Final #1

Mark the correct answer in each part of the following questions.

4. Reuven and Shimon hold a backgammon match of 300 games. The probability of Reuven to win each game is 2/3 and that of Shimon is 1/3. For $k = 0, 1, \ldots, 300$, denote by X_k and Y_k the number of wins of Reuven and of Shimon, respectively, out of the first k games. (For example, if Reuven wins the first two games and Shimon wins the third, then $X_0 = 0, X_1 = 1, X_2 = X_3 = 2, Y_0 = Y_1 = Y_2 = 0, Y_3 = 1$.) Put $M = \max_{0 \le k \le 300} (X_k - Y_k)$ and $L = \min_{0 \le k \le 300} (X_k - Y_k)$.

   (a) $P(X_{100} - Y_{100} = 20) =$
      (i) $\binom{100}{20} \left(\frac{1}{3}\right)^{20} \left(\frac{2}{3}\right)^{80}$.
      (ii) $\binom{100}{20} \left(\frac{2}{3}\right)^{20} \left(\frac{1}{3}\right)^{80}$.
      (iii) $\binom{100}{40} \left(\frac{2}{3}\right)^{60} \left(\frac{1}{3}\right)^{40}$.
      (iv) none of the above.

   (b) $P(X_{300} = 150, L = 0) =$
      (i) $\frac{\binom{300}{150} \cdot 2^{150}}{3^{300}}$.
      (ii) $\frac{\binom{300}{150} \cdot 2^{150}}{151 \cdot 3^{300}}$.
      (iii) $\frac{\binom{300}{150} \cdot 2^{150}}{301 \cdot 3^{300}}$.
      (iv) none of the above.

   (c) $P(M = 0 | X_{300} = 100)$ lies in the interval:
      (i) $[0, 1/e]$.
      (ii) $(1/e, 0.5]$.
      (iii) $(0.5, 1 - 1/e]$.
      (iv) $(1 - 1/e, 1]$.

(d) $E\left((X_{300} - Y_{300})^2\right)$ is:

(i) 10000.

(ii) $\frac{30800}{3}$.

(iii) larger than 30800.

(iv) none of the above.

(e) $P(L = -300) =$

(i) $\left(\frac{2}{3}\right)^{300}$.

(ii) $\left(\frac{1}{3}\right)^{300}$.

(iii) $\frac{\binom{300}{150}}{3^{300}}$.

(iv) none of the above.

5. An urn contains a white ball and a black ball. We draw out of the urn a ball in random, and return it along with another ball of the same color. The process is continued until at least one white ball and at least one black ball have been drawn. Let X be the number of the stage at which the first white ball has been drawn and Y the number of the stage at which the first black ball has been drawn. (For example, if three times a black ball has been drawn, and returned along with another black ball, and at the fourth trial the white ball has been drawn, then $X = 4, Y = 1$.)

(a) $P(X = 100) =$

(i) 1/10100.

(ii) 1/10000.

(iii) 1/9900.

(iv) none of the above.

(b) $E(X) =$

(i) 2.

(ii) e.

(iii) ∞ (i.e., X does not have an expectation).

(iv) none of the above.

(c) $P(X = 100 \,|\, Y = 1) =$

(i) $1/100$.

(ii) $1/5050$.

(iii) $1/10000$.

(iv) none of the above.

(d) $E\left(\frac{X+1}{2^X}\right) =$

(i) $\ln 2$.

(ii) 1.

(iii) $2 \ln 2$.

(iv) none of the above.

(e) Consider the random variables X, Y.

(i) They are dependent, geometrically distributed with the same parameter.

(ii) They are independent, geometrically distributed with the same parameter.

(iii) They are dependent, identically distributed with a non-geometric distribution.

(iv) None of the above.

6. The life length (in hours) of light bulbs of type A is distributed Exp(1) and that of light bulbs of type B – Exp(1/2). We turn on first a light bulb of type A, and when it burns out turn on a light bulb of type B. Denote by T the amount of time since the first bulb is lit until the second burns out.

(a) For $t \geq 0$:

(i) $f_T(t) = \frac{1}{2}e^{-3t/2}$.

(ii) $f_T(t) = e^{-t/2} - e^{-t}$.

(iii) $f_T(t) = e^{-t} + \frac{1}{2}e^{-3/2}$.

(iv) none of the above.

(b) $E(T) =$

(i) 2.

(ii) 3.

(iii) 5.

(iv) none of the above.

(c) $V(T) =$

(i) 5.

(ii) 6.

(iii) 12.

(iv) none of the above.

(d) Suppose we light 2000 rooms in this manner, namely first by a type A light bulb, and when it burns out – by a type B bulb. For $1 \leq i \leq 2000$ denote by T_i the amount of time until the second bulb at the ith room burns out. Then:

(i) $P\left(\sum_{i=1}^{2000} T_i \geq 5900\right) \approx 0.16$.

(ii) $P\left(\sum_{i=1}^{2000} T_i \geq 5900\right) \approx 0.5$.

(iii) $P\left(\sum_{i=1}^{2000} T_i \geq 5900\right) \approx 0.84$.

(iv) none of the above.

(e) Now suppose we light one room by one type A light bulb, and when it burns out we put on simultaneously two type B bulbs. Denote the latter two light bulbs by B_1 and B_2. Let T^* and T^{**} be the time from the beginning until B_1 burns out and the time until B_2 burns out. (For example, if the first light bulb lasted one hour, B_1 lasted 6 hours and B_2 4 lasted hours, then $T^* = 7$ and $T^{**} = 5$.)

(i) $\rho(T^*, T^{**}) = -1/5$.

(ii) $\rho(T^*, T^{**}) = 0$.

(iii) $\rho(T^*, T^{**}) = 1/5$.

(iv) none of the above.

Solutions

4. (a) For an arbitrary fixed $k = 0, 1, \ldots, 300$ we have $X_k + Y_k = k$, and hence $X_k - Y_k = 2X_k - k$. Therefore

$$P(X_{100} - Y_{100} = 20) = P(2X_{100} - 100 = 20) = P(X_{100} = 60).$$

Obviously, $X_k \sim B\left(k, \frac{2}{3}\right)$ (and $Y_k \sim B\left(k, \frac{1}{3}\right)$), and hence

$$P(X_{100}-Y_{100} = 20)=\binom{100}{60}\left(\frac{2}{3}\right)^{60}\left(\frac{1}{3}\right)^{40}=\binom{100}{40}\left(\frac{2}{3}\right)^{60}\left(\frac{1}{3}\right)^{40}.$$

Thus, (iii) is true.

(b) We have:

$$P(X_{300} = 150, L = 0) = P\left(X_{300} = 150\right) P(L = 0|X_{300} = 150).$$

Clearly,

$$P\left(X_{300} = 150\right) = \binom{300}{150}\left(\frac{2}{3}\right)^{150}\left(\frac{1}{3}\right)^{150} = \frac{\binom{300}{150}2^{150}}{3^{300}}.$$

Now:

$$\{L = 0\} = \{X_k \geq Y_k,\ k = 0, 1, \ldots, 300\}.$$

Given that $X_{300} = 150$, namely that Reuven and Shimon win 150 games each, the event $L = 0$ becomes the event in the Ballot Problem, considered in class. Namely, if in the ballot the first candidate scores $m = 150$ votes and the second scores $n = 150$ votes, then the probability that throughout the counting the second candidate never leads is $\frac{m-n+1}{m+1} = \frac{1}{151}$. Therefore, $P(L = 0|X_{300} = 150) = \frac{1}{151}$. Hence, the probability of the event in question is $\frac{\binom{300}{150}2^{150}}{151 \cdot 3^{300}}$.

Thus, (ii) is true.

(c) We proceed similarly to the preceding part. The event $\{X_{300} = 100\}$ amounts to 100 wins of Reuven and 200 of Shimon. The event $\{M = 0\}$, whereby Reuven never leads, given that $X_{300} = 100$, is equivalent to the particular instance of the Ballot Problem with $m = 200$ and $n = 100$. It follows that

$$P(M = 0|Y_{300} = 200) = \frac{200 - 100 + 1}{200 + 1} = \frac{101}{201}.$$

Hence, the probability of the event in question belongs to $(0.5, 1 - 1/e]$.

Thus, (iii) is true.

(d) Employing the decomposition $X_k - Y_k = 2X_k - k$ from part (a) and the fact that $X_k \sim B(k, \frac{2}{3})$, we obtain:

$$\begin{aligned} E((X_{300} - Y_{300})^2) &= E((2X_{300} - 300)^2) \\ &= E(4X_{300}^2 - 4 \cdot 300X + 300^2) \qquad (1) \\ &= 4E(X_{300}^2) - 1200E(X_{300}) + 300^2. \end{aligned}$$

Now

$$E(X_{300}) = 300 \cdot \frac{2}{3} = 200$$

and

$$V(X_{300}) = 300 \cdot \frac{2}{3} \cdot \frac{1}{3} = \frac{200}{3},$$

so that

$$E(X_{300}^2) = V(X_{300}) + E^2(X_{300}) = \frac{120200}{3}.$$

Substituting in (1), we get $E((X_{300} - Y_{300})^2) = \dfrac{30800}{3}$.

Thus, (ii) is true.

(e) Obviously:

$$\begin{aligned} P(L = -300) &= P(2X_{300} - 300 = -300) \\ &= P(X_{300} = 0) = \left(\tfrac{1}{3}\right)^{300}. \end{aligned}$$

Thus, (ii) is true.

5. (a) For an arbitrary fixed positive integer k, the event $\{X = k\}$ occurs if and only if at the k-th trial the white ball is drawn, while in all previous trials a black ball has been drawn, (and returned along with another black ball). Therefore:

$$P(X = k) = \left(\prod_{i=1}^{k-1} \frac{i}{i+1} \right) \frac{1}{k+1} = \frac{(k-1)!}{(k+1)!} = \frac{1}{k(k+1)}. \qquad (2)$$

In particular, for $k = 100$:

$$P(X = 100) = \frac{1}{10100}.$$

Thus, (i) is true.

(b) By (2):

$$E(X) = \sum_{k=1}^{\infty} kP(X = k) = \sum_{k=1}^{\infty} \frac{k}{k(k+1)} = \sum_{k=1}^{\infty} \frac{1}{k+1} = \infty,$$

i.e., X does not have an expectation.

Thus, (iii) is true.

(c) We have:

$$P(X = 100|Y = 1) = \frac{P(X = 100, Y = 1)}{P(Y = 1)} = \frac{P(X = 100)}{P(Y = 1)}.$$

Obviously, $P(Y = 1) = 1/2$, so that by part (a):

$$P(X = 100|Y = 1) = \frac{2}{10100} = \frac{1}{5050}.$$

Thus, (ii) is true.

(d) By (2) we have:

$$
\begin{aligned}
E\left(\frac{X+1}{2^X}\right) &= \sum_{k=1}^{\infty} \frac{k+1}{2^k} P(X=k) \\
&= \sum_{k=1}^{\infty} \frac{k+1}{2^k} \cdot \frac{1}{k(k+1)} \\
&= \sum_{k=1}^{\infty} \frac{1}{k} \cdot \left(\frac{1}{2}\right)^k \\
&= -\ln(1-1/2) = \ln 2.
\end{aligned}
$$

Thus, (i) is true.

(e) Obviously, X and Y are dependent. Indeed, $P(X=1) = P(Y=1) = 1/2$, while $P(X=1, Y=1) = 0$, so that

$$P(X=1, Y=1) \neq P(X=1) \cdot P(Y=1).$$

We claim that X is not geometrically distributed. Indeed, if X was geometrically distributed, then we would have in particular

$$P(X=2) = P(X=1)(1-P(X=1)) = 1/2 \cdot 1/2 = 1/4.$$

However, by (2):

$$P(X=2) = \frac{1}{6} \neq \frac{1}{4}.$$

By symmetry, it is clear that X and Y are identically distributed.

Thus, (iii) is true.

6. (a) Denote by T_A the life length (in hours) of the first light bulb and by T_B that of the second. With these notations, $T = T_A + T_B$. Employing the fact that T_A and T_B are independent and $T_A \sim \text{Exp}(1)$ and $T_B \sim \text{Exp}(1/2)$, we obtain that the density function of T vanishes for $t < 0$, while for $t \geq 0$:

$$
\begin{aligned}
f_T(t) &= \textstyle\int_{-\infty}^{\infty} f_{T_A}(y) \cdot f_{T_B}(t-y)dy \\
&= \textstyle\int_0^t e^{-y} \cdot \frac{1}{2} e^{-\frac{1}{2}(t-y)} dy \\
&= \left[-e^{-y/2-t/2}\right]_0^t \\
&= e^{-t/2} - e^{-t}.
\end{aligned} \tag{3}
$$

Thus, (ii) is true.

(b) Employing the above decomposition of T and the formula for the expectation of an exponentially distributed random variable, we get:

$$E(T) = E(T_A) + E(T_B) = 1 + 2 = 3.$$

Thus, (ii) is true.

(c) Employing again the above decomposition, as well as the independence of T_A and T_B, we obtain:

$$V(T) = V(T_A) + V(T_B) = 1^2 + 2^2 = 5. \tag{4}$$

Thus, (i) is true.

(d) Similarly to part (a), denote by T_{A_i} the life length of the light bulb of type A, and by T_{B_i} – of the light bulb of type B, in the i-th room. With these notations, $T_i = T_{A_i} + T_{B_i}$ for $1 \le i \le 2000$. Obviously, the variables T_i, $1 \le i \le 2000$, are independent identically distributed random variables with density function given by (3). Hence $E(T_i) = 3$ and $V(T_i) = 5$ for $1 \le i \le 2000$. By the Central Limit Theorem we obtain:

$$\begin{aligned}
P\left(\sum_{i=1}^{2000} T_i \ge 5900\right) =& P\left(\frac{\sum_{i=1}^{2000} T_i - 2000E(T_1)}{\sqrt{2000V(T_1)}} \ge \frac{5900 - 2000E(T_1)}{\sqrt{2000V(T_1)}}\right) \\
=& P\left(\frac{\sum_{i=1}^{2000} T_i - 2000 \cdot 3}{\sqrt{2000 \cdot 5}} \ge \frac{5900 - 2000 \cdot 3}{\sqrt{2000 \cdot 5}}\right) \\
=& P\left(\frac{\sum_{i=1}^{2000} T_i - 2000 \cdot 3}{\sqrt{2000 \cdot 5}} \ge -1\right) \\
\approx& 1 - \Phi(-1) = \Phi(1) \approx 0.84.
\end{aligned}$$

Thus, (iii) is true.

(e) Similarly to part (a), denote by T_A the life length of the light bulb of type A, and by T'_{B_i} the life length of B_i, $\;i = 1, 2$. With these notations, $T^* = T_A + T'_{B_1}$ and $T^{**} = T_A + T'_{B_2}$. As in (4), $V(T^*) = V(T^{**}) = 5$. Due to the independence of the pair T_A and T'_{B_1}, the pair T_A and T'_{B_2}, and the pair T'_{B_1} and T'_{B_2}, we have:

$$\begin{aligned}
\mathrm{Cov}(T^*, T^{**}) &= \mathrm{Cov}(T_A + T'_{B_1}, T_A + T'_{B_2}) \\
&= \mathrm{Cov}(T_A, T_A) + \mathrm{Cov}(T_A, T'_{B_1}) \\
&\quad + \mathrm{Cov}(T_A, T'_{B_2}) + \mathrm{Cov}(T'_{B_1}, T'_{B_2}) \\
&= \mathrm{Cov}(T_A, T_A) = V(T_A) = 1.
\end{aligned}$$

It follows that

$$\rho(T^*, T^{**}) = \frac{\mathrm{Cov}(T^*, T^{**})}{\sqrt{V(T^*) \cdot V(T^{**})}} = \frac{1}{5}.$$

Note that the positivity of $\rho(T^*, T^{**})$ is very intuitive, since both variables contain the common term T_A.

Thus, (iii) is true.