Midterm

Mark the correct answer in each part of the following questions.

1. Let $V = C[5,6]$ be the vector space of all continuous complex-valued functions from $[5,6]$ to $\mathbf{C}$, with the inner product $\langle \cdot, \cdot \rangle$ given by

$$\langle f, g \rangle = \int_5^6 f(x)\overline{g(x)}dx$$

and the norm $\| \cdot \|_2$ induced by this inner product. Consider the following two claims:
I. When applying the Cauchy-Schwarz inequality to the two vectors f_1 and g_1, defined by $f_1(x) = x^{10}$ and $g_1(x) = x^{11}$ for $x \in [5,6]$, we actually get an equality.
II. When applying the Cauchy-Schwarz inequality to the two vectors $f_2(x) = \log(x^{10})$ and $g_2(x) = \log(x^{11})$ for $x \in [5,6]$, we actually get an equality.

   (a) Both claims are true.

   (b) Claim I is true, while claim II is false.

   (c) Claim I is false, while claim II is true.

   (d) Both claims are false.

2. Let $v = (1, -1/2, 1/3, -1/4, \ldots) \in \ell_2$ and let $W = \text{span}\{(1, 1/2, 1/3, 1/4, \ldots)\} \subset \ell_2$. Let $w^* \in W$ be the best approximation of v in W. Then $\|w^* - v\| =$

   (a) 0.

   (b) $\pi/4$.

(c) $\pi/\sqrt{8}$.

(d) $\pi/\sqrt{6}$.

(e) None of the above.

3. Consider the spaces ℓ_1, ℓ_2, and ℓ_∞ of all sequences $\mathbf{x} = (x_1, x_2, \ldots)$ of complex numbers, satisfying the condition $\sum_{n=1}^{\infty} |x_n| < \infty$, the condition $\sum_{n=1}^{\infty} |x_n|^2 < \infty$, and the condition that $\sup_{1 \leq n < \infty} |x_n| < \infty$, respectively. The norms on these spaces are given by:

$$\|\mathbf{x}\|_1 = \sum_{n=1}^{\infty} |x_n|, \qquad \mathbf{x} \in \ell_1,$$

$$\|\mathbf{x}\|_2 = \sqrt{\sum_{n=1}^{\infty} |x_n|^2}, \qquad \mathbf{x} \in \ell_2,$$

$$\|\mathbf{x}\|_\infty = \sup_{1 \leq n < \infty} |x_n|, \qquad \mathbf{x} \in \ell_\infty.$$

Now define a sequence $\left(\mathbf{v}^{(n)}\right)_{n=1}^{\infty}$ as follows. For each n, the first n terms are $1/n$ each, and all other terms are 0. (For example, $\mathbf{v}^{(5)} = (1/5, 1/5, 1/5, 1/5, 1/5, 0, 0, \ldots)$.) Note that the sequence $\left(\mathbf{v}^{(n)}\right)_{n=1}^{\infty}$ resides in all three spaces ℓ_1, ℓ_2, and ℓ_∞.

(a) The sequence converges in ℓ_2, but not in ℓ_1 and ℓ_∞.

(b) The sequence converges in ℓ_1, but not in ℓ_2 and ℓ_∞.

(c) The sequence converges in ℓ_∞, but not in ℓ_1 and ℓ_2.

(d) The sequence converges in ℓ_2 and ℓ_∞, but not in ℓ_1.

(e) None of the above.

4. Let V be an inner product space, $\{e_n\}_{n=1}^{\infty}$ an orthonormal system in V and $v \in V$. It is known that for each positive integer n there exist n complex numbers $\alpha_{n1}, \alpha_{n2}, \ldots, \alpha_{nn}$ such that

$$\|v - (\alpha_{n1}e_1 + \alpha_{n2}e_2 + \ldots + \alpha_{nn}e_n)\| \leq 1/\sqrt{n}.$$

(a) We necessarily have $v \in \operatorname{span}\{e_n\}_{n=1}^{\infty}$.

(b) We necessarily have $v \in \overline{\operatorname{span}\{e_n\}_{n=1}^{\infty}}$, but not necessarily $v \in \operatorname{span}\{e_n\}_{n=1}^{\infty}$.

(c) There exists a sequence of complex numbers $(\alpha_n)_{n=1}^{\infty}$ such that

$$\|v - (\alpha_1 e_1 + \alpha_2 e_2 + \ldots + \alpha_n e_n)\| \underset{n\to\infty}{\longrightarrow} 0,$$

but there may not exist a sequence of complex numbers $(\alpha_n)_{n=1}^{\infty}$ such that

$$\|v - (\alpha_1 e_1 + \alpha_2 e_2 + \ldots + \alpha_n e_n)\| \leq 1/\sqrt{n}$$

for each n.

(d) There exists a sequence of complex numbers $(\alpha_n)_{n=1}^{\infty}$ such that

$$\|v - (\alpha_1 e_1 + \alpha_2 e_2 + \ldots + \alpha_n e_n)\| \leq 1/n$$

for each n.

(e) None of the above.

5. The two vectors $\cos 2x$ and $\sin 3x$ have been removed from the closed orthonormal system $\{1/\sqrt{2}, \cos x, \sin x, \cos 2x, \sin 2x, \ldots\}$. Instead of them, the two vectors $v_1 = \alpha_1 \cos 2x + \beta_1 \sin 3x$ and $v_2 = \alpha_2 \cos 2x + \beta_2 \sin 3x$, where $\alpha_1, \beta_1, \alpha_2, \beta_2$ are some constants, have been added to the system.

(a) If $\alpha_1 = \frac{1+i}{2},\ \beta_1 = \frac{1-i}{2},\ \alpha_2 = 0.6,\ \beta_2 = 0.8$, then the new system is also orthonormal and closed.

(b) If $\alpha_1 = \frac{5i}{13},\ \beta_1 = \frac{12}{13},\ \alpha_2 = \frac{12}{13},\ \beta_2 = \frac{-5i}{13}$, then the new system is also orthonormal and closed.

(c) If $\alpha_1 = \frac{15}{17},\ \beta_1 = \frac{4(1+i)\sqrt{2}}{17},\ \alpha_2 = \frac{4(-1+i)\sqrt{2}}{17},\ \beta_2 = \frac{15}{17}$, then the new system is also orthonormal and closed.

(d) If $\alpha_1 = 0.3,\ \beta_1 = 0.7,\ \alpha_2 = 0.4,\ \beta_2 = 0.6$, then the new system is also orthonormal and closed.

(e) None of the above.

6. A function $f \in L^2_{\mathrm{PC}}[-\pi, \pi]$ is given. The series

$$\frac{a_0}{2} + \sum_{n=1}^{\infty} (a_n \cos nx + b_n \sin nx)$$

is the Fourier series of f if and only if:

(a) For every $\varepsilon > 0$ there exists an N such that for every $n > N$ and $x \in [-\pi, \pi]$ we have

$$\left| f(x) - \left(\frac{a_0}{2} + \sum_{k=1}^{n} (a_k \cos kx + b_k \sin kx) \right) \right| < \varepsilon.$$

(b) For every $x \in [-\pi, \pi]$ and $\varepsilon > 0$ there exists an N such that for every $n > N$ we have

$$\left| f(x) - \left(\frac{a_0}{2} + \sum_{k=1}^{n} (a_k \cos kx + b_k \sin kx) \right) \right| < \varepsilon.$$

(c) For every $x \in [-\pi, \pi]$ and $\varepsilon > 0$ there exists an N such that for every $n > N$ there exists some $\delta > 0$ such that

$$\int_{x-\delta}^{x+\delta} \left| f(t) - \left(\frac{a_0}{2} + \sum_{k=1}^{n} (a_k \cos kt + b_k \sin kt) \right) \right|^2 dt < \varepsilon.$$

(d) For every $\varepsilon > 0$ there exists an N such that for every $n > N$ we have

$$\int_{-\pi}^{\pi} \left| f(t) - \left(\frac{a_0}{2} + \sum_{k=1}^{n} (a_k \cos kt + b_k \sin kt) \right) \right|^2 dt < \varepsilon.$$

(e) None of the above.

7. We expand the function $f(x) = (1+i)x + \cos 2x$ into a complex Fourier series

$$\sum_{n=-\infty}^{\infty} c_n e^{inx}.$$

Then:

(a) $c_2 = i$ and $c_{-2} = -i$.

(b) $c_2 = -i$ and $c_{-2} = i$.

(c) $c_2 = i/2$ and $c_{-2} = 1 - i/2$.

(d) $c_2 = -i/2$ and $c_{-2} = 1 + i/2$.

(e) None of the above.

Solutions

1. The Cauchy-Schwarz inequality becomes an equality if and only if the two vectors are linearly dependent. Here, note that

$$f_2(x) = \log\left(x^{10}\right) = 10\log x = \frac{10}{11}\cdot 11\log x = \frac{10}{11}\log\left(x^{11}\right) = \frac{10}{11}g_2(x),$$

but f_1, g_1 are clearly linearly independent since any non-trivial linear combination of the two functions is a non-zero polynomial function, and thus vanishes for at most a finite number of points.

Thus, (c) is true.

2. Denote $w = \left(1, \frac{1}{2}, \frac{1}{3}, \frac{1}{4}, \ldots\right)$. Since

$$w^* = \frac{\langle v, w\rangle}{\langle w, w\rangle} w,$$

we have

$$\|w^*\|_2 = \frac{|\langle v, w\rangle|}{\|w\|_2}.$$

Thus, it suffices to find:

$$\begin{aligned}
\langle v, v\rangle &= \sum_{n=1}^{\infty}\left|\frac{(-1)^{n-1}}{n}\right|^2 = \sum_{n=1}^{\infty}\frac{1}{n^2} = \frac{\pi^2}{6},\\
\langle w, w\rangle &= \sum_{n=1}^{\infty}\left|\frac{1}{n}\right|^2 = \frac{\pi^2}{6},\\
\langle v, w\rangle &= \sum_{n=1}^{\infty}\frac{(-1)^{n-1}}{n^2} = \sum_{1\le n \text{ odd}}\frac{1}{n^2} - \sum_{1\le n \text{ even}}\frac{1}{n^2}\\
&= \sum_{n=1}^{\infty}\frac{1}{n^2} - 2\sum_{1\le n \text{ even}}\frac{1}{n^2} = \sum_{n=1}^{\infty}\frac{1}{n^2} - 2\sum_{n=1}^{\infty}\frac{1}{(2n)^2}\\
&= \left(1-\frac{2}{4}\right)\sum_{n=1}^{\infty}\frac{1}{n^2} = \frac{1}{2}\cdot\frac{\pi^2}{6} = \frac{\pi^2}{12}.
\end{aligned}$$

Hence:

$$\|v-w^*\|_2 = \sqrt{\|v\|_2^2 - \|w^*\|_2^2} = \sqrt{\frac{\pi^2}{6} - \frac{\pi^4/144}{\pi^2/6}} = \pi\sqrt{\frac{1}{6}-\frac{1}{24}} = \frac{\pi}{\sqrt{8}}.$$

Thus, (c) is true.

3. We begin by noting that, since each coordinate of $\mathbf{v}^{(n)}$ is either 0 or $1/n$, the sequence converges coordinate-wise to the $\mathbf{0}$ vector. Now:

$$\begin{aligned}
\left\|\mathbf{v}^{(n)}\right\|_1 &= \sum_{k=1}^{n}\left|\frac{1}{n}\right| = 1,\\
\left\|\mathbf{v}^{(n)}\right\|_2 &= \sqrt{\sum_{k=1}^{n}\left|\frac{1}{n}\right|^2} = \frac{1}{\sqrt{n}} \xrightarrow[n\to\infty]{} 0,\\
\left\|\mathbf{v}^{(n)}\right\|_\infty &= \frac{1}{n} \xrightarrow[n\to\infty]{} 0.
\end{aligned}$$

Hence the sequence converges to $\mathbf{0}$ in ℓ_2 and ℓ_∞, but not in ℓ_1.

Thus, (d) is true.

4. For all n, the vector $\langle v, e_1 \rangle e_1 + \langle v, e_2 \rangle e_2 + \ldots + \langle v, e_n \rangle e_n$ is the best approximation to v in span $\{e_k\}_{k=1}^n$. Hence the given inequality implies that

$$\|v - (\langle v, e_1 \rangle e_1 + \langle v, e_2 \rangle e_2 + \ldots + \langle v, e_n \rangle e_n)\|_2 \leq \frac{1}{\sqrt{n}}.$$

It also implies that v can be approximated to within any desired accuracy in span $\{e_n\}_{n=1}^\infty$, so that

$$v \in \overline{\text{span}\,\{e_n\}_{n=1}^\infty}.$$

To show that (d) is false, we construct a counter-example by requiring that

$$\|v - (\langle v, e_1 \rangle e_1 + \langle v, e_2 \rangle e_2 + \ldots + \langle v, e_n \rangle e_n)\|_2 = \frac{1}{\sqrt{n}},$$

implying that

$$\frac{1}{n} = \left\| \sum_{k=n+1}^{\infty} \langle v, e_k \rangle e_k \right\|_2^2 = |\langle v, e_{n+1} \rangle|^2 + \left\| \sum_{k=n+2}^{\infty} \langle v, e_k \rangle e_k \right\|_2^2 = |\langle v, e_{n+1} \rangle|^2 + \frac{1}{n+1}.$$

This motivation leads us to require that

$$\langle v, e_{n+1} \rangle = \sqrt{\frac{1}{n} - \frac{1}{n+1}} = \sqrt{\frac{1}{n(n+1)}},$$

or

$$v = \sum_{n=2}^{\infty} \frac{1}{\sqrt{(n-1)n}} e_n.$$

To be specific, this happens when choosing

$$v = \left(0, \frac{1}{\sqrt{1 \cdot 2}}, \frac{1}{\sqrt{2 \cdot 3}}, \frac{1}{\sqrt{3 \cdot 4}}, \ldots\right) \in \ell_2$$

and $\{e_n\}_{n=1}^\infty$ as the “standard basis” vectors.

Thus, (b) is true.

5. One readily verifies that v_1, v_2 are orthogonal to every element of the given orthonormal sequence (other than $\cos 2x$, $\sin 3x$). For v_1, v_2 to be orthogonal we need

$$\langle v_1, v_2 \rangle = \alpha_1 \overline{\alpha_2} + \beta_1 \overline{\beta_2} = 0,$$

and to be unit vectors we need

$$\langle v_1, v_1 \rangle = |\alpha_1|^2 + |\beta_1|^2 = 1$$

and

$$\langle v_2, v_2 \rangle = |\alpha_2|^2 + |\beta_2|^2 = 1.$$

Substituting the values of $\alpha_1, \beta_1, \alpha_2, \beta_2$ suggested in the various options, we find that only those of (c) satisfy all requirements.

Since the original system is closed, if the vectors v_1, v_2 are independent, which certainly is the case when the new system is orthonormal, the new system is closed as well.

Thus, (c) is true.

6. The series

$$\frac{a_0}{2} + \sum_{n=1}^{\infty} (a_n \cos nx + b_n \sin nx)$$

is the Fourier series of f if and only if it converges in $\|\cdot\|_2$ to f. This is exactly the condition in (d). The condition in (a) amounts to saying that the Fourier series converges pointwise to the function, which, as we know, is not necessary. The condition in (b) amounts to saying that the Fourier series converges uniformly to the function, which is therefore also not necessarily the case. The condition in (c) requires very little, because the fact that we integrate on an arbitrarily short interval lets the approximation hold even if the function and the Fourier series differ by much. For example, if f is any function in $L^2_{\mathrm{PC}}[-\pi, \pi]$, then the identically 0 Fourier series satisfies the condition, as can be seen by choosing $\delta < \frac{\varepsilon}{2\|f\|_\infty^2}$.

Thus, (d) is true.

7. Recall that the Fourier series of the function $g(x) = x$ is

$$\sum_{n=1}^{\infty} \frac{2 \cdot (-1)^{n+1}}{n} \sin nx.$$

Since $f(x) = (1+i)g(x) + \cos 2x$, the relevant Fourier coefficients of f are:

$$a_2 = 1, \qquad b_2 = -(1+i).$$

Hence

$$c_2 = \frac{a_2 - ib_2}{2} = \frac{1 + i(1+i)}{2} = \frac{i}{2},$$

and

$$c_{-2} = \frac{a_2 + ib_2}{2} = \frac{1 - i(1+i)}{2} = 1 - \frac{i}{2}.$$

Thus, (c) is true.