

Midterm

Mark the correct answer in each part of the following questions.

1. An urn contains $2n$ cards, n of which are red and n blue. The cards of each color are marked by the numbers $1, 2, \dots, n$. We will refer to the red cards as r_1, r_2, \dots, r_n and to the blue ones as b_1, b_2, \dots, b_n . All cards are drawn from the urn one by one in a random order. For $1 \leq i \leq n$, denote by R_i the number of the stage at which r_i is drawn and by B_i the number of the stage at which b_i is drawn. Put:

$$N = |\{1 \leq i \leq n : R_i < B_i\}|.$$

Let W denote the number of even stages at which the card drawn has an even mark. (For example, if $n = 3$ and the order of drawing is $r_2, b_3, b_1, r_3, b_2, r_1$, then $R_1 = 6, R_2 = 1, R_3 = 4, B_1 = 3, B_2 = 5, B_3 = 2, N = 1, W = 0$.)

- (a) The distributions of the three random variables $B_1 - R_1, N$ and W satisfy:
 - (i) $B_1 - R_1$ is uniformly distributed, N is binomially distributed, W is hypergeometrically distributed.
 - (ii) $B_1 - R_1$ is not uniformly distributed, N is binomially distributed, W is hypergeometrically distributed.
 - (iii) $B_1 - R_1$ is not uniformly distributed, N is binomially distributed, W is not hypergeometrically distributed.
 - (iv) $B_1 - R_1$ is uniformly distributed, N is not binomially distributed, W is hypergeometrically distributed.
 - (v) $B_1 - R_1$ is uniformly distributed, N is binomially distributed, W is not hypergeometrically distributed.
 - (vi) None of the above.

- (b) Suppose $n = 5$. Then $P(N = 3 | B_3 = 7)$ is:
- (i) $1/4$.
 - (ii) $7/24$.
 - (iii) $1/3$.
 - (iv) $5/12$.
 - (v) $1/2$.
 - (vi) None of the above.
- (c) The probability that $|R_i - B_i| = 1$ for $1 \leq i \leq n$ is:
- (i) $\frac{2^n n!}{(2n)!}$.
 - (ii) $\frac{1}{n^n}$.
 - (iii) $\frac{1}{(2n-1)^n}$.
 - (iv) $\frac{1}{n!}$.
 - (v) $\frac{1}{3^{n-1}}$.
 - (vi) None of the above.
- (d) Let $n = 5$, and suppose we repeat the whole experiment 512 times. The probability that in no two of the experiments will the sequence of drawings be exactly the same is:
- (i) Very close to 0.
 - (ii) Very close to $1/2$.
 - (iii) $(1 - \frac{1}{10!})^{511}$.
 - (iv) $(1 - \frac{1}{10!})^{\binom{511}{2}}$.
 - (v) $\frac{(10!)!}{(10! - 512)! \cdot 10!^{512}}$.
 - (vi) None of the above.
- (e) Let $n = 10$, and suppose we repeat the whole experiment 512 times. Let M be the number of times we get $N = 10$. Then $P(M \leq 1)$ is approximately:
- (i) $\frac{3}{2e}$.
 - (ii) $\frac{2}{3\sqrt{e}}$.
 - (iii) $\frac{3}{4e}$.
 - (iv) $\frac{3}{2\sqrt{e}}$.
 - (v) 1.

(vi) None of the above.

2. Reuven and Shimon hold a dreidel game. A dreidel is rolled over and over until all of the letters N, G, H and P appear in this order (as a subsequence, not necessarily in succession). Let X be the number of times the dreidel is rolled, and Y the number of times until all letters appear at least once in any order. (For example, if the sequence of results is G, N, N, G, P, N, H, G, P, then at this point the game ends, $X = 9$ and $Y = 7$.) Reuven then pays Shimon $2Y - X$ shekels. (If this amount is negative, Shimon pays Reuven $X - 2Y$ shekels.)

(a) The distributions of the three random variables X, Y and $X + Y$ satisfy:

(i) X is negative binomial, Y cannot be expressed as a sum of geometric random variables, $X + Y$ can be expressed as a sum of geometric random variables.

(ii) X is not negative binomial, Y and $X + Y$ can be expressed as sums of geometric random variables.

(iii) X is not negative binomial, Y can be expressed as a sum of geometric random variables, $X + Y$ cannot be expressed as a sum of geometric random variables.

(iv) X is negative binomial, Y and $X + Y$ can be expressed as sums of geometric random variables.

(v) X is negative binomial, Y and $X + Y$ are binomial.

(vi) None of the above.

(b) $P(X = 4|Y = 4)$ is:

(i) $1/48$.

(ii) $1/24$.

(iii) $1/16$.

(iv) $1/12$.

(v) $1/6$.

(vi) None of the above.

(c) $P(Y = 4|X = 5)$ is:

- (i) $1/18$.
 - (ii) $1/12$.
 - (iii) $1/6$.
 - (iv) $1/4$.
 - (v) $1/3$.
 - (vi) None of the above.
- (d) The expected amount Reuven pays Shimon (taking into account possible negative payments) lies in:
- (i) $(-\infty, -2)$.
 - (ii) $[-2, 0]$.
 - (iii) $(0, 1/2]$.
 - (iv) $(1/2, 5)$.
 - (v) $[5, 10]$.
 - (vi) None of the above.
- (e) Consider the distribution functions F_X and F_Y . We have:
- (i) $F_X(4) = 1/24, F_X(5) = 41/768, F_Y(4) = 3/32, F_Y(5) = 15/64$.
 - (ii) $F_X(4) = 1/12, F_X(5) = 73/768, F_Y(4) = 3/32, F_Y(5) = 15/64$.
 - (iii) $F_X(4) = 1/48, F_X(5) = 25/768, F_Y(4) = 5/32, F_Y(5) = 19/64$.
 - (iv) $F_X(4) = 1/24, F_X(5) = 41/768, F_Y(4) = 7/32, F_Y(5) = 23/64$.
 - (v) $F_X(4) = 1/18, F_X(5) = 37/768, F_Y(4) = 1/32, F_Y(5) = 9/64$.
 - (vi) None of the above.
3. An experiment with a random outcome X is performed, where X may assume non-negative integer values only. Then a die is tossed X times. Let Y be the number of times the die shows “6”.
- (a) If $X \sim B(n, p)$, then for $m = 0, 1, \dots, n$:

- (i) $P(Y = m) = \binom{n}{m} \left(\frac{p}{6}\right)^m \left(1 - \frac{5p}{6}\right)^{n-m}$.
- (ii) $P(Y = m) = \binom{n}{m} p^m (1 - p)^{n-m}$.
- (iii) $P(Y = m) = \binom{n}{m} \left(\frac{p}{6}\right)^m \left(1 - \frac{p}{6}\right)^{n-m}$.
- (iv) $P(Y = m) = \binom{n}{m} \left(\frac{p}{6}\right)^m (1 - p)^{n-m}$.
- (v) $P(Y = m) = \binom{n}{m} \left(\frac{p}{6}\right)^m \left(\frac{5(1-p)}{6}\right)^{n-m}$.
- (vi) None of the above.

(b) If $X \sim U[a, b]$ (discrete uniform), where $b \geq a \geq 1$, then the number of discontinuity points of F_Y is:

- (i) $a + 1$.
- (ii) $b + 1$.
- (iii) $b - a + 1$.
- (iv) $b - a$.
- (v) $a + b + 1$.
- (vi) None of the above.

(c) If $X \sim G(p)$ then:

- (i) $P(Y = 10) = \frac{6p(1-p)^{11}}{5^{10}(5+p)^{11}}$.
- (ii) $P(Y = 10) = \frac{6p(1-p)^{10}}{5^{10}(5+p)^{10}}$.
- (iii) $P(Y = 10) = \frac{6p(1-p)^{11}}{5^{10}(5+p)^{10}}$.
- (iv) $P(Y = 10) = \frac{6p(1-p)^{10}}{5^{11}(5+p)^{11}}$.
- (v) $P(Y = 10) = \frac{6p(1-p)^{11}}{5^{11}(5+p)^{11}}$.
- (vi) None of the above.

(d) If X itself is the result of one toss of a die, then the probability that at the stage of tossing the die X times all results between 1 and 6 will be obtained is:

- (i) $1/720$.
- (ii) $5/324$.
- (iii) $1/64$.
- (iv) $3/32$.
- (v) $5/54$.
- (vi) None of the above.

- (e) If $X \sim P(\lambda)$ then the probability of obtaining the same outcome at all X tosses of the die is:
- (i) $e^{-\lambda/6} - e^{-\lambda}$.
 - (ii) $e^{-\lambda/6} - e^{-\lambda}/6$.
 - (iii) $e^{-5\lambda/6} - e^{-\lambda}$.
 - (iv) $6e^{-5\lambda/6} - 5e^{-\lambda}$.
 - (v) $7e^{-\lambda/6} - 6e^{-\lambda}$.
 - (vi) None of the above.

Solutions

1. (a) Let us start with the distribution of $B_1 - R_1$. We claim that $B_1 - R_1$ is not uniformly distributed. In fact, $B_1 - R_1$ assumes $4n - 2$ possible values, namely, $-(2n - 1), -(2n - 2), \dots, -1, 1, \dots, 2n - 2, 2n - 1$. Given the two extreme value, $-(2n - 1)$ and $2n - 1$, if $B_1 - R_1$ was uniformly distributed we would have $B_1 - R_1 \sim U[-(2n - 1), (2n - 1)]$. Since the value 0 is not assumed, the distribution is not uniform. (We mention in passing that the values of $B_1 - R_1$ does assume, are not assumed with equal probabilities. For example, $P(B_1 - R_1 = 2n - 1)$ is the probability that the first card to be drawn is r_1 and the last is b_1 . Therefore:

$$P(B_1 - R_1 = 2n - 1) = \frac{(2n - 2)!}{(2n)!} = \frac{1}{2n(2n - 1)}.$$

Thus $P(B_1 - R_1 = 2n - 1) \neq \frac{1}{4n - 2}$, so that $B_1 - R_1$ does not assume all possible values with equal probabilities.)

Obviously, N is a binomially distributed random variable. Indeed, by the symmetry:

$$P(R_i < B_i) = P(R_i > B_i) = 1/2.$$

Refer to the events $\{R_i < B_i\}$ as “successes”. In these terms, N is the number of successes out of n “trials”. Obviously, the events

$\{R_i < B_i\}$, $1 \leq i \leq n$, are independent, and hence $N \sim B(n, \frac{1}{2})$.

Next we claim that W is hypergeometrically distributed. Indeed, there are $2\lfloor \frac{n}{2} \rfloor$ cards with even marks and $2\lceil \frac{n}{2} \rceil$ with odd ones. We pick n of the cards to be drawn at even stages. Hence $W \sim H(n, 2\lfloor \frac{n}{2} \rfloor, 2\lceil \frac{n}{2} \rceil)$.

Thus, (ii) is true.

(b) By the law of total probability we have:

$$\begin{aligned} P(N = 3|B_3 = 7) &= P(R_3 > 7|B_3 = 7)P(N = 3|B_3 = 7, R_3 > 7) \\ &\quad + P(R_3 < 7|B_3 = 7)P(N = 3|B_3 = 7, R_3 < 7) \\ &= \frac{3}{9} \cdot P(N = 3|B_3 = 7, R_3 > 7) \\ &\quad + \frac{6}{9} \cdot P(N = 3|B_3 = 7, R_3 < 7) \\ &= \frac{3}{9} \cdot \frac{\binom{n-1}{3}}{2^{n-1}} + \frac{6}{9} \cdot \frac{\binom{n-1}{2}}{2^{n-1}} \\ &= \frac{3}{9} \cdot \frac{4}{2^4} + \frac{6}{9} \cdot \frac{6}{2^4} = \frac{1}{3}. \end{aligned}$$

Thus, (iii) is true.

(c) Clearly, the event in question occurs if and only if each two cards marked by the same number are drawn in succession from the urn. Denote this event by A . The possibilities comprising A differ in the order of the numbers of the cards drawn and the order between r_i and b_i for each i , and therefore:

$$P(A) = \frac{2^n n!}{(2n)!}.$$

Thus, (i) is true.

(d) There are $10!$ possible sequence of drawings. The probability that

all 512 sequences are distinct is:

$$\begin{aligned}
P &= \prod_{i=0}^{511} \frac{10! - i}{10!} \\
&= \frac{1}{10!^{512}} \cdot \prod_{i=0}^{511} (10! - i) \\
&= \frac{1}{10!^{512}} \cdot \frac{(\prod_{i=0}^{511} (10! - i)) (10! - 512)!}{(10! - 512)!} \\
&= \frac{(10!)!}{10!^{512} (10! - 512)!}.
\end{aligned}$$

Thus, (v) is true.

- (e) First, we consider $P(N = 10)$. By (a) we have $N \sim B(n, \frac{1}{2})$, which, in particular, yields:

$$P(N = 10) = \frac{1}{2^{10}}.$$

Obviously, $M \sim B(512, p)$, where $p = P(N = 10) = \frac{1}{2^{10}}$. It follows that M is approximately Poisson distributed with parameter $\lambda = 512 \cdot 1/2^{10} = 1/2$. Hence:

$$P(M \leq 1) = P(M = 0) + P(M = 1) \approx e^{-\frac{1}{2}} + \frac{1}{2}e^{-\frac{1}{2}} = \frac{3}{2}e^{-\frac{1}{2}}.$$

Thus, (iv) is true.

2. (a) Define a success as a roll in which the result is the letter required to appear now (N in the beginning, G after N has appeared, and so forth). The probability of a success at each roll (until all letters appeared in the required order) is $\frac{1}{4}$, and X is the number of drawings until 4 successes are obtained. Therefore $X \sim \bar{B}(4, \frac{1}{4})$.

For $i = 0, 1, 2, 3, 4$, let W_i be the number of rolls until i distinct letters appear (where $W_0 = 0$). Clearly, $W_i - W_{i-1} \sim G(\frac{5-i}{4})$ for $1 \leq i \leq 4$. Now

$$Y = (W_1 - W_0) + (W_2 - W_1) + (W_3 - W_2) + (W_4 - W_3),$$

so that Y is a sum of 4 geometrically distributed random variables. Finally, since $X \sim \overline{B}\left(4, \frac{1}{4}\right)$, it can be also considered as a sum of 4 geometrically distributed random variables, and hence $X + Y$ can be expressed as a sum of 8 geometric random variables.

Thus, (iv) is true.

- (b) The event $\{Y = 4\}$ consists of $4! = 24$ possibilities, corresponding to all permutations of N,G,H,P. Out of these, only for the identity permutation we have $X = 4$. Hence:

$$P(X = 4|Y = 4) = \frac{1}{24}.$$

Thus, (ii) is true.

- (c) Suppose that $\{X = 5\}$. Under this condition we have a new sample space Ω' . All outcomes in Ω' have a P at the fifth roll. Splitting them into two subsets depending on whether the letter at the second roll is N or not, we see that $\Omega' = \{w_1, w_2, \dots, w_{12}\}$, where:

$$\begin{aligned} w_1 &= \text{NGGHP}, & w_4 &= \text{NGPHP}, & w_7 &= \text{NHGHP}, & w_{10} &= \text{GNGHP}, \\ w_2 &= \text{NGHGP}, & w_5 &= \text{NGHNP}, & w_8 &= \text{NPGHP}, & w_{11} &= \text{HNGHP}, \\ w_3 &= \text{NGHHP}, & w_6 &= \text{NGNHP}, & w_9 &= \text{NNGHP}, & w_{12} &= \text{PNGHP}. \end{aligned}$$

Therefore:

$$P(Y = 4|X = 5) = P(\{w_4, w_8, w_{12}\}|X = 5) = \frac{3}{12} = \frac{1}{4}.$$

Thus, (iv) is true.

- (d) Employing the decomposition of Y from part (a) and the fact that $X \sim \overline{B}\left(4, \frac{1}{4}\right)$ we obtain:

$$\begin{aligned}
E(2Y - X) &= 2E(Y) - E(X) \\
&= 2 \sum_{i=1}^4 E(W_i - W_{i-1}) - E(X) \\
&= 2 \sum_{i=1}^4 \frac{4}{5-i} - \frac{4}{1/4} = \frac{2}{3}.
\end{aligned}$$

Thus, (iv) is true.

(d) By part (a) we have $X \sim \bar{B}(4, \frac{1}{4})$, and in particular

$$\begin{aligned}
P(X = 4) &= \frac{1}{4^4} = \frac{1}{256}, \\
P(X = 5) &= 4 \cdot \frac{3}{4^5} = \frac{3}{256}.
\end{aligned}$$

Hence

$$F_X(4) = P(X = 4) = \frac{1}{256}$$

and

$$F_X(5) = P(X = 4) + P(X = 5) = \frac{1}{64}.$$

Since the possibilities comprising the event $\{Y = 4\}$ differ in the order of the letters, we have:

$$P(Y = 4) = \frac{4!}{4^4} = \frac{3}{32}.$$

Clearly, in the outcomes comprising $\{Y \leq 5\}$, one of the letters N, G, H, P appears exactly twice in the first five rolls, while the other three letters appear exactly one time. Therefore:

$$P(Y \leq 5) = \frac{4 \cdot \binom{5}{2} \cdot 3!}{4^5} = \frac{15}{64},$$

where the factor 4 in the numerator indicates the number of possibilities for choosing the letter appearing twice, $\binom{5}{2}$ – at which

rolls this letter was obtained, and $3!$ corresponds to the different orders of the other three letters. Hence

$$F_Y(4) = P(Y = 4) = \frac{3}{32},$$

and

$$F_Y(5) = P(Y \leq 5) = \frac{15}{64}.$$

Thus, (vi) is true.

3. (a) By the law of total probability, for each $m = 0, 1, \dots, n$, we have:

$$\begin{aligned} P(Y = m) &= \sum_{k=m}^n \binom{n}{k} p^k (1-p)^{n-k} \cdot \binom{k}{m} \left(\frac{1}{6}\right)^m \left(\frac{5}{6}\right)^{k-m} \\ &= \binom{n}{m} \left(\frac{p}{6}\right)^m \sum_{k=m}^n \binom{n-m}{k-m} \left(\frac{5p}{6}\right)^{k-m} (1-p)^{n-m-(k-m)} \\ &= \binom{n}{m} \left(\frac{p}{6}\right)^m \left(1-p + \frac{5p}{6}\right)^{n-m} \\ &= \binom{n}{m} \left(\frac{p}{6}\right)^m \left(1 - \frac{p}{6}\right)^{n-m}. \end{aligned}$$

One can also reach the same conclusion almost without computations. In fact, since the die is tossed as many times as we succeed in the first stage of the experiment, we may view the whole experiment as follows: We perform a sequence of n independent trials with success probability p in each, and each time we succeed we also toss a die. Now Y is the number of times we both succeed and the die shows a “6”. In other words, considering as successes only those trials where we both have a success in terms of the first stage of the experiment and the die shows a “6”, we perform a sequence of n independent trials with success probability $p/6$ in each. Hence $Y \sim B(n, p/6)$, so that $P(Y = m) = \binom{n}{m} \left(\frac{p}{6}\right)^m \left(1 - \frac{p}{6}\right)^{n-m}$ for each m .

Thus, (iii) is true.

- (b) Since X assumes a value between a and b , the number of times the die shows a “6” is between 0 and b . Hence F_Y is discontinuous at the points $0, 1, \dots, b$.

Thus, (ii) is true.

- (c) By the law of total probability, for each $m \geq 0$ we have:

$$\begin{aligned} P(Y = m) &= \sum_{n=1}^{\infty} (1-p)^{n-1} p \cdot \binom{n}{m} \left(\frac{1}{6}\right)^m \left(\frac{5}{6}\right)^{n-m} \\ &= \frac{p}{(1-p) \cdot 5^m} \sum_{n=1}^{\infty} \binom{n}{m} \left(\frac{5(1-p)}{6}\right)^n. \end{aligned}$$

Consequently,

$$P(Y = 0) = \frac{p}{1-p} \cdot \frac{\frac{5(1-p)}{6}}{1 - \frac{5(1-p)}{6}} = \frac{5p}{1+5p},$$

and for $m \geq 1$

$$P(Y = m) = \frac{p}{(1-p)5^m} \cdot \frac{\left(\frac{5(1-p)}{6}\right)^m}{\left(1 - \frac{5(1-p)}{6}\right)^{m+1}} = \frac{6p(1-p)^{m-1}}{(1+5p)^{m+1}}.$$

In particular:

$$P(Y = 10) = \frac{6p(1-p)^9}{(1+5p)^{11}}.$$

Thus, (vi) is true.

- (d) We cannot get all the results between 1 and 6 at the second stage unless the first die shows a “6”. In case it does, there are at the second stage 6^6 possible sequences of results, out of which only $6!$ satisfy the required condition. Hence the probability of the event in question is

$$\frac{1}{6} \cdot \frac{6!}{6^6} = \frac{5}{1944}.$$

Thus, (vi) is true.

- (e) If X assumes the value 0, the condition is trivially satisfied. If X assumes a positive value, then the event in question occurs if all $X - 1$ tosses of the die after the first toss have the same outcome as the first. Hence the required probability is

$$\begin{aligned} P &= \frac{\lambda^0}{0!} e^{-\lambda} + \sum_{k=1}^{\infty} \frac{\lambda^k}{k!} e^{-\lambda} \frac{6}{6^k} = e^{-\lambda} + 6 \sum_{k=1}^{\infty} \frac{(\lambda/6)^k}{k!} e^{-\lambda} \\ &= e^{-\lambda} + 6e^{-\lambda}(e^{\lambda/6} - 1) = 6e^{-5\lambda/6} - 5e^{-\lambda}. \end{aligned}$$

Thus, (iv) is true.