Fourier Analysis

Exercises

1 Review Questions in Linear Algebra

1. Determine which of the following subsets of $\mathbf{R}^2$ form linear subspaces:

(a) $\{(x, y) : x = y\}$.

(b) $\{(x, y) : x < y\}$.

(c) $\{(x, y) : x \leq y\}$.

(d) $\{(x, y) : x^2 = y\}$.

(e) $\{(x, y) : x = 1\}$.

(f) $\{(x, y) : x = 0\}$.

(g) $\{(x, y) : 4.7x + 6.8y = 0\}$.

2. Determine which of the following subsets of $\mathbf{C}^3$ form linear subspaces:

(a) $\{(x, y, z) : y = z\}$.

(b) $\{(x, y, z) : x + y = z\}$.

(c) $\{(x, y, z) : |y| = |z|\}$.

(d) $\{(x, y, z) : xy = z\}$.

(e) $\{(x, y, z) : (2 + i)x - (3 + 4i)y - (7 + i)z = 0\}$.

(f) $\{(x, y, z) : (2 + i)x - (3 + 4i)y - (7 + i)z = 1\}$.

3. Let $\mathbf{R}[x]$ be the vector space of all polynomials over $\mathbf{R}$. Determine which of the following subsets of $\mathbf{R}[x]$ form linear subspaces:

(a) All polynomials of degree at most 2.

(b) All polynomials of degree exactly 2.

(c) $\{a_0 + a_1 x + a_2 x^2 + \ldots + a_n x^n : n \in \mathbf{N}, a_{n-i} = a_i \forall 0 \leq i \leq n\}$.

(d) $\{P \in \mathbf{R}[x] : P(3) = 0\}$.

(e) $\{P \in \mathbf{R}[x] : P(2+i) = 0\}$.

(f) $\{P \in \mathbf{R}[x] : P(i) = i\}$.

4. Denote by $C[a,b]$ the vector space of all continuous complex-valued functions on the interval $[a,b]$. Determine which of the following subsets of $C[-1,1]$ form linear subspaces:

(a) $\{f \in C[-1,1] : f(1/2) = i\}$.

(b) $\{f \in C[-1,1] : f(1/e) = 0\}$.

(c) $\{f \in C[-1,1] : f(0) = (f(1) + f(-1))/2\}$.

(d) $\{f \in C[-1,1] : |f(x)| \leq 10, x \in [-1,1]\}$.

(e) $\{f \in C[-1,1] : f(1/4) = f^2(1/2)\}$.

2 Inner Product Spaces

5. Determine which of the following define inner products on $\mathbf{R}^n$. For those that are not, indicate which of the required properties fail to hold:

(a) $\langle \mathbf{x}, \mathbf{y} \rangle = (x_1 + x_2 + \ldots + x_n)(y_1 + y_2 + \ldots + y_n)$ for $\mathbf{x} = (x_1, x_2, \ldots, x_n), \mathbf{y} = (y_1, y_2, \ldots, y_n)$.

(b) $\langle \mathbf{x}, \mathbf{y} \rangle = \sum_{k=1}^{n} k x_k y_k$.

(c) $\langle \mathbf{x}, \mathbf{y} \rangle = \sum_{k=1}^{n} 2^k x_k y_k$.

(d) $\langle \mathbf{x}, \mathbf{y} \rangle = \sum_{k=1}^{n} x_k y_{n+1-k}$.

6. Determine which of the following define inner products on $C[-1,1]$. For those that are not, indicate which of the required properties fail to hold:

(a) $\langle f, g \rangle = f(0) + \bar{g}(0)$ for $f, g \in C[-1,1]$.

(b) $\langle f, g \rangle = f(0)\bar{g}(0)$.

(c) $\langle f, g \rangle = \int_{-1}^{1} f(t)dt + \int_{-1}^{1} \bar{g}(t)dt$.

(d) $\langle f, g\rangle = \int_{-1}^{1} f(t)dt \int_{-1}^{1} \bar{g}(t)dt$.

(e) $\langle f, g\rangle = \int_{-1}^{1} f(t)dt \int_{-1}^{1} \bar{g}(t)dt + f(-1/2)\bar{g}(-1/2)$.

(f) $\langle f, g\rangle = \sum_{k=-d}^{d} f(k/d)\bar{g}(k/d)$, where d is an arbitrary fixed positive integer.

7. Let V be an inner product space and $\|\cdot\|$ be the norm induced by the inner product. Prove the so-called parallelogram identity:

$$\|u+v\|^2 + \|u-v\|^2 = 2(\|u\|^2 + \|v\|^2).$$

3 Normed Spaces

8. Let $V = C[-1, 1]$ be the vector space of all continuous functions from $[-1, 1]$ to $\mathbf{C}$. Determine which of the following functions define norms on V.

(a) $\|f\| = \max\limits_{-1\leq x\leq 0} |f(x)| + \max\limits_{0\leq x\leq 1} |f(x)|$.

(b) $\|f\| = \int_{-1}^{1} |f(x)| \cdot x^2\, dx$.

(c) $\|f\| = \max\limits_{-1\leq x\leq 1} |f^2(x)|$.

9. Let $(V, \|\cdot\|)$ be a normed vector space. Assume that $\lim_{n\to\infty} v_n = v$. Prove that $\lim_{n\to\infty} \|v_n\| = \|v\|$ using only the triangle inequality. (Note that this proves that the norm forms a continuous function from V to $\mathbf{R}$.)

10. Let V be an inner product space. Assume that $\lim_{n\to\infty} u_n = u$ and $\lim_{n\to\infty} v_n = v$. Using the Cauchy-Schwarz inequality, prove that $\lim_{n\to\infty}\langle u_n, v_n\rangle = \langle u, v\rangle$.

11. Let $V = C[0, 1]$ and $h_1, h_2 \in V$ be such that $h_1(x) \geq h_2(x) > 0$ for all x. Define $\|f(x)\|_{1,h_1} = \int_0^1 |f(x)| \cdot h_1(x)dx$ and $\|f(x)\|_{1,h_2} = \int_0^1 |f(x)| \cdot h_2(x)dx$. Suppose that $\lim_{n\to\infty} f_n(x) = f(x)$ in the norm $\|.\|_{1,h_1}$. Prove that $\lim_{n\to\infty} f_n(x) = f(x)$ in the norm $\|.\|_{1,h_2}$.

12. Let $g_n : \mathbf{R} \to \mathbf{R}$ be a sequence of piecewise continuous functions. Assume that $|g_n(x)| \leq M$ for all $n \in \mathbf{N}$ and $x \in \mathbf{R}$, that $\int_{-\infty}^{\infty} |g_n(x)|\, dx < \infty$ and that $g_n \overset{\|\cdot\|_1}{\to} g$. Prove that $g_n \overset{\|\cdot\|_2}{\to} g$.

13.

(a) Employ Exercise 7 to prove that the $\|\cdot\|_1$ norm on $\mathbf{C}^n$ is not induced by any inner product.

(b) Same for $\|\cdot\|_\infty$.

4 Orthogonal Systems

14. Let $v_1, v_2, \ldots, v_n$ be an orthogonal system in an inner product space V. Show that:

$$\left\| \sum_{k=1}^{n} v_k \right\|^2 = \sum_{k=1}^{n} \|v_k\|^2.$$

15. Consider $\mathbf{C}^n$ with the standard inner product. Put $\zeta = e^{2\pi i/n}$. Show that the vectors

$$v_k = (1, \zeta^k, \zeta^{2k}, \ldots, \zeta^{(n-1)k}), \qquad k = 0, 1, \ldots, n-1,$$

form an orthogonal basis of the space.

16. Let V be the space of polynomials over $\mathbf{R}$.

(a) Define $\langle \cdot, \cdot \rangle$ by:

$$\langle P, Q \rangle = \int_0^\infty P(x)Q(x)e^{-x}dx, \qquad P, Q \in V.$$

Show that $\langle \cdot, \cdot \rangle$ forms an inner product on V.

(b) Show that $\{1, x-1, x^2/2 - 2x + 1\}$ is an orthonormal basis of the subspace of V, consisting of all polynomials of degree not exceeding 2.

(c) Prove by induction that:

$$\int_0^\infty x^n e^{-x} dx = n!, \qquad n = 0, 1, 2, \ldots.$$

(d) Employ the previous part to show directly that the Cauchy-Schwarz inequality holds for the vectors x^m and x^n.

17. Consider the space $C^1[-1,1]$ of all continuously differentiable complex-valued functions over $[-1,1]$.

(a) Define $\langle \cdot, \cdot \rangle$ by:

$$\langle f, g \rangle = f(0)\overline{g(0)} + \int_{-1}^{1} f'(x)\overline{g'(x)}dx, \qquad f, g \in C^1[-1, 1].$$

Show that $\langle \cdot, \cdot \rangle$ forms an inner product on $C^1[-1, 1]$.

(b) Characterize the subspace of those functions that are orthogonal to the constants.

18. Consider the space $C[-a, a]$ with the inner product defined by:

$$\langle f, g \rangle = \int_{-a}^{a} f(x)\overline{g(x)}dx, \qquad f, g \in C[-a, a].$$

Suppose we perform the Gram-Schmidt process on the space of polynomials of degree not exceeding n, starting with the vectors $v_1 = 1, v_2 = x, v_3 = x^2, \ldots, v_{n+1} = x^n$. Show that in the basis $\{e_1, e_2, \ldots, e_{n+1}\}$ we obtain, the polynomials $e_1, e_3, e_5, \ldots$ are even, while the others are odd.

19. We run the Gram-Schmidt process twice, once starting from a basis $\{v_1, v_2, \ldots, v_n\}$, and once starting from a basis $\{u_1, u_2, \ldots, u_n\}$, related to the former basis by the equalities

$$u_k = \alpha_{k1}v_1 + \alpha_{k2}v_2 + \ldots + \alpha_{kk}v_k, \qquad 1 \le k \le n,$$

where $(\alpha_{kl})_{k,l=1,1}^{n,k}$ is a triangular array of scalars. What is the relation between the orthonormal bases obtained by the two processes?

5 Best Approximations

20. Let V be a normed space, $v \in V$ and W a linear subspace of V. Prove that the set of all vectors $w^* \in W$, that are closest to v within W, is convex. (Namely, if $w_1^*, w_2^* \in W$ are closest to v within W, then so is the vector $\alpha w_1^* + (1 - \alpha)w_2^*$ for every $\alpha \in [0, 1]$.)

21. Find the nearest point(s) to $v = (-2, 3)$ within the subspace $W = \{(x, -3x) : x \in \mathbf{R}\}$ of $V = \mathbf{R}^2$ in the:

(a) $\|\cdot\|_1$-norm.

(b) $\|\cdot\|_\infty$-norm.

(c) $\|\cdot\|_2$-norm.

22. Find the best approximation(s) for the point $(x_1, x_2, \ldots, x_n) \in \mathbf{R}^n$ within the subspace spanned by the vector $(1, 1, \ldots, 1)$ in the:

(a) $\|\cdot\|_1$-norm.

(b) $\|\cdot\|_\infty$-norm.

(c) $\|\cdot\|_2$-norm.

In each case, explain when we get a unique answer and when multiple answers.

23. In $\mathbf{R}^4$, let S be the span of the vectors $v_1 = (1, 1, -, 1, -1)$ and $v_2 = (1, 1, 1, 1)$.

(a) Find the orthogonal projection of $x = (1, 2, 3, 4)$ to S.

(b) Find the distance from x to the plane S.

24. Find the distances between the following pairs of vector and subspace (in the $\|\cdot\|_2$-norm):

(a) $f(x) = x^3,\ W = \operatorname{span}\{1, x, x^2\} \subseteq C[-1, 1]$.

(b) $f(x) = \cos^2 x,\ W = \operatorname{span}\{1, \cos 2x\} \subseteq C[-\pi, \pi]$.

(c) $f(x) = x,\ W = \operatorname{span}\{1, \cos x, \sin x\} \subseteq C[-\pi, \pi]$.

25. Find the complex numbers α, β, γ minimizing the integral

$$\int_{-1}^{1} |1 + ix - (\alpha \cos \pi x + \beta \sin \pi x + \gamma)|^2 \, dx.$$

26. Let V be the inner product space considered in Exercise 16. Find the polynomial closest to x^n in the subspace of all polynomials of degree not exceeding 2.

27. In l_2, denote:

$$g_\lambda = (\lambda, \lambda^2, \lambda^3, \ldots), \qquad |\lambda| < 1.$$

(a) Find the orthogonal projection of $g_{1/3}$ to $\operatorname{span}\{g_{1/2}\}$.

(b) Find the orthogonal projection of $g_{1/3}$ to $\operatorname{span}\{g_{1/2}, g_{1/4}\}$.

6 Convergence in Normed Spaces

28. Consider the sequence of functions

$$f_n(x) = n\sqrt{|x|}e^{-\frac{n^2x^2}{2}}, \qquad n \geq 1,\ x \in [-1,1]$$

in $C[-1,1]$.

(a) Does f_n converge pointwise to some function $f \in C[-1,1]$?

(b) Does f_n converge uniformly to some function $f \in C[-1,1]$?

(c) Does f_n converge in $\|\cdot\|_2$ to the zero function?

29.

(a) Prove that for every $f \in PC[a,b]$ and $x \in [a,b]$:

$$\left|\int_a^x f(t)\,dt\right| \leq \sqrt{x-a}\,\|f\|_2\,.$$

(Hint: Use the Cauchy-Schwarz inequality.)

(b) Prove that, if $f_n \xrightarrow[n\to\infty]{} f$ in $\|\cdot\|_2$ in $PC[a,b]$, then f_n also converges to f in $\|\cdot\|_1$.

(c) Does convergence in $\|\cdot\|_1$ imply convergence in $\|\cdot\|_2$ in $PC[a,b]$? If yes, prove it. If not, give a counter-example.

30. Let V be the space of continuous functions f from $[0,\infty)$ to $\mathbf{C}$, satisfying

$$\int_0^\infty |f(x)|^2 e^{-x}dx < \infty,$$

with the norm:

$$\|f\| = \sqrt{\int_0^\infty |f(x)|^2 e^{-x}dx}.$$

(a) Prove that, if $(f_n)_{n=1}^\infty$ in V converges uniformly to f, then $f \in V$ and f_n converges to f in norm.

(b) Find a sequence of functions $(f_n)_{n=1}^\infty$ in V such that f_n converges in norm but does not converge uniformly on $[0,\infty)$.

(c) Find a sequence of functions $(f_n)_{n=1}^\infty$ in V such that f_n converges pointwise but not in norm.

31. Let V be an inner-product space, and $\{e_n\}_{n=1}^\infty \subset V$ an orthonormal system.

(a) Is there a $u \in V$ such that $\langle u, e_n \rangle = \frac{1}{\sqrt{n}}$?

(b) Assume $\{e_n\}_{n=1}^{\infty}$ is a closed orthonormal system. Let $u, v \in V$ such that $\langle u, e_n \rangle = \frac{1}{n}$ and $\langle v, e_n \rangle = \frac{1}{n+1}$. Calculate $\langle u, v \rangle$.

(c) Let $u \in V$ be such that $\langle u, e_n \rangle = \frac{1}{\sqrt{n(n+2)}}$. Find the best approximations u_1, u_2, u_3 of u in span $\{e_1\}$, span $\{e_1, e_2\}$, span $\{e_1, e_2, e_3\}$, respectively.

(d) Assume $\{e_n\}_{n=1}^{\infty}$ is a closed orthonormal system. Calculate $\|u - u_1\|$, $\|u - u_2\|$, $\|u - u_3\|$, where u_1, u_2, u_3 are the best approximations from the previous part.

32. Let V be an inner-product space. Prove or disprove the following:

(a) Let $\{e_n\}_{n=1}^{\infty}$ be an orthonormal system. Then for every $u \in V$ we have $\lim_{n\to\infty} \langle u, e_n \rangle = 0$.

(b) Let $\{e_n\}_{n=1}^{\infty}$ be an orthonormal system. Then for every $u \in V$ we have $\lim_{n\to\infty} |\langle u, e_n \rangle|^2 = 0$.

(c) Let $W = \text{span}\{e_1, \ldots, e_n\}$, where $\{e_i\}_{i=1}^{n}$ is an orthogonal system, $u \in V$ and $\tilde{u} = \sum_{i=1}^{n} \frac{\langle u, e_i \rangle}{\|e_i\|} e_i$. Then for every $w \in W$ we have $\|u - \tilde{u}\| \le \|u - w\|$.

(d) Let $W = \text{span}\{e_1, \ldots, e_n\}$, where $\{e_i\}_{i=1}^{n}$ is an orthonormal system, $u \in V$ and $\tilde{u} = \sum_{i=1}^{n} \frac{\langle u, e_i \rangle}{\|e_i\|} e_i$. Then for every $w \in W$ we have $\|u - \tilde{u}\| \le \|u - w\|$.

(e) Let $W = \text{span}\{e_1, \ldots, e_n\}$, where $\{e_i\}_{i=1}^{n}$ is an orthonormal system, $u \in V$ and $\tilde{u} = \sum_{i=1}^{n} \langle u, e_i \rangle e_i$. Then for every $w \in W$ except for $\tilde{u}$ we have $\|u - \tilde{u}\| < \|u - w\|$.

(f) Let $W = \text{span}\{e_1, \ldots, e_n\}$, where $\{e_i\}_{i=1}^{n}$ is any set of vectors. Let $u \in V$ and let $\tilde{u} \in W$ be such that for every $w \in W$ we have $\|u - \tilde{u}\| \le \|u - w\|$. Then $\langle u - \tilde{u}, w \rangle = 0$ for all $w \in W$.

(g) Let $W = \text{span}\{e_1, \ldots, e_n\}$, where $\{e_i\}_{i=1}^{n}$ is an orthonormal system, $u \in V$ and let $\tilde{u} \in W$ be such that for every $w \in W$ we have $\|u - \tilde{u}\| \le \|u - w\|$. Then $\tilde{u} = \sum_{i=1}^{n} \langle u, e_i \rangle e_i$.

33. Let V be the space of piecewise continuous functions f from $[0, \infty)$ to $\mathbf{R}$ (functions with at most a finite number of discontinuities, all of which are either removable or of type I) such that $\int_0^{\infty} |f(x)|^2 \, dx < \infty$. Find a sequence in V that converges uniformly to 0 but does not converge in the $\|\cdot\|_2$ norm.

34. Let $\{e_n\}_{n=1}^{\infty} \subset L^2_{PC}[0,1]$ be a closed orthonormal system. Prove that:

$$\sum_{n=1}^{\infty} \left| \int_0^a e_n(x)\,dx \right|^2 = a, \quad a \in [0,1].$$

35. Let $V = L^2_{PC}[0,1]$. Find a sequence $(f_n)_{n=1}^{\infty}$ in V such that f_n converges in norm to zero, but does not converge pointwise for any $x \in [0,1]$.

7 Fourier Series

36. Find the real Fourier series for each of the following functions:

(a) $f(x) = 9\cos x + 7\sin 2x + 11\cos 3x,\ x \in [-\pi,\pi]$.

(b) $f(x) = \begin{cases} \sin x, & 0 < x \le \pi, \\ \cos x, & -\pi \le x \le 0. \end{cases}$

37. Find the real Fourier series of the function $f(x) = |x|^3,\ x \in [-\pi,\pi]$.

38. Find the complex Fourier series for each of the following functions:

(a) $f(x) = \sin \frac{x}{2},\ x \in [-\pi,\pi]$.

(b) $f(x) = \pi - x^2,\ x \in [-\pi,\pi]$.

39. Find the complex Fourier series of the function $f(x) = e^{i|x|}$, $x \in [-\pi,\pi]$.

40. Let $f, g : \mathbf{R} \to \mathbf{C}$ be piecewise continuous functions with a period of 2π with Fourier series:

$$f(x) = \sum_{n=-\infty}^{\infty} \gamma_n e^{inx}, \qquad g(x) = \sum_{n=-\infty}^{\infty} \gamma'_n e^{inx}.$$

Find the complex Fourier series of $h(x) = \frac{1}{2\pi}\int_{-\pi}^{\pi} f(x-t)\,g(t)\,dt$.

41.

(a) Prove that for $0 < r < 1$,

$$\sum_{n=-\infty}^{\infty} r^{|n|} e^{inx} = \frac{1-r^2}{1-2r\cos x + r^2}.$$

(b) Denote $P_r(x) = \frac{1-r^2}{1-2r\cos x+r^2}$. ($P_r$ is the *Poisson kernel*.) Let $f(x)$ be a piecewise continuous function from $[-\pi,\pi]$ to $\mathbf{C}$ with a Fourier series $f(x) = \sum_{n=-\infty}^{\infty} c_n e^{inx}$. Prove that

$$\frac{1}{2\pi}\int_{-\pi}^{\pi} f(x-t)\,P_r(t)\,dt = \sum_{n=-\infty}^{\infty} c_n r^{|n|} e^{inx},$$

and that the series converges absolutely and uniformly for x.

(c) Prove the following properties of the Poisson kernel:

(i) $P_r(x) \geq 0$ for every $x \in [-\pi,\pi]$ and $r \in (0,1)$.

(ii) $P_r(x) \xrightarrow[r\to 1^-]{} 0$ uniformly for x on $[-\pi,-\delta]\cup[\delta,\pi]$ for any $\delta > 0$.

(iii) $\frac{1}{2\pi}\int_{-\pi}^{\pi} P_r(x)\,dx = 1$ for every $r \in (0,1)$.

(d) Let f be a continuous function $f : [-\pi,\pi] \to \mathbf{C}$ with $f(-\pi) = f(\pi)$ and Fourier series $f(x) = \sum_{n=-\infty}^{\infty} c_n e^{inx}$. Prove that

$$\lim_{r\to 1^-} \sum_{n=-\infty}^{\infty} c_n r^{|n|} e^{inx} = f(x)$$

uniformly over x.

Remark: this exercise represents an alternative way to Fejer's theorem to "deal" with the fact that a Fourier series of a continuous function does not necessarily converge. In Fejer's theorem, we improve the convergence by looking at arithmetic averages instead of partial sums. Here we improve the convergence by multiplying the series' elements by $r^{|n|}$.

42. Let $f : [-\pi,\pi] \to \mathbf{C}$ be continuously differentiable $k-1$ times with $f^{(j)}(-\pi) = f^{(j)}(\pi)$, $j = 0,\ldots,k-1$, and piecewise continuously differentiable k times. Let c_n be the Fourier coefficients of $f(x)$. Prove that $\lim_{n\to\infty} n^k c_n = 0$.

43.

(a) Find the real Fourier series of $f(x) = \sin\frac{px}{2}$ $p \neq 0$ $x \in [-\pi,\pi]$, where $p \in \mathbf{R}\setminus\{0\}$.

(b) Using Parseval's identity, prove that:

$$\sum_{n=1}^{\infty} \frac{n^2}{\left(1-4n^2\right)^2} = \frac{\pi^2}{64}.$$

44.

(a) Let $h \in (-\pi, \pi) \setminus \{0\}$. Find the Fourier series of:

$$f(x) = \begin{cases} 0, & -\pi \leq x < h, \\ h^2, & h \leq x \leq \pi. \end{cases}$$

(b) Write Parseval's identity for the series and calculate $\sum_{n=1}^{\infty} \frac{(1-(-1)^n \cos(2n))}{n^2}$.

45. Let $\sum_{n=-\infty}^{\infty} c_n e^{inx}$ be the Fourier series of $f(x)$. Find the Fourier series of the following functions:

(a) $g(x) = f(x + a)$.

(b) $h(x) = e^{imx} f(x)$, where m is an integer.

46. Let $(a_n)_{n=-\infty}^{\infty}$ and $(b_n)_{n=-\infty}^{\infty}$ be sequences of complex numbers such that $\sum_{n=-\infty}^{\infty} |a_n| < \infty$ and $\sum_{n-\infty}^{\infty} |b_n| < \infty$. Let $f(x) = \sum_{n=-\infty}^{\infty} a_n e^{inx}$ and $g(x) = \sum_{n=-\infty}^{\infty} b_n e^{inx}$.

(a) Show that the series $\sum_{n=-\infty}^{\infty} a_{m-n} b_n$ converges for every $m \in \mathbf{Z}$ and that

$$\sum_{n=-\infty}^{\infty} |c_n| \leq \sum_{n=-\infty}^{\infty} |a_n| \cdot \sum_{n=-\infty}^{\infty} |b_n|$$

where $c_m = \sum_{n=-\infty}^{\infty} a_{m-n} b_n$.

(b) Let $h(x) = \sum_{n=-\infty}^{\infty} c_n e^{inx}$. Show that $h(x) = f(x) g(x)$.

(c) Show that:

$$\|h\|_{\infty} \leq \sum_{n=-\infty}^{\infty} |a_n| \cdot \sum_{n=-\infty}^{\infty} |b_n|$$

47. Use the real Fourier series of the function $f(x) = \cos ax$ on the interval $[-\pi, \pi]$, where a is not an integer, to show that:

(a)

$$\frac{1}{\sin a\pi} = \frac{1}{a\pi} + \sum_{n=1}^{\infty} (-1)^n \left(\frac{1}{a\pi + n\pi} + \frac{1}{a\pi - n\pi} \right)$$

(b)

$$\cot a\pi = \frac{1}{a\pi} + \sum_{n=1}^{\infty} \left(\frac{1}{a\pi + n\pi} + \frac{1}{a\pi - n\pi} \right)$$

48. Let $\sum_{n=-\infty}^{\infty} c_n$ be an absolutely convergent series of complex numbers.

(a) Prove that the series $\sum_{n=-\infty}^{\infty} c_n e^{-inx}$ converges uniformly on $[-\pi, \pi]$ to a continuous function $f(x)$ (with equal values at $-\pi$ and π).

(b) Show that f is not necessarily piecewise continuously differentiable.

49. Consider the function f defined by

$$f(x) = e^{e^{ix}}, \qquad -\pi \leq x \leq \pi.$$

(a) Find the Fourier series of f.

(b) Show that f is a solution of the differential equation:

$$y' = ie^{ix}y.$$

(c) Show directly that the series obtained by term-by-term differentiation of the series you have found in part (a) satisfies the same differential equation.

50. Prove that the following two normed spaces are not complete:

(a) $L^2_{\mathrm{PC}}[-1, 1]$.

(b) The space of all bounded complex-valued continuous functions on $[0, \infty)$, equipped with the norm:

$$\|f\| = \int_0^\infty |f(x)| \cdot e^{-x} dx.$$

51. Prove that the normed space ℓ_∞, consisting of all infinite sequences $\mathbf{x} = (x_1, x_2, \ldots)$ of complex numbers which are bounded (i.e., satisfy the condition $\sup_{n\geq 1} |x_n| < \infty$), with the norm

$$\|\mathbf{x}\|_\infty = \sup_{n\geq 1} |x_n| < \infty,$$

is complete.

8 The Fourier Transform

52. Compute the Fourier transform of the following functions:

(a) $f(x) = x^4 e^{-|x|}$.

(b) $f(x) = \begin{cases} 3xe^{-x}, & x \geq 0, \\ 0, & x < 0. \end{cases}$

(c) $f(x) = 8x^3 e^{\frac{-4(x+1)^2+5}{3}}$.

(d) $f(x) = \sin 2x \cdot e^{-x^2}$.

(e) $f(x) = \begin{cases} 1 - |x|, & -1 \leq x \leq 1, \\ 0, & \text{otherwise.} \end{cases}$

53. Consider the function f defined by:

$$f(x) = e^{-x^2} \int_0^x e^{t^2} dt, \qquad -\infty < x < \infty.$$

(a) Show that there exists a constant $C > 0$ such that $f(x) > C/x$ for all sufficiently large x. (Hint: Estimate the integrand from below in the interval $[x - 1/x, x]$.)

(b) Conclude that $f \notin L^1_{\text{PC}}(-\infty, \infty)$.

54. Let V be the vector space of all complex-valued continuous functions f on $\mathbf{R}$, satisfying $\int_{-\infty}^{\infty} |f(x)|^2 dx < \infty$, with the inner product $\langle \cdot, \cdot \rangle$ given by

$$\langle f, g \rangle = \int_{-\infty}^{\infty} f(x)\overline{g(x)} dx$$

and the norm $\| \cdot \|_2$ induced by it. Let v and w be the functions $v : x \mapsto e^{-x^2}$ and $w : x \mapsto e^{-2x^2}$. Find the orthogonal projection of v in the subspace spanned by w.

55. Consider the differential equation:

$$y'' + xy' + y = 0.$$

(a) Employing the Fourier transform, find the absolutely integrable and twice continuously differentiable solution of the equation, satisfying the initial conditions:

$$y(0) = 1, \qquad y'(0) = 0.$$

(b) Without employing the Fourier transform, find the general solution of the equation.

56.

(a) Compute the Fourier transform of the function f defined by:

$$f(x) = \begin{cases} 1 - x^2, & |x| \leq 1, \\ 0, & |x| > 1. \end{cases}$$

(b) Employ the Fourier inversion formula to calculate the integrals:

(i)

$$\int_0^\infty \frac{\sin x - x\cos x}{x^3} \cos x dx.$$

(ii)

$$\int_0^\infty \frac{\sin x - x\cos x}{x^3} \cos \frac{x}{2} dx.$$

57.

(a) Compute the Fourier transform of the function f defined by

$$f(x) = \begin{cases} -1, & -a \leq x \leq 0, \\ 1, & 0 < x \leq a, \\ 0, & \text{otherwise}, \end{cases}$$

where $a > 0$ is a constant.

(b) Calculate the integral

$$\int_0^\infty \frac{\cos ax - 1}{x} \sin bx dx$$

for any constants $a, b > 0$.

58. Utilize Plancherel's theorem to calculate the integral

$$\int_{-\infty}^\infty \frac{x^2 dx}{(a^2 + x^2)(b^2 + x^2)}.$$