Final #1

Mark the correct answer in each part of the following questions.

1. The sequence $(d_k)_{k=1}^{\infty}$ is defined by:

$$d_k = \begin{cases} 50, & k \leq 20, \\ 3, & k > 20. \end{cases}$$

Denote $D_n = (d_1, d_2, \ldots, d_n)$ for every n.

(i) The sequence D_n is graphic for infinitely many values of n and non-graphic for infinitely many other values of n. However, for no n does there exist a tree whose sequence of vertex degrees is D_n.

(ii) The sequence D_n is graphic for every sufficiently large n.

(iii) There exists exactly one value of n for which D_n is the sequence of vertex degrees of some tree.

(iv) For every constant C, there exists an n such that D_n is the sequence of vertex degrees of some graph G with $\omega(G) > C$.

(v) None of the above.

2. Let $(d_i)_{i=1}^{n}$ be a sequence of integers with $d_1 \geq d_2 \geq \ldots \geq d_n \geq 1$, where $n \geq 5$.

(i) The sequence is graphic if and only if there exist indices i, j, with $1 \leq i < j \leq n$, such that the sequence $d_1, d_2, \ldots, d_{i-1}, d_i - 1, d_{i+1}, \ldots, d_{j-1}, d_j - 1, d_{j+1}, \ldots, d_n$ is graphic.

(ii) If $d_{n-1} = d_n = 1$, then the sequence is graphic if and only if the sequence $d_1, d_2, \ldots, d_{n-3}, d_{n-2}$ is.

(iii) If $d_{n-3} = d_{n-2} = d_{n-1} = d_n = 3$, then the sequence is graphic if and only if the sequence $d_1, d_2, \ldots, d_{n-5}, d_{n-4}$ is.

(iv) The sequence is graphic if and only if the sequence $d_1 - 1, d_2 - 1, d_3, \ldots, d_n$ is graphic and $d_1 < n$.

(v) None of the above.

3. Let A be a set of size 6. Consider the graph $G = (V, E)$, where $V = 2^A$ (i.e., the set of all subsets of A) and

$$E = \{(B, C) : B, C \subseteq A, |B \triangle C| \equiv 1 \pmod 2\}.$$

$$E = \{(B, C) : B, C \subseteq A, |B \triangle C| \equiv 1 \, (\text{mod } 2)\}.$$

(Here we denote by $\triangle$ the symmetric difference of sets, namely $B \triangle C = (B - C) \cup (C - B)$.)

(a) The independence number of G is

(i) $\alpha(G) = 2^2$.

(ii) $\alpha(G) = 2^3$.

(iii) $\alpha(G) = 2^4$.

(iv) $\alpha(G) = 2^5$.

(v) none of the above.

(b) The chromatic number of G is

(i) $\chi(G) = 2$.

(ii) $\chi(G) = 6$.

(iii) $\chi(G) = 7$.

(iv) $\chi(G) = 32$.

(v) none of the above.

(c) The number of spanning trees of G is

(i) $\tau(G) = 2^{32}$.

(ii) $\tau(G) = 2^{64}$.

(iii) $\tau(G) = 2^{310}$.

(iv) $\tau(G) = 2^{372}$.

(v) none of the above.

4. Define (for the purpose of this question only) a *sub-Latin square with memory* r to be an $n \times n$ square, satisfying the same requirements as does a Latin square, except for the following change regarding the columns: Instead of requiring that all entries in each column be distinct, we require only that each entry be distinct from the r entries immediately above it, but it may equal any of the entries above them. (For example, for $r = 3$, denoting by a_{ij} the (i, j)-th entry, we require that $a_{10,5} \neq a_{7,5}, a_{8,5}, a_{9,5}$ but allow any of the equalities $a_{10,5} = a_{1,5},\ a_{10,5} = a_{2,5}, \ldots, a_{10,5} = a_{6,5}$.) We employ the methods, used in class to bound from below and from above the number of Latin squares, to accomplish the same for sub-Latin squares with memory $r = 3$.

   (a) The lower bound our method yields is:
      (i) $n!^n(1 - 3/n)^{n^2-n}$.
      (ii) $n!^n(1 - 1/n)^n(1 - 2/n)^n(1 - 3/n)^{n^2-3n}$.
      (iii) $n!^n(1 - 1/n)(1 - 2/n)(1 - 3/n)^{n^2-n-2}$.
      (iv) $n!^n(1 - 1/n)(1 - 3/n)^{n^2-n-1}$.
      (v) none of the above.

   (b) The upper bound our method yields is:
      (i) $(n-0)!^{\frac{n}{n-1}}(n-1)!^{\frac{n-1}{n-2}}(n-2)!^{\frac{n-2}{n-3}}(n-3)!^{n-4}$.
      (ii) $(n-0)!(n-1)!(n-2)!(n-3)!^n$.
      (iii) $(n-0)!^{\frac{n}{n-0}}(n-1)!^{\frac{n}{n-1}}(n-2)!^{\frac{n}{n-2}}(n-3)!^n$.
      (iv) $(n-0)!^{1+1/n}(n-1)!^{1+2/n}(n-2)!^{1+3/n}(n-3)!^n$.
      (v) none of the above.

5. Let $G = (V, E)$ be a graph, with $V = \{v_1, v_2, \ldots, v_n\}$ for some even number n. For a positive integer k, denote by $\tilde{\chi}(G, 2k)$ the number of proper colorings of G in the $2k$ colors $1, 2, \ldots, 2k$, in which each vertex v_i with an even i is colored in an even color (namely, one of the colors $2, 4, \ldots, 2k$), while each v_i with an odd i is colored in an odd color. Denote by E_{00} (resp. E_{11}, E_{01}) the set of edges (v_i, v_j) with both i and j even (resp. both i and j odd, i even and j odd).

(i) The function $\tilde{\chi}(G, 2k)$ is a polynomial function of degree n in k. The coefficient of k^{n-1} is $-|E_{01}|$.

(ii) The function $\tilde{\chi}(G, 2k)$ is a polynomial function of degree n in k. The coefficient of k^{n-1} is $-|E_{00}| - |E_{11}|$.

(iii) The function $\tilde{\chi}(G, 2k)$ is a polynomial function of degree n in k. The coefficient of k^{n-1} is $-2^n|E_{01}|$.

(iv) The function $\tilde{\chi}(G, 2k)$ is a polynomial function of degree n in k. The coefficient of k^{n-1} is $-2^n(|E_{00}| + |E_{11}|)$.

(v) None of the above.

6. Let $K_{m,n} = (A \cup B, A \times B)$ be a complete bi-partite graph, with parts $A = \{a_1, a_2, \ldots, a_m\}$ and $B = \{b_1, b_2, \ldots, b_n\}$. Consider the graph $G = (A \cup B, A \times B \cup \{(b_1, b_2)\})$. We have $\tau(G) =$

   (i) $m^{n-1}n^{m-1} + 2m^{n-1}n^{m-2}$.

   (ii) $m^{n-1}n^{m-1} + 2m^{n-2}(n-1)^{m-1}$.

   (iii) $m^{n-1}n^{m-1} + 2m^{n-2}n^{m-1}$.

   (iv) $m^{n-1}n^{m-1} + 2m^{n-1}(n-1)^{m-1}$.

   (v) None of the above.

7. In the matrix-tree theorem and its proof we have defined matrices Q, C, that satisfied the equality $CC^T = Q$. Denoting by $\text{abs}(M)$, for any matrix M, the matrix obtained from M by replacing each entry by its absolute value, the matrix $\text{abs}(C)\text{abs}(C^T)$

   (i) is $\text{abs}(Q)$.

   (ii) is a matrix with determinant 1.

   (iii) is a matrix with at least one eigenvalue that is not real (for $n \geq 3$).

   (iv) is a matrix, the sum of all whose entries is the same as the sum of all entries of Q.

   (v) enjoys none of the above properties.

Solutions

1. We use the Erdős-Gallai Theorem. It is easy to see that, if $n \geq 21$ is odd, then so is $\sum_{i=1}^{n} d_i$. Therefore, for such n, the sequence D_n is not a graphic sequence.

   We now claim that for sufficiently large even n, the sequence is graphic. The condition that $\sum_{i=1}^{n} d_i$ is even is certainly satisfied for such n. We have to show that

$$\sum_{i=1}^{k} d_i \leq k(k-1) + \sum_{j=k+1}^{n} \min\{k, d_j\}, \qquad 1 \leq k \leq n.$$

   Indeed, if $k \leq 50$, then for $n \geq 1140$,

$$\sum_{i=1}^{k} d_i \leq 20 \cdot 50 + 30 \cdot 3 \leq \sum_{j=51}^{1140} \min\{k, d_j\} < k(k-1) + \sum_{j=k+1}^{n} \min\{k, d_j\}.$$

   If $k \geq 51$, then for every $n (\geq k)$,

$$\sum_{i=1}^{k} d_i \leq k \cdot 50 \leq k(k-1) \leq k(k-1) + \sum_{j=k+1}^{n} \min\{k, d_j\}.$$

   D_n cannot be the sequence of vertex degrees of a tree because $\sum_{i=1}^{n} d_i > 2n-2$. (Alternatively, since a graph with this sequence of vertex degrees has no leaves.)

   Thus, (i) is true.

2. The condition in (i) is necessary. In fact, suppose that G is a graph with vertices $v_1, v_2, \ldots, v_n$, and corresponding degrees $d_1, d_2, \ldots, d_n$. Pick two adjacent vertices v_i and v_j. Omitting the edge (v_i, v_j) from G, we see that the sequence $d_1, d_2, \ldots, d_{i-1}, d_i - 1, d_{i+1}, \ldots, d_{j-1}, d_j - 1, d_{j+1}, \ldots, d_n$ is graphic. However, the condition is not sufficient. For example, take $d_1 = d_2 = n, d_3 = d_4 = \ldots = d_n = n - 1$. The sequence

is not graphic, as a graph with this sequence of vertex degrees requires v_1 to have more neighbors than the maximum possible. Yet reducing d_1 and d_2 by 1 each, we obtain the sequence of vertex degrees of K_n.

In (ii), if the sequence $d_1, d_2, \ldots, d_{n-3}, d_{n-2}$ is graphic, then indeed so is the original sequence; take a graph with vertex degrees $d_1, d_2, \ldots, d_{n-2}$ and add two vertices v_{n-1}, v_n, neighboring each other but no other vertices. However, the converse is false. For example, let $n = 4, d_1 = d_2 = 2, d_3 = d_4 = 1$. A path on 4 vertices has these vertex degrees but, omitting d_3 and d_4, we are clearly left with a non-graphic sequence.

In (iii), the situation is similar. Namely, if the sequence $d_1, d_2, \ldots, d_{n-4}$ is graphic, then, adding to a graph with these vertex degrees a K_4, we see that so is the original sequence. To see that the converse is false, note that the sequence given by $d_1 = d_2 = \ldots = d_6 = 3$ is graphic (being the sequence of vertex degrees of $K_{3,3}$, for example), but the subsequence consisting of the first two elements is not such.

The condition in (iv) is necessary. First, we must have $d_1 < n$ since each vertex may have at most $n - 1$ neighbors. From the proof of the Havel-Hakimi Theorem (one may use the theorem itself instead), it follows that, if the original sequence is graphic, then there exists a graph with this sequence of vertex degrees, such that the vertex v_1 with degree d_1 neighbors exactly the vertices $v_2, v_3, \ldots, v_{k+1}$, with degrees $d_2, d_3, \ldots, d_{k+1}$, where $k = d_1$. Removing the edge (v_1, v_2), we obtain a graph whose vertex degrees are $d_1 - 1, d_2 - 1, d_3, \ldots, d_n$. However, the condition is not sufficient. For example, the sequence defined by $d_1 = d_2 = 3, d_3 = d_4 = 1$ is clearly non-graphic, but, reducing the first two d_i's by 1 each, we obtain a graphic sequence.

Thus, (v) is true.

3. One verifies readily that two subsets are adjacent if and only if their sizes are of different parities. As A has 32 subsets of even size and 32 of odd size, G is isomorphic to $K_{32,32}$.

   (a) Each of the two parts of the graph forms an independent set, and clearly there is no larger independent set. Hence $\alpha(G) = 32$.

   Thus, (iv) is true.

(b) Since the graph is bi-partite and non-empty, $\chi(G) = 2$.
Thus, (i) is true.

(c) According to the formula for the number of spanning trees of complete bi-partite graphs, we have $\tau(G) = 32^{32-1} \cdot 32^{32-1} = 2^{310}$.
Thus, (iii) is true.

4. In both parts of the question, the approach is identical to that we took for Latin squares. The only difference with respect to the situation there is the number of possibilities we have for each entry when filling in any row. With the constraints in this question, the first four rows behave the same as for Latin squares. However, for each row $k \geq 5$, instead of having to avoid at each entry the symbols used at all $k-1$ entries above it, we only have to avoid the three right above it (that are different from each other). Namely, at each entry by itself we have $n-3$ options.

   (a) For filling in the first row we have exactly $n!$ possibilities, for the second row – at least $(n-1)^n \cdot \frac{n!}{n^n}$ possibilities, for the third row – at least $(n-2)^n \cdot \frac{n!}{n^n}$ possibilities, and for each of the following $n-3$ rows – at least $(n-3)^n \cdot \frac{n!}{n^n}$ possibilities. Hence the total number of possibilities is bounded below by

$$n! \cdot (n-1)^n \frac{n!}{n^n} \cdot (n-2)^n \frac{n!}{n^n} \left((n-3)^n \cdot \frac{n!}{n^n} \right)^{n-3}.$$

   Thus, (ii) is true.

   (b) For filling in the first row we have exactly $n!$ possibilities, for the second row – at most $(n-1)!^{\frac{n}{n-1}}$ possibilities, for the third row – at most $(n-2)!^{\frac{n}{n-2}}$ possibilities, and for each of the following $n-3$ rows – at most $(n-3)!^{\frac{n}{n-3}}$ possibilities. Hence the total number of possibilities is bounded above by

$$(n-0)!^{\frac{n}{n-0}} (n-1)!^{\frac{n}{n-1}} (n-2)!^{\frac{n}{n-2}} (n-3)!^{n}.$$

   Thus, (iii) is true.

5. A proper coloring of G is actually made up by two independent proper colorings of subgraphs of G. We have to choose a proper coloring of the subgraph G_0, induced by the vertices $v_2, v_4, \ldots, v_n$, using the colors $2, 4, \ldots, 2k$, and a proper coloring of the subgraph G_1, induced by the vertices $v_1, v_3, \ldots, v_n$, using the colors $1, 3, \ldots, 2k-1$. Now, the set of edges of G_0 is E_{00}, and that of G_1 is E_{11}. Hence, denoting $m = n/2$, the number of colorings of G_0 is

$$\chi(G_0, k) = k^m - |E_{00}|k^{m-1} + \sum_{i=2}^{m} a_i k^{m-i}$$

and that of G_1 is

$$\chi(G_1, k) = k^m - |E_{11}|k^{m-1} + \sum_{i=2}^{m} b_i k^{m-i}$$

for appropriate integers a_i, b_i. Consequently:

$$\tilde{\chi}(G, 2k) = \chi(G_0, k)\chi(G_1, k) = k^n - (|E_{00}| + |E_{11}|)\, k^{n-1} + \sum_{i=2}^{n} c_i k^{n-i}$$

for appropriate integers c_i.

Thus, (ii) is true.

6. The natural approach to this problem is to employ the formula

$$\tau(G) = \tau(G-e) + \tau(G \cdot e),$$

where $e = (b_1, b_2)$. Now, $G - e$ is simply $K_{m,n}$, so that

$$\tau(G-e) = m^{n-1}n^{m-1}.$$

The graph $G \cdot e$ is "approximately" $K_{m,n-1}$, the difference being that all edges, one of whose endpoints is the vertex merged from b_1 and b_2, are double edges. Thus, after rearranging the indices, our problem is to find $\tau(G')$, where $G' = (V', E')$ is the multi-graph with $V' = \{a_1, a_2, \ldots, a_m, b_1, b_2, \ldots, b_{n-1}\}$ and E' consisting of double edges between all a_i's and b_1 and single edges between all a_i's and b_j's with $j \geq 2$. We proceed as in the proof of the formula for the number of spanning trees of a complete bi-partite graph. Recall that, for sizes m and $n-1$ of the parts of the graph, our proof shows that, for each t between 0 and $m-1$ and each choice of parents for $b_2, b_3, \ldots, b_{n-1}$, there are $\binom{m-1}{t}(n-2)^t$ possibilities of assigning parents to t of the a_i's out of the vertices b_j with $j \geq 2$, and letting the remaining $m-t$ of the a_i's be children of b_1. The only difference in our case is that, for each of these $m-t$ vertices, there are two possibilities of connecting it to b_1, so that we have an additional factor of 2^{m-t}. It follows that:

$$\begin{aligned}
\tau(G') &= m^{n-2} \sum_{t=0}^{m-1} \binom{m-1}{t} (n-2)^t \cdot 2^{m-t} \\
&= m^{n-2} \cdot 2^m \sum_{t=0}^{m-1} \binom{m-1}{t} \left(\frac{n-2}{2}\right)^t \\
&= m^{n-2} \cdot 2^m \left(\frac{n-2}{2} + 1\right)^{m-1} \\
&= 2m^{n-2} n^{m-1}.
\end{aligned}$$

Altogether:

$$\tau(G) = m^{n-1} n^{m-1} + 2m^{n-2} n^{m-1}.$$

Thus, (iii) is true.

7. Recall that each entry of C is either 0 or ± 1. Moreover, at each column of C there is a single 1 and a single -1. Thus, when calculating the entry at any location (i,i) along the main diagonal of CC^T, we sum terms that are either 0^2 or $(\pm 1)^2$ each. The sum is non-negative, and the sum obtained for the product $\mathrm{abs}(C)\mathrm{abs}(C^T)$ coincides with it. An off-diagonal entry of CC^T, at location (i,j) with $i \neq j$, is non-zero if and only if the vertices corresponding to the i-th and j-th vertices of the graph are adjacent. Namely, all terms in the sum are of the form $0 \cdot 0$, with a single possible exception of a $-1 \cdot 1$ product. Clearly, in the first case the corresponding entry of $\mathrm{abs}(C)\mathrm{abs}(C^T)$ vanishes as well, and in the second case the $-1 \cdot 1$ product is replaced by $1 \cdot 1$. Hence $\mathrm{abs}(C)\mathrm{abs}(C^T) = \mathrm{abs}(CC^T) = \mathrm{abs}(Q)$.

In particular, the sum of all entries of $\mathrm{abs}(C)\mathrm{abs}(C^T)$ is the same as the sum of all entries of Q only if G is an empty graph.

If the graph G is K_2, then

$$\mathrm{abs}(Q) = \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix},$$

whose determinant is clearly 0.

The matrix Q is symmetric, and hence so is $\mathrm{abs}(Q)$ as well. Thus, all eigenvalues are necessarily real.

Thus, (i) is true.