Review Questions

Mark the correct answer in each part of the following questions.

1. Consider the space $V = C[0,1]$ with the inner product $\langle \cdot\,, \cdot \rangle$ given by

$$\langle f, g \rangle = \int_0^1 f(x)\overline{g(x)}dx.$$

Let f be the vector defined by $f(x) = \frac{1}{2+x}$ for $x \in [0,1]$ and $W = \text{span}\{g\}$, where g is the vector defined by $g(x) = \frac{1}{1+x}$ for $x \in [0,1]$. Let d be the distance between the vector f and the vector $w^* \in W$ which is the closest to it within W. Then $d =$

(a) $\sqrt{\log 2}$.

(b) $\sqrt{1 - \log(e-1)}$.

(c) $\sqrt{\frac{1}{6} - 2\log^2 \frac{4}{3}}$.

(d) $\sqrt{\frac{1}{3} - \sin^2 \frac{\pi}{12}}$.

(e) None of the above.

2. A function $f \in L^2_{\text{PC}}[-\pi, \pi]$ is given. It is known that $\|f\| = 3$. Let

$$\sum_{n=-\infty}^{\infty} c_n e^{inx}$$

be the Fourier series of f. For $n = 1, 2, 3, \ldots$, consider the following functions on $[-\pi, \pi]$:

$$g_n(x) = \sum_{k=0}^{n} c_k e^{ikx}, \qquad h_n(x) = \sum_{k=1}^{n} c_{-k} e^{-ikx}.$$

Denote:

$$r_n = \|g_n\| \cdot \|h_n\|, \qquad n = 1, 2, 3, \ldots.$$

(a) $\lim_{n\to\infty} r_n = 0$.

(b) $\lim_{n\to\infty} r_n \le 3$. Moreover, the bound 3 is the best possible, namely for every $\delta > 0$ we can find an f as above with $\lim_{n\to\infty} r_n > 3 - \delta$.

(c) $\lim_{n\to\infty} r_n \le 9/2$. Moreover, the bound 9/2 is the best possible, namely for every $\delta > 0$ we can find an f as above with $\lim_{n\to\infty} r_n > 9/2 - \delta$.

(d) $\lim_{n\to\infty} r_n \le 9$. Moreover, the bound 9 is the best possible, namely for every $\delta > 0$ we can find an f as above with $\lim_{n\to\infty} r_n > 9 - \delta$.

(e) None of the above.

3. Let V be an inner product space, $\{e_n\}_{n=1}^\infty$ an orthonormal system in V and $v \in V$. It is known that for two sequences $(\alpha_n)_{n=1}^\infty$ and $(\beta_n)_{n=1}^\infty$ of complex numbers we have:

$$v = \sum_{n=1}^{\infty} \alpha_n e_n = \sum_{n=1}^{\infty} \beta_n e_n.$$

(a) We necessarily have $\sum_{n=1}^\infty |\alpha_n|^2 = \sum_{n=1}^\infty |\beta_n|^2$, but not necessarily $\sum_{n=1}^\infty \alpha_n^2 = \sum_{n=1}^\infty \beta_n^2$.

(b) We necessarily have $\sum_{n=1}^\infty \alpha_n^2 = \sum_{n=1}^\infty \beta_n^2$, but not necessarily $|\alpha_n| = |\beta_n|$ for each n.

(c) We necessarily have $|\alpha_n| = |\beta_n|$ for each n, but not necessarily $\alpha_n = \beta_n$ for each n.

(d) We necessarily have $\alpha_n = \beta_n$ for each n.

(e) None of the above.

4. Consider the sequence $(f_n)_{n=1}^\infty$ in $C[-1, 1]$, given by:

$$f_n(x) = \begin{cases} n^{2/3}\left(1 - n^2x^2\right), & |x| \le 1/n, \\ 0, & \text{otherwise.} \end{cases}$$

(a) The sequence converges to 0 in $\|\cdot\|_\infty$.

(b) The sequence converges to 0 pointwise, in $\|\cdot\|_1$, and in $\|\cdot\|_2$, but not in $\|\cdot\|_\infty$.

(c) The sequence converges to 0 in $\|\cdot\|_1$ and in $\|\cdot\|_2$, but neither pointwise nor in $\|\cdot\|_\infty$.

(d) The sequence converges to 0 in $\|\cdot\|_1$, but neither pointwise nor in $\|\cdot\|_2$.

(e) None of the above.

5. We expand the function $f(x) = \ln(x^2+7)$ into a real Fourier series

$$\frac{a_0}{2} + \sum_{n=1}^{\infty} (a_n \cos nx + b_n \sin nx)\,.$$

Then $b_3 - b_1 - b_2 =$

(a) 0.

(b) $-\pi \cdot \ln \frac{11}{2}$.

(c) $\frac{\pi}{2} \cdot \ln \frac{11}{2}$.

(d) $\pi \cdot \ln \frac{11}{2}$.

(e) None of the above.

6. We expand the function $f(x) = \frac{1}{1-ie^{ix}/2}$ into a Fourier series

$$\frac{a_0}{2} + \sum_{n=1}^{\infty} (a_n \cos nx + b_n \sin nx)\,.$$

Then $a_{90} =$

(a) $1/2^{90}$.

(b) $-1/2^{90}$.

(c) $i/2^{90}$.

(d) $-i/2^{90}$.

(e) None of the above.

(Hint: You may use the fact that

$$1 + x + x^2 + \ldots \underset{n\to\infty}{\longrightarrow} \frac{1}{1-x},$$

where the convergence is uniform for $|x| \le 1-\delta$ for every fixed $\delta > 0$.)

7. Let $v = (1/2, 1/3, 1/4, \ldots) \in \ell_2$ and let $W = \operatorname{span}\{w\} \subset \ell_2$, where $w = (1, 1/2, 1/3, \ldots)$. Let $w^* \in W$ be the best approximation of v in W. Then $\|w^* - v\| =$

   (a) $\sqrt{\pi^2/6 - 1}$.

   (b) $\sqrt{\pi^2/12 - 1/2}$.

   (c) $\sqrt{\pi^2/6 - 1 - 6/\pi^2}$.

   (d) $\sqrt{\pi^2/6 - 6/\pi^2}$.

   (e) None of the above.

8. Let V be the vector space consisting of all continuous functions $f : [1, \infty) \to \mathbf{R}$, satisfying the condition $\int_1^\infty |f(x)|^2 dx < \infty$. Let $\langle \cdot, \cdot \rangle$ be the inner product defined on V by:

$$\langle f, g \rangle = \int_1^\infty f(x)g(x)dx, \qquad f, g \in V.$$

Applying the Cauchy-Schwarz inequality to the functions f, g given by $f(x) = x^{-a}$ and $g(x) = x^{-b}$ for $x \in [1, \infty)$, where $a, b > 1/2$, yields the inequality:

   (a) $\frac{1}{a+b} \le \sqrt{\frac{1}{4ab}}$.

   (b) $\frac{1}{a+b} \le \sqrt{\frac{1}{ab}}$.

   (c) $\frac{1}{a+b-1} \le \sqrt{\frac{1}{2(2a-1)^2} + \frac{1}{2(2b-1)^2}}$.

(d) $\frac{1}{a+b-1} \leq \sqrt{\frac{1}{(2a-1)(2b-1)}}$.

(e) None of the above.

9. Let V be an inner product space, v a vector in V, and $\{e_n\}_{n=1}^{\infty}, \{e'_n\}_{n=1}^{\infty}$ two othonormal systems in V, the first of which is closed and the second not. It is known that

$$v = \sum_{n=1}^{\infty} \alpha_n e_n = \sum_{n=1}^{\infty} \beta_n e'_n,$$

where $\alpha_n = 1/2^n$ for $n \geq 1$ and $\beta_n = 1/3^n$ for $n \geq 2$, but β_1 is not known.

(a) $|\beta_1| > 1/3$.

(b) $\beta_1 = 1/3$.

(c) $|\beta_1| < \sqrt{23/72}$.

(d) $|\beta_1|$ may assume any value in the interval $\left[0, \sqrt{23/72}\right]$, but no value outside this interval.

(e) None of the above.

10. We expand the function

$$f(x) = \begin{cases} 3i, & -\pi \leq x \leq 0, \\ 5, & 0 < x \leq \pi, \end{cases}$$

into a real Fourier series $\frac{a_0}{2} + \sum_{n=1}^{\infty} (a_n \cos nx + b_n \sin nx)$.

(a) $a_5 = 0,\ b_5 = \frac{10-6i}{5\pi}$.

(b) $a_5 = \frac{10-6i}{5\pi},\ b_5 = 0$.

(c) $a_5 = \frac{6-10i}{5\pi},\ b_5 = \frac{10-6i}{5\pi}$.

(d) $a_5 = \frac{10-6i}{5\pi},\ b_5 = \frac{6-10i}{5\pi}$.

(e) None of the above.

Solutions

1. We have:

$$\|g\|^2 = \int_0^1 |g(x)|^2 dx = \int_0^1 \frac{1}{(1+x)^2} dx = \frac{-1}{1+x}\Big|_0^1 = \frac{1}{2}.$$

Hence $\|g\| = 1/\sqrt{2}$, so that a unit vector in the direction of g is the vector $\sqrt{2}g$. Now:

$$\begin{aligned}\langle f, g\rangle &= \textstyle\int_0^1 \frac{1}{(2+x)(1+x)} dx \\ &= \textstyle\int_0^1 \left(\frac{1}{1+x} - \frac{1}{2+x}\right) dx \\ &= \log(1+x) - \log(2+x)\Big|_0^1 = \log\frac{4}{3}.\end{aligned}$$

It follows that

$$w^* = \langle f, \sqrt{2}g\rangle \cdot \sqrt{2}g = \log\frac{4}{3} \cdot \sqrt{2} \cdot \sqrt{2}g,$$

and therefore:

$$\|w^*\| = \log\frac{4}{3} \cdot \sqrt{2}$$

Since $f - w^*$ and w^* are orthogonal we have

$$\|f\|^2 = \|w^*\|^2 + \|f - w^*\|^2.$$

Now

$$\|f\|^2 = \int_0^1 \frac{1}{(2+x)^2} dx = \frac{1}{6},$$

and thus:

$$d = \sqrt{\|f\|^2 - \|w^*\|^2} = \sqrt{\frac{1}{6} - 2\log^2\frac{4}{3}}.$$

Thus, (c) is true.

2. Since $\|f\| = 3$, we have:

$$\sum_{k=-\infty}^{\infty} |c_k|^2 = 3^2 = 9.$$

Now

$$\|g_n\| = \sqrt{\sum_{k=0}^{n} |c_k|^2}, \qquad n = 0, 1, \dots,$$

and

$$\|h_n\| = \sqrt{\sum_{k=-n}^{-1} |c_k|^2}, \qquad n = 1, 2, \dots,$$

which clearly implies that the sequences of norms are non-decreasing. As both sequences are bounded above by $\sum_{k=-\infty}^{\infty} |c_k|^2$, they converge, say,

$$\|g_n\| \underset{n\to\infty}{\longrightarrow} a, \qquad \|h_n\| \underset{n\to\infty}{\longrightarrow} b.$$

Clearly, $a^2 + b^2 = 9$, and therefore

$$\lim_{n\to\infty} r_n = ab \le \frac{a^2 + b^2}{2} = \frac{9}{2}.$$

The result is the best possible as we can choose f so that $a = b = \sqrt{9/2}$, in which case we have equality. For example, we may take $f(x) = \sqrt{9/2}\,(e^{ix} + e^{-ix})$.

Thus, (c) is true.

3. For each k we have:

$$\alpha_k = \left\langle \sum_{n=1}^{\infty} \alpha_n e_n, e_k \right\rangle = \langle v, e_k \rangle = \left\langle \sum_{n=1}^{\infty} \beta_n e_n, e_k \right\rangle = \beta_k.$$

Thus, (d) is true.

4. For each n we have $f_n(0) = n^{2/3}$, so that $f_n(0) \underset{n\to\infty}{\longrightarrow} \infty$ and our sequence cannot converge pointwise, and certainly not uniformly. Now

$$\begin{aligned} \|f_n - 0\|_1 &= 2\int_0^{1/n} n^{2/3}\,(1-n^2x^2)\,dx \\ &= 2n^{2/3}x - 2\tfrac{n^{8/3}x^3}{3}\Big|_0^{1/n} \\ &= 4n^{-1/3}/3 \underset{n\to\infty}{\longrightarrow} 0, \end{aligned}$$

while

$$\begin{aligned} \|f_n - 0\|_2^2 &= 2\int_0^{1/n} n^{4/3}\,(1-n^2x^2)^2\,dx \\ &= 2n^{4/3}\left(x - \tfrac{2n^2x^3}{3} + \tfrac{n^4x^5}{5}\right)\Big|_0^{1/n} \\ &= \tfrac{16}{15}n^{1/3} \underset{n\to\infty}{\longrightarrow} \infty. \end{aligned}$$

Consequently, the sequence converges to 0 in $\|\cdot\|_1$ but not in $\|\cdot\|_2$.

Thus, (d) is true.

5. The given function is even, and hence $b_n = 0$ for each n.

Thus, (a) is true.

6. We have

$$\frac{1}{1 - ie^{ix}/2} = 1 + \frac{i}{2}e^{ix} + \left(\frac{i}{2}\right)^2 e^{2ix} + \left(\frac{i}{2}\right)^3 e^{3ix} + \ldots,$$

where the convergence of the series on the right-hand side is uniform. In particular, the convergence holds in norm, so that the series is the Fourier series of f. Hence, $c_{90} = (i/2)^{90}$ and $c_{-90} = 0$, which implies $a_{90} = c_{90} + c_{-90} = i^{90}/2^{90} = -1/2^{90}$.

Thus, (b) is true.

7. We have

$$\|w\|^2 = \sum_{n=1}^{\infty} \frac{1}{n^2} = \frac{\pi^2}{6},$$

so that $\|w\| = \pi/\sqrt{6}$. Hence, a unit vector in the direction of w is the vector $e_1 = \frac{\sqrt{6}}{\pi} w$. Now:

$$\langle v, w \rangle = \sum_{n=1}^{\infty} \frac{1}{(n+1)n} = \sum_{n=1}^{\infty} \left(\frac{1}{n} - \frac{1}{n+1} \right) = 1.$$

It follows that

$$w^* = \langle v, e_1 \rangle e_1 = \frac{\sqrt{6}}{\pi} \langle v, w \rangle e_1 = \frac{\sqrt{6}}{\pi} e_1,$$

and therefore:

$$\|w^*\| = \frac{\sqrt{6}}{\pi}.$$

Since $v - w^*$ and w^* are orthogonal we have

$$\|v\|^2 = \|w^*\|^2 + \|v - w^*\|^2.$$

Now

$$\|v\|^2 = \sum_{n=2}^{\infty} \frac{1}{n^2} = \frac{\pi^2}{6} - 1,$$

which yields:

$$\|w^* - v\| = \sqrt{\|v\|^2 - \|w^*\|^2} = \sqrt{\pi^2/6 - 1 - 6/\pi^2}.$$

Thus, (c) is true.

8. We need to calculate the inner product of the two vectors and their norms. Since in general for $c > 1$

$$\int_1^{\infty} x^{-c} dx = \left. \frac{x^{1-c}}{1-c} \right|_1^{\infty} = \frac{1}{c-1},$$

we obtain:

$$\langle f, g\rangle = \int_1^\infty x^{-(a+b)}dx = \frac{1}{a+b-1},$$
$$\|f\|^2 = \int_1^\infty x^{-2a}dx = \frac{1}{2a-1},$$
$$\|g\|^2 = \int_1^\infty x^{-2b}dx = \frac{1}{2b-1}.$$

The Cauchy-Schwarz inequality yields:

$$\frac{1}{a+b-1} \leq \sqrt{\frac{1}{(2a-1)(2b-1)}}.$$

Thus, (d) is true.

9. By Parseval's identity we have:

$$\|v\|^2 = \sum_{n=1}^\infty |\alpha_n|^2 = \sum_{n=1}^\infty |\beta_n|^2.$$

(Note that the question whether the system is closed or not is irrelevant.) Now

$$\sum_{n=1}^\infty |\alpha_n|^2 = \sum_{n=1}^\infty \frac{1}{4^n} = \frac{1}{3},$$

and therefore

$$|\beta_1|^2 = \frac{1}{3} - \sum_{n=2}^\infty |\beta_n|^2 = \frac{1}{3} - \frac{1}{72} = \frac{23}{72},$$

which implies

$$|\beta_1| = \sqrt{\frac{23}{72}}.$$

Thus, (a) is true.

10. Denoting by g the function given by

$$g(x) = \begin{cases} 0, & -\pi \leq x \leq 0, \\ 1, & 0 < x \leq \pi, \end{cases}$$

we readily see that $f = 3i \cdot 1 + (5-3i) \cdot g$. Hence the Fourier coefficients c_n of f are the coefficients d_n of g, multiplied by $5 - 3i$ (except for c_0, or a_0, where we need to take into account also the contribution of the $3i \cdot 1$ addend). Now the Fourier coefficients of g have been calculated in class, and in particular $d_5 = \frac{-i}{5\pi}$ and $d_{-5} = \frac{i}{5\pi}$. It follows that $c_5 = \frac{-3-5i}{5\pi}$ and $c_{-5} = \frac{3+5i}{5\pi}$. Consequently, the corresponding coefficients in the real Fourier series are

$$a_5 = c_5 + c_{-5} = 0, \qquad b_5 = ic_5 - ic_{-5} = \frac{10 - 6i}{5\pi}.$$

Thus, (a) is true.