frac{(-1)^n \left(e^{-2ix}\right)^{2n+1}}{(2n+1)!} = \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+1)!} e^{-2(2n+1)ix}$$

converges uniformly to f on $[-\pi, \pi]$. It is therefore its Fourier series, i.e., $c_{-2(2n+1)} = \frac{(-1)^n}{(2n+1)!}$ for all $n \geq 0$, and $c_m = 0$ for all m which is not of that form. In particular, for $-10 = -2(2 \cdot 2 + 1)$, we have

$$c_{-10} = \frac{(-1)^2}{(2 \cdot 2 + 1)!} = \frac{1}{5!}.$$

Thus, (b) is true.

6. The sequence $(f_n(x))_{n=1}^{\infty}$ is identically 1 for $x = 0$, and in particular converges. Let $x \neq 0$, and suppose that the sequence converges at x, say $f_n(x) \underset{n\to\infty}{\longrightarrow} \alpha$ for some α. Since $|f_n(x)| = 1$ for each n, we have $|\alpha| = 1$. Clearly, $f_{n+1}(x) \underset{n\to\infty}{\longrightarrow} \alpha$ as well, while on the other hand

$$f_{n+1}(x) = e^{ix} f_n(x) \underset{n\to\infty}{\longrightarrow} e^{ix}\alpha.$$

Thus, $e^{ix} = 1$, contradicting the fact that $x \neq 0$.

Now:

$$g_n(x) = \frac{1}{n+1}\sum_{k=0}^{n}\left(e^{ix}\right)^n = \begin{cases} \dfrac{1}{n+1}\cdot\dfrac{1-e^{i(n+1)x}}{1-e^{ix}}, & x \neq 0, \\ 1, & x = 0. \end{cases}$$

For each $x \neq 0$, the second factor on the right-hand side is fixed, and therefore $g_n(x) \underset{n\to\infty}{\longrightarrow} 0$. Since the pointwise limit of the sequence (g_n) is therefore discontinuous at $x = 0$, while each member of the sequence is continuous, the convergence cannot possibly be uniform. However, for any fixed $\delta \in (0, \pi)$ and all $x \in [-\pi, -\delta] \cup [\delta, \pi]$ we have

$$\left|1 - e^{ix}\right| = \left|e^{-ix/2} - e^{ix/2}\right| = 2\left|\sin\frac{x}{2}\right| \geq 2\sin\frac{\delta}{2},$$

and therefore

$$|g(x)| \leq \frac{\left|1 - e^{i(n+1)x}\right|}{(n+1)\cdot 2\sin\frac{\delta}{2}} \leq \frac{1}{\sin\frac{\delta}{2}}\cdot\frac{1}{n+1}.$$

It follows that $g_n(x) \underset{n\to\infty}{\longrightarrow} 0$ uniformly on $[-\pi, -\delta] \cup [\delta, \pi]$.

Thus, (c) is true.

7. Since f is piecewise continuously differentiable,

$$\begin{aligned} S_n(-\pi) \underset{n\to\infty}{\longrightarrow} \frac{f(-\pi+) + f(\pi-)}{2} &= \frac{f(-\pi) + f(\pi)}{2} \\ = \frac{(-\pi-1)^2 + (\pi-1)^2}{2} &= \pi^2 + 1. \end{aligned}$$

This is exactly the meaning of (a), and thereby shows that (b) is false. Since $f(-\pi) \neq \pi^2 + 1$, claim (d) is also false.

If the sequence (S_n) converged uniformly in $(-\pi, 0]$, then, since it converges at the point $-\pi$ also, it would converge uniformly on $[-\pi, 0]$. But then the limit would be continuous, which it is not, as it is $(x-1)^2$ for $x \in (-\pi, 0]$ and $(\pi+1)^2$ for $x = -\pi$.

Thus, (a) is true.

8. For $\omega \neq 0$:

$$\begin{aligned}
\widehat{f}(\omega) &= \frac{1}{2\pi}\sum_{k=0}^{\infty}\int_k^{k+1} f(x)e^{-i\omega x}dx = \frac{1}{2\pi}\sum_{k=0}^{\infty}2^{-k}\int_k^{k+1} e^{-i\omega x}dx \\
&= \frac{1}{2\pi}\sum_{k=0}^{\infty}2^{-k}\frac{e^{-ki\omega}-e^{-(k+1)i\omega}}{i\omega} = \frac{1-e^{-i\omega}}{2\pi i\omega}\sum_{k=0}^{\infty}2^{-k}e^{-ki\omega} \\
&= \frac{1-e^{-i\omega}}{2\pi i\omega}\sum_{k=0}^{\infty}\left(\frac{e^{-i\omega}}{2}\right)^k = \frac{1}{2\pi i\omega}\cdot\frac{1-e^{-i\omega}}{1-e^{-i\omega}/2}.
\end{aligned}$$

Since the factor $1 - e^{-i\omega}/2$ in the denominator is bounded away from 0 (namely, its absolute value is at least $1/2$), the transform is bounded above in absolute value by $C/|\omega|$ for some constant C. Hence $\widehat{f} \in L^2_{\mathrm{PC}}(-\infty,\infty)$.

Thus, (a) is true.

9. The function $g(x) = e^{-|x|}$ belongs to $L^1_{\mathrm{PC}}(-\infty,\infty)$, but its product with e^{-x} is identically 1 on the negative half-line, so that it does not belong to $L^1_{\mathrm{PC}}(-\infty,\infty)$. However, in our case

$$f(x) = e^{-x-x^2/2} = e^{1/2}e^{-(x+1)^2/2},$$

so that f is obtained by shifting and multiplying by a constant the function $g(x) = e^{-x^2/2}$. Since g belongs to $L^1_{\mathrm{PC}}(-\infty,\infty)$, so does f. Moreover,

$$\begin{aligned}
\widehat{f}(\omega) &= \mathcal{F}\left\{e^{1/2}e^{-(x+1)^2/2}\right\}(\omega) = e^{1/2}\mathcal{F}\left\{e^{-(x+1)^2/2}\right\}(\omega) = \\
&= e^{1/2}e^{i\omega}\mathcal{F}\left\{e^{-x^2/2}\right\}(\omega) = e^{1/2}e^{i\omega}\tfrac{1}{\sqrt{2\pi}}e^{-\omega^2/2} = \sqrt{\tfrac{e}{2\pi}}e^{i\omega-\omega^2/2}.
\end{aligned}$$

Thus, (d) is true.