Review Questions

Mark the correct answer in each of the following questions.

1. Let V be a vector space and $\|\cdot\|$ a norm on V. Define (for the sake of this question only) a set of vectors $\{v_n\}_{n=1}^{\infty}$ as *generating* if for every vector $v \in V$ and $\varepsilon > 0$ there exist a positive integer n and complex numbers $\alpha_1, \alpha_2, \ldots, \alpha_n$ such that $\|v - (\alpha_1 v_1 + \alpha_2 v_2 + \ldots + \alpha_n v_n)\| < \varepsilon$.

   Now consider the space $L^2_{\mathrm{PC}}[0,1]$ with the $\|\cdot\|_1$-norm, the $\|\cdot\|_2$-norm, and the $\|\cdot\|_\infty$-norm. Let $\mathbf{Q}'$ be the set of rational numbers in $[0,1]$. For $a, b \in \mathbf{Q}'$ with $a \leq b$, denote by $1_{[a,b]}$ the indicator function of $[a,b]$ (which is 1 on $[a,b]$ and 0 outside). Consider the system $\{1_{[a,b]} : a, b \in \mathbf{Q}', a \leq b\}$.

   (a) The system is orthogonal but not orthonormal.

   (b) The system is generating with respect to the $\|\cdot\|_2$-norm.

   (c) The system is not generating with respect to the $\|\cdot\|_1$-norm.

   (d) The system is generating with respect to the $\|\cdot\|_\infty$-norm.

   (e) None of the above.

2. Let $(f_n)_{n=1}^{\infty}$ and f be continuous functions from $\mathbf{R}$ to $\mathbf{R}$, all belonging to $L^1_{\mathrm{PC}}(-\infty,\infty) \cap L^2_{\mathrm{PC}}(-\infty,\infty)$.

   (a) If $f_n \underset{n\to\infty}{\longrightarrow} f$ uniformly on $\mathbf{R}$, then the same holds in $\|\cdot\|_2$.

   (b) If $f_n \underset{n\to\infty}{\longrightarrow} f$ in $\|\cdot\|_2$, then the same holds uniformly.

   (c) Suppose $|f(x)| \leq 1$ for every $x \in \mathbf{R}$ and $|f_n(x)| \leq 1$ for every $1 \leq n < \infty$ and $x \in \mathbf{R}$. If $f_n \underset{n\to\infty}{\longrightarrow} f$ in $\|\cdot\|_2$, then the same holds in $\|\cdot\|_1$.

(d) Suppose $|f(x)| \le 1$ for every $x \in \mathbf{R}$ and $|f_n(x)| \le 1$ for every $1 \le n < \infty$ and $x \in \mathbf{R}$. If $f_n \underset{n\to\infty}{\longrightarrow} f$ in $\|\cdot\|_1$, then the same holds in $\|\cdot\|_2$.

(e) None of the above.

3. Let f be the function given by:

$$f(x) = e^{-x^2} \cdot \int_{-\infty}^{x} e^{-t^2}\,dt, \qquad -\infty < x < \infty,$$

and let $y = \hat{f}$. Then the function y satisfies the differential equation:

(a)
$$y' + \frac{\omega}{2} y = \frac{i}{4\sqrt{2\pi}} e^{-\omega^2/8}.$$

(b)
$$y' + \frac{\omega}{2} y = \frac{i}{8\sqrt{2\pi}} e^{-\omega^2/8}.$$

(c)
$$y' + \frac{\omega}{2} y = \frac{i}{12\sqrt{2\pi}} e^{-\omega^2/8}.$$

(d)
$$y' + \frac{\omega}{2} y = \frac{i}{16\sqrt{2\pi}} e^{-\omega^2/8}.$$

(e) None of the above.

4. Consider the reasoning and obviously false conclusion of the following paragraph.

In class we have found that the Fourier transform of the function f, defined by

$$f(x) = e^{-|x|}, \qquad x \in \mathbf{R},$$

is given by:

$$\hat{f}(\omega) = \frac{1}{\pi(\omega^2+1)}, \qquad \omega \in \mathbf{R}.$$

Now f actually coincides with e^{-x} on $(0,\infty)$ and with e^x on $(-\infty,0)$. Hence f is piecewise differentiable of every order (the only problematic point being 0, where f "switches" from e^x to e^{-x}). Also, f'' is identical to f, except for the point 0, where f'' is undefined. Employing the rules for transforms, we find that

$$\widehat{f''}(\omega) = (i\omega)^2 \hat{f}(\omega) = -\omega^2 \hat{f}(\omega). \tag{1}$$

It follows that

$$\hat{f}(\omega) = \widehat{f''}(\omega) = -\omega^2 \hat{f}(\omega),$$

which implies

$$\left(\omega^2+1\right) \hat{f}(\omega) = 0,$$

so that $\hat{f}$ is identically 0, and hence so is f itself.

The error in the reasoning is:

(a) The function f is not twice differentiable at 0.

(b) The function f'' does not belong to $L^1_{\mathrm{PC}}(-\infty,\infty)$.

(c) To use the rule in (1), the function f has to be continuously differentiable, not just piecewise continuously differentiable.

(d) The transform of a function tends to 0 as the argument tends to $\pm\infty$. In our case, we had to check that the function $(i\omega)^2\hat{f}(\omega)$ satisfies this condition before we could use the rule.

(e) None of the above.

5. Consider the sequence $(f_n)_{n=1}^{\infty}$ of functions from $\mathbf{R}$ to itself, defined by:

$$f_n = 1_{[-1,1]} - \frac{1}{2} \cdot 1_{[-2,2]} + \ldots + \frac{(-1)^{n-1}}{n} \cdot 1_{[-n,n]}, \qquad n = 1, 2, \ldots.$$

Let $\widehat{f_n}$ be the Fourier transform of f_n for each n.

(a) The sequence of numbers $(\widehat{f}_n(1/2))$ is the sequence of partial sums of a conditionally convergent series. Hence the sequence of functions $(\widehat{f}_n)$ does not converge uniformly on the interval $[1/2-\delta, 1/2+\delta]$ for any $\delta > 0$.

(b) If $\omega = m\pi$, where π is a non-zero integer, then the sequence of numbers $(\widehat{f}_n(\omega))$ is identically 0, and in particular it converges to 0. However, the sequence converges for no other value of ω.

(c) The sequence of functions $(\widehat{f}_n)$ converges on the interval $[1/2, \pi/2]$, but not uniformly.

(d) The sequence of functions $(\widehat{f}_n)$ converges uniformly on the interval $[\delta, \pi-\delta]$ for every $\delta > 0$.

(e) None of the above.

6. Let $[\cdot]$ denote the *integer part* function, defined by:

$$[x] = m, \qquad m \leq x < m+1,\ m \in \mathbf{Z}.$$

(For example, $[5] = [5.99] = 5,\ [-5.01] = -6$.) Define the function f by:

$$f(x) = \begin{cases} e^{-[x]}, & x \geq 0, \\ 0, & x < 0. \end{cases}$$

(a) We have $\hat{f} \in L^2_{\mathrm{PC}}(-\infty, \infty)$ and

$$\hat{f}(\pi) = \frac{e}{i\pi^2(e+1)}.$$

(b) We have $\hat{f} \in L^2_{\mathrm{PC}}(-\infty, \infty)$ and

$$\hat{f}(\pi) = \frac{e}{i\pi(e+1)}.$$

(c) We have $\hat{f} \notin L^2_{\mathrm{PC}}(-\infty, \infty)$ and

$$\hat{f}(\pi) = \frac{e}{i\pi^2(e+1)}.$$

(d) We have $\hat{f} \notin L^2_{\mathrm{PC}}(-\infty, \infty)$ and
$$\hat{f}(\pi) = \frac{e}{i\pi(e+1)}.$$

(e) None of the above.

7. We proved in class that the Fourier series of a piecewise continuously differentiable function converges uniformly to the function away from the discontinuities; moreover, at the discontinuities the series converges to the average of the one-sided limits of the function. Recall that the first part of the proof was devoted to show that the above holds for the specific function ϕ, defined by $\phi(x) = x$ for $\pi \leq x < \pi$.

   (a) If, instead of proving the theorem first for ϕ, we prove it for some other non-zero function $\psi \in L^2_{\mathrm{PC}}(-\pi, \pi)$, then the second part of the proof still works as before.

   (b) The previous claim is false. However, if, instead of proving the theorem first for ϕ, we prove it for some non-zero piecewise continuously differentiable function ψ, with exactly one discontinuity in $[-\pi, \pi]$, whether a removable discontinuity or a jump discontinuity, then the second part of the proof still works as before.

   (c) The previous claims are false. However, if, instead of proving the theorem first for ϕ, we prove it for some non-zero piecewise continuously differentiable function ψ, with exactly one jump discontinuity in $[-\pi, \pi]$, then the second part of the proof still works as before.

   (d) The previous claims are false. However, if, instead of proving the theorem first for ϕ, we prove it for some non-zero continuously differentiable function ψ, then the second part of the proof still works as before.

   (e) None of the above.

   Remark: We consider a function f as continuous at π if and only if $f(-\pi) = f(-\pi+) = f(\pi-) = f(\pi)$. If it is not, we count the points $-\pi$ and π as a single discontinuity.

Solutions

1. The product of two functions $1_{[a,b]}$ and $1_{[a',b']}$ from our system is the indicator function of the intersection of the two intervals. It follows easily that the functions are orthogonal if and only if the underlying intervals do not intersect.

   The subspace of the space of all piecewise continuous functions, spanned by all indicator functions, is that of piecewise constant functions. We have seen that every piecewise continuous function can be uniformly approximated by piecewise constant functions, and in particular can be arbitrarily well approximated in $\|\cdot\|_2$-norm and in $\|\cdot\|_1$-norm.

   In our case, since we start with indicator functions of intervals with rational endpoints only, we get by using linear combinations only piecewise constant functions that change from one value to another at rational points. This does not affect approximation in $\|\cdot\|_2$-norm and in $\|\cdot\|_1$-norm. Indeed, consider the two functions $\sum_{j=1}^k \alpha_j 1_{[a_j,b_j]}$ and $\sum_{j=1}^k \alpha_j 1_{[a'_j,b'_j]}$. If all the a_j and b_j are close to the corresponding a'_j and b'_j, then the two functions are close in $\|\cdot\|_2$-norm and in $\|\cdot\|_1$-norm. Since every point in $[0,1]$ can be approximated arbitrarily well by a rational point, linear combinations of indicator functions of intervals with rational endpoints can approximate arbitrarily well any piecewise continuous function in $\|\cdot\|_2$-norm and in $\|\cdot\|_1$-norm.

   However, this is not the case with uniform approximation. For example, take the function $f = 1_{[0,s]}$, where $s \in [0,1] - \mathbf{Q}'$. We claim that any function g of the form $\sum_{j=1}^k \alpha_j 1_{[a_j,b_j]}$ with rational a_j and b_j is at a distance of at least $1/2$ from f in $\|\cdot\|_\infty$-norm. In fact, let c be the largest number in $\{a_1, b_1, a_2, b_2, \ldots, a_k, b_k\}$ that is smaller than s, and d the smallest number in the same set that is larger than s. (Here $c = 0$ if no number in the set is smaller than s, and $d = 1$ in the analogous situation to the right of s.) Hence, g is constant on $[c,d]$, while f assumes both the values 0 and 1. It follows that $\|f-g\|_\infty \geq 1/2$.

   Thus, (b) is true.

2. To disprove (a), we construct $(f_n)_{n=1}^{\infty}$ as follows. We let f_n vanish outside $[-n^2, n^2]$. Next we take $f_n(n^2) = f_n(-n^2) = 0$ and $f_n(0) = 1/n$, and let f_n vary linearly on $[-n^2, 0]$ and on $[0, n^2]$. Clearly,

$$\|f_n - 0\|_\infty = \sup_{x \in \mathbf{R}} |f_n(x)| = 1/n \underset{n\to\infty}{\longrightarrow} 0,$$

so that f_n converges uniformly to 0 as $n \to \infty$. On the other hand,

$$\int_{-\infty}^{\infty} |f_n(x) - 0|^2 dx = 2 \int_0^{n^2} \left(\frac{1}{n} - \frac{1}{n^2} x \right)^2 dx = \frac{2}{3},$$

so that f_n does not converge to 0 in $\|\cdot\|_2$.

To disprove (b), we take f_n to vanish outside $[n, n+2/n]$, assume the value 1 at the point $n + 1/n$, and vary linearly on $[n, n+1/n]$ and on $[n+1/n, n+2/n]$. Then

$$\int_{-\infty}^{\infty} |f_n(x) - 0|^2 dx = 2 \int_n^{n+1/n} (n(x-n))^2 \, dx = 2 \int_0^{1/n} (nx)^2 \, dx = \frac{2}{3n} \underset{n\to\infty}{\longrightarrow} 0,$$

so that f_n converges to 0 in $\|\cdot\|_2$ as $n \to \infty$. On the other hand, we clearly have $\|f_n - 0\|_\infty = 1$ for each n, namely f_n does not converge uniformly to 0. Note that

$$\int_{-\infty}^{\infty} |f_n(x) - 0| dx = 2 \int_0^{1/n} nx dx = 1,$$

which means that f_n does not converge to 0 in $\|\cdot\|_1$ either, and thus disproves (d) as well.

To disprove (c), take f_n vanish outside $[-n, n]$, let $f_n(n) = f_n(-n) = 0$ and $f_n(0) = 1/n$, and let f_n vary linearly on $[-n, 0]$ and on $[0, n]$. One checks easily that f_n converges to 0 in $\|\cdot\|_2$, but not in $\|\cdot\|_1$.

Under the conditions of (d), we have $|f_n(x) - f(x)| \le 2$ for every n and x, and therefore

$$\int_{-\infty}^{\infty} |f_n(x) - f(x)|^2 dx \le 2 \int_{-\infty}^{\infty} |f_n(x) - f(x)| dx,$$

so that the convergence $f_n \underset{n\to\infty}{\longrightarrow} f$ in $\|\cdot\|_1$ implies the same convergence in $\|\cdot\|_2$.

Thus, (d) is true.

3. The second factor in the product defining f is bounded by $\int_{-\infty}^{\infty} e^{-t^2} dt$, and hence $f \in L^1_{\mathrm{PC}}$. Differentiating by sides the equality, we obtain:

$$f'(x) = e^{-x^2} \cdot (-2x) \cdot \int_{-\infty}^{x} e^{-t^2} dt + e^{-x^2} \cdot e^{-x^2} = -2xf(x) + e^{-2x^2}. \quad (2)$$

According to the rules for calculating Fourier transforms,

$$-\widehat{2xf(x)}(\omega) = -2i\hat{f}'(\omega).$$

Defining g and h by $g(x) = e^{-x^2/2}$ and $h(x) = e^{-2x^2}$, we note that $h(x) = g(2x)$, and therefore

$$\hat{h}(\omega) = \frac{1}{2}\hat{g}(\omega/2) = \frac{1}{2} \cdot \frac{1}{\sqrt{2\pi}} e^{-(\omega/2)^2/2} = \frac{1}{2\sqrt{2\pi}} e^{-\omega^2/8}.$$

Passing to Fourier transforms in (2), we arrive at

$$i\omega\hat{f}(\omega) = -2i\hat{f}'(\omega) + \frac{1}{2\sqrt{2\pi}} e^{-\omega^2/8},$$

which yields after simplification:

$$\hat{f}'(\omega) + \frac{\omega}{2} y = -\frac{i}{4\sqrt{2\pi}} e^{-\omega^2/8}.$$

Thus, (e) is true.

4. To prove the rule for $\widehat{f'}(\omega)$, we required that f be continuous. In our example, we have used it twice, so that actually f' needs to be continuous as well, namely f has to be continuously differentiable (and its derivative needs to be piecewise differentiable). It is not required that f be twice differentiable.

The function f'', being the same as f (except at 0), certainly belongs to $L^1_{\mathrm{PC}}(-\infty, \infty)$.

It is true that the transform of a function tends to 0 as the argument tends to $\pm\infty$, so that if our reasoning was valid then $(i\omega)^2\hat{f}(\omega)$ would have to satisfy this condition. However, the condition follows from the conditions on f itself, and does not need to be checked.

Thus, (c) is true.

5. We have shown in class that, if $f = 1_{[-b,b]}$, then

$$\hat{f}(\omega) = \frac{\sin \omega b}{\pi\omega}.$$

It follows by the linearity of the transform that in our case:

$$\widehat{f_n}(\omega) = \sum_{k=1}^{n} (-1)^{k-1} \frac{\sin k\omega}{\pi k\omega} = \frac{1}{2\pi\omega} \sum_{k=1}^{n} 2 \cdot (-1)^{k-1} \frac{\sin k\omega}{k}. \tag{3}$$

The sum on the right-hand side is exactly the partial sum S_n of the Fourier series of the identity function $g(x) = x$ (in our case, written with the argument ω). We have shown that the partial sums converge uniformly to g on $[-\pi + \delta, \pi - \delta]$ for every $\delta > 0$.

In our case, on the right-hand side of (3) we have an extra factor of $\frac{1}{2\pi\omega}$. This factor is bounded above by $\frac{1}{2\pi\delta}$ on $[\delta, \pi - \delta]$, which implies that $\widehat{f_n}$ converges uniformly on this interval.

Thus, (d) is true.

6. First we calculate $\hat{f}$:

$$\begin{aligned}
\hat{f}(\omega) &= \tfrac{1}{2\pi}\int_{-\infty}^{\infty} f(x)e^{-i\omega x}dx \\
&= \tfrac{1}{2\pi}\int_{0}^{\infty} e^{-[x]}e^{-i\omega x}dx \\
&= \tfrac{1}{2\pi}\sum_{n=0}^{\infty} e^{-(n+i\omega n)}\int_{n}^{n+1} e^{-i\omega(x-n)}dx \\
&= \tfrac{1}{2\pi}\sum_{n=0}^{\infty} e^{-(1+i\omega)n}\int_{0}^{1} e^{-i\omega x}dx \\
&= \tfrac{1}{2\pi}\tfrac{1}{1-e^{-(1+i\omega)}}\tfrac{1-e^{-i\omega}}{i\omega} \\
&= \tfrac{1}{2\pi i\omega}\tfrac{1-e^{-i\omega}}{1-e^{-(1+i\omega)}}.
\end{aligned}$$

(Note that, in fact, the replacement of the integral over $[0,\infty)$ by the infinite sum of integrals over all intervals $[n,n+1]$ requires a justification. The integral over $[0,\infty)$ is the limit of the integral over $[0,t]$ as $t\to\infty$, whereas our sum is the same limit as $t\to\infty$ over integers only. If the integrand does not go to 0 as the argument goes to ∞, we may have problems; for example, consider $\int_0^\infty (x-[x]-1/2)dx$. In our case the integrand goes to 0, so the replacement is correct.)

In particular:

$$\hat{f}(\pi) = \frac{1}{2\pi i\pi}\frac{1-(-1)}{1+e^{-1}} = \frac{e}{i\pi^2(e+1)}.$$

Since

$$\left|1-e^{-(1+i\omega)}\right| \geq 1-e^{-1}, \qquad \omega\in\mathbf{R},$$

we have $|\hat{f}(\omega)| < \frac{C}{|\omega|}$ for some $C>0$ and all sufficiently large $|\omega|$, and therefore $\hat{f}\in L^2_{\mathrm{PC}}(-\infty,\infty)$.

Thus, (a) is true.

7. The contribution of the first part of the proof is that it lets you "remove" the discontinuities of the given function f. Since the function

ϕ has exactly one discontinuity, which is a jump, we can remove one discontinuity of f by adding to it an appropriate scalar multiple of an appropriate shift of ϕ, without introducing new discontinuities in the process. Hence, any piecewise continuously differentiable function, with exactly one jump discontinuity in $[-\pi, \pi]$, would do instead of ϕ (if we could indeed prove the theorem for that function).

A ψ with more than one discontinuity would fail in general, as while fixing one discontinuity, it would create others. A ψ with a removable discontinuity (same as a continuous ψ) would not help either, as it could not be used to remove discontinuities of f.

Thus, (c) is true.