Review Questions

1. Let $d_1 \geq d_2 \geq \ldots \geq d_n \geq 1$ be integers. Consider the following two claims:

   I. If $d_1, d_2, d_3, d_4, \ldots, d_n$ is a graphic sequence, then so is also the sequence $d_1 - 1, d_2 - 1, d_3, d_4, \ldots, d_n$.

   II. If $d_1 - 1, d_2 - 1, d_3, d_4, \ldots, d_n$ is a graphic sequence, then so is also the sequence $d_1, d_2, d_3, d_4, \ldots, d_n$.

   (i) Both claims are true.

   (ii) Claim I is true, while claim II is false.

   (iii) Claim I is false, while claim II is true.

   (iv) Both claims are false.

2. Let $G = (V, E)$, where $V = \{2, 3, 4, \ldots, 30\}$, and $u, v \in V$ are adjacent if they have a non-trivial common divisor. (For example, $(12, 18) \in E$ since 6 divides both 12 and 18, but $(10, 21) \notin E$ since 1 is the only common divisor of 10 and 21.)

   (a) The clique number $\omega(G)$ of G is

      (i) 8.

      (ii) 9.

      (iii) 10.

      (iv) 11.

      (v) None of the above.

(b) The independence number $\alpha(G)$ of G is

(i) 1.

(ii) 6.

(iii) 7.

(iv) 8.

(v) None of the above.

(c) The coloring number $\chi(G)$ of G is

(i) 9.

(ii) 10.

(iii) 11.

(iv) 12.

(v) None of the above.

3. A *Latin cube* is an $n \times n \times n$ cube, at each of whose entries one of the numbers $1, 2, \ldots, n$ is written, in such a way that every pair of entries seeing each other (along lines parallel to the x-axis, the y-axis, or the z-axis) contain different numbers. Let $L_{3,n}$ be the 3-dimensional analogue of the graph L_n introduced in class in regard to Latin squares. Then the coefficient of k^{n^3-1} in $\chi(L_{3,n}, k)$ is:

(i) $-\dfrac{3n(n-1)(2n-1)^2}{8}$.

(ii) $-\dfrac{3n(n-1)^3}{2}$.

(iii) $-\dfrac{3n^2(n-1)^2}{2}$.

(iv) $-\dfrac{3n^3(n-1)}{2}$.

(v) None of the above.

4. We use the greedy algorithm to color a tree T (without the improvement whereby we order the vertices according to non-increasing degrees). Let k be the number of colors in the resulting coloring.

(i) We necessarily have $k = \chi(T)$.

(ii) We necessarily have either $k = \chi(T)$ or $k = \chi(T) + 1$. Both cases may occur.

(iii) There exists a constant C such that we necessarily have $k \leq C$, but the former two claims are false.

(iv) The gap $k - \chi(T)$ may be arbitrarily large.

(v) None of the above.

Solutions

1. Claim I is true. One way to see it is by using the Havel-Hakimi proof that the sequence $d_2 - 1, d_3 - 1, \ldots, d_{k+1} - 1, d_{k+2}, \ldots, d_n$ (where $k = d_1$) is graphic. In that proof, we have seen that there exists a graph $G = (V, E)$, with vertices $v_1, v_2, \ldots, v_n$ and corresponding degrees $d_1, d_2, \ldots, d_n$, in which v_1 neighbors the vertices $v_2, v_3, \ldots, v_{k+1}$. Removing the edge (v_1, v_2) from this graph, we obtain a graph with vertex degrees $d_1 - 1, d_2 - 1, d_3, d_4, \ldots, d_n$.

   Claim II is false. In fact, let $d_1 - 1, d_2 - 1, d_3, d_4, \ldots, d_n$ be the sequence of vertex degrees of any graph in which one of the vertices is adjacent to all others (such as K_n or S_n). Then $d_1 - 1 = n - 1$, so that $d_1 = n$, and the sequence $d_1, d_2, \ldots, d_n$ cannot possibly be graphic.

   Thus, (ii) is true.

2. (a) The set $\{2, 4, \ldots, 30\}$ is a clique (since every two of its elements have the number 2 as a common divisor). As its size is 15, we have $\omega(G) \geq 15$.

   On the other hand, if $A \subseteq V$ is any set of size 16 or more, it must contain two consecutive integers. Such integers certainly have no non-trivial common divisor, and hence A is not a clique. Hence,

$\omega(G) \leq 15$.

Altogether, $\omega(G) = 15$.

Thus, (v) is true.

(b) Two elements of V have no non-trivial divisor if and only if the sets of primes dividing them are disjoint. Now up to 30 there are 10 primes – the numbers $2, 3, 5, 7, 11, 13, 17, 19, 23, 29$. If A is an independent set, then the sets of primes attached to its elements are pairwise disjoint subsets of this set of primes. Hence, $\alpha(G) \leq 10$.

On the other hand, the set of all primes in V is certainly independent, so that $\alpha(G) \geq 10$.

Altogether, $\alpha(G) = 10$.

Thus, (v) is true.

(c) Since in general $\chi(G) \geq \omega(G)$, we have $\chi(G) \geq 15$ (which suffices to answer this question, but we will find $\chi(G)$ exactly anyway).

Recall that the improved coloring algorithm, presented in class, starts by ordering the vertices according to their degrees, so that $d(v_1) \geq d(v_2) \geq \ldots \geq d(v_n)$. It provides a coloring using at most $\max_{1 \leq i \leq n} \min\{d(v_i) + 1, i\}$ colors. In our case, the vertices $2, 4, \ldots, 30$ will appear first in the ordering, as their degrees are all at least 14. (The degrees of some are quite higher; the maximum is obtained for 30, whose degree is 21.) These vertices, forming a clique, will be colored using colors 1 through 15. However, none of the other vertices has a degree exceeding 14. In fact, the maximum for these vertices is obtained for the vertex 15, whose degree is 13. Hence we obtain this way a proper coloring using 15 colors.

Altogether, $\chi(G) = 15$.

Thus, (v) is true.

3. As $L_{3,n}$ has n^3 vertices, $\chi(L_{3,n}, k)$ is a polynomial of degree n^3. The coefficient of k^{n^3-1} is $-|E|$, where E is the number of edges of the graph. Each vertex is adjacent to $n-1$ other vertices in each of the three directions – altogether $3(n-1)$ vertices. Hence the number of edges is

$$|E| = \frac{n^3 \cdot 3(n-1)}{2} = \frac{3n^3(n-1)}{2}.$$

Thus, (iv) is true.

4. Recall that the coloring number of every tree with at least 2 vertices is 2. We claim that, for each positive integer k, there exists a tree, for which the greedy algorithm, for some ordering of the vertices, will provide a coloring by k colors.

We proceed by induction. For $k = 1$, take a tree with a single vertex. For the induction step, suppose we have already constructed trees $T_1, T_2, \ldots, T_k$, such that each T_j, for an appropriate ordering of the vertices, is colored by j colors. Take the graph $T_1 + T_2 + \ldots + T_k$. (Namely, we take the union of the sets of vertices and the union of the sets of edges of $T_1, T_2, \ldots, T_k$.) Add to this graph one vertex v. For each $j \leq k$, connect v to one of the vertices of T_j who have been colored by color j. Obviously, the graph we obtain is a tree. If we color this tree by the greedy algorithm, coloring first each T_j according to the order specified above, and finally coloring v, we obtain a coloring by $k+1$ colors. Indeed, since there are no edges between the various T_j's, when we get to color any T_j, the colors assigned to the vertices of other T_i's do not matter. In particular, the neighbors of v in the graph we have constructed are colored in all colors $1, 2, \ldots, k$, so that v will have to be colored in color $k+1$.

Thus, (iv) is true.