Final #2 – Part I

Mark the correct answer in each part of the following questions.

1. An urn contains n balls, enumerated by $1, 2, \ldots, n$. The balls are drawn out of the urn one by one without replacement, in a random order, as follows. At the first stage, we continue drawing until we draw ball #1. At the second stage, we draw until we get the ball with the minimal number out of those remaining in the urn after the first stage. At the third stage, we draw until we get the ball with the minimal number out of those remaining in the urn after the first two stages, and so forth. For $1 \leq i \leq n$, let A_i denote the event whereby ball $\#i$ is the last to be drawn at one of the stages. For $1 \leq j \leq n$, let X_j denote the number of balls drawn at stage j (if the process has not ended previously; if it did, $X_j = 0$). Let T be the number of stages until the process ends. (For example, suppose that $n = 10$, at the first stage the balls drawn were $5, 8, 2, 3, 1$, at the second stage – $10, 4$, and at the third stage – $9, 7, 6$. Then A_1, A_4, and A_6 occur, while $A_2, A_3, A_5, A_7, A_8, A_9$, and A_{10} do not; $X_1 = 5, X_2 = 2, X_3 = 3, X_4 = X_5 = \ldots = X_{10} = 0$, and $T = 3$.)

   (a) For $1 \leq i \leq n$, we have $P(A_j) =$

   (i) $1/i!$.

   (ii) $1/2^{i-1}$.

   (iii) $1/i$.

   (iv) $1/2$.

   (v) none of the above.

   (b) $V(X_1)$ is

   (i) exactly $n/4$ for each n.

   (ii) approximately en for large n.

(iii) approximately πn for large n.

(iv) exactly $(n^2 - 1)/12$ for each n.

(v) none of the above.

(c) For $1 \leq k \leq n-1$ we have $P(X_2 = k) =$

(i) $\frac{1}{n}\sum_{r=k+1}^{n-1}\frac{1}{r}$.

(ii) $\frac{1}{n}\sum_{r=k}^{n-1}\frac{1}{r}$.

(iii) $\frac{1}{n}\sum_{r=k+1}^{n}\frac{1}{r}$.

(iv) $\frac{1}{n}\sum_{r=k}^{n}\frac{1}{r}$.

(v) none of the above.

(d) $E(T|X_1 = n-2) =$

(i) 2.

(ii) 5/2.

(iii) 3.

(iv) 7/2.

(v) none of the above.

(e) As $n \to \infty$

(i) $E(T)$ is bounded above but does not converge.

(ii) $E(T)$ converges to a finite limit.

(iii) $E(T) \to \infty$ but $\frac{E(T)}{n} \to 0$.

(iv) $\frac{E(T)}{n} \to \alpha$, where $0 < \alpha < 1$.

(v) None of the above.

2. In this question we consider a version of the problem regarding the queue at the cinema house, as follows. In the beginning of the process, the cashier has no change. A ticket costs 50 shekels. Throughout the generations, an infinite number of people will arrive. Each of them will give the cashier either a 50 shekel bill with a probability of 3/4 or a 100 shekel bills with probability 1/4.

(a) Let Y be the number of people, out of the first 1000 people to visit, who will give the cashier a 100 shekel bill, and Z the number of people in that group who will give a 50 shekel bill. Then $P(Z - Y = 200) =$

(i) $\binom{1000}{200} \cdot \frac{3^{200}}{4^{1000}}$.

(ii) $\binom{1000}{400} \cdot \frac{3^{400}}{4^{1000}}$.

(iii) $\binom{1000}{600} \cdot \frac{3^{600}}{4^{1000}}$.

(iv) $\binom{1000}{800} \cdot \frac{3^{800}}{4^{1000}}$.

(v) none of the above.

(b) Let T be a discrete uniformly distributed random variable, $T \sim U[1, 199]$. Let X be the number of people, out of the first T visitors, who give a 100 shekel bill. Then $\rho(T, X) \in$

(i) $[-1, -0.6]$.

(ii) $(-0.6, -0.2]$.

(iii) $(-0.2, 0.2]$.

(iv) $(0.2, 0.6]$.

(v) $(0.6, 1]$.

(c) The probability that the cashier will never get a 100 shekel bill when there is no 50 shekel bill to give as change is

(i) 5/12.

(ii) 1/2.

(iii) 2/3.

(iv) 3/4.

(v) none of the above.

Solutions

1. (a) Clearly, $|\Omega| = n!$. The event A_i takes place if all balls marked by the numbers smaller than i are drawn prior to ball number i. By symmetry, each of the balls $1, 2, \ldots, i$ may be the last to appear among these balls, whence $P(A_i) = \frac{1}{i}$.

   Let us prove this equality also in a more computational way. Suppose that ball number i will be chosen at the j'th drawing for some $i \leq j \leq n$. Then, out of the $j-1$ drawings prior to that, the $i-1$ smaller-numbered balls had to be chosen, where the order matters, yielding $\binom{j-1}{i-1}(i-1)!$ possibilities. The remaining $j-i$ drawings will be of arbitrary balls whose numbers are larger than i, yielding $\binom{n-i}{j-i}(j-i)!$ possibilities. The rest of the balls provide $(n-j)!$ possibilities. Therefore, by the product principle, the required probability is

$$\begin{aligned} P(A_i) &= \sum_{j=i}^{n} \frac{\binom{j-1}{i-1}(i-1)! \cdot \binom{n-i}{j-i}(j-i)! \cdot (n-j)!}{n!} \\ &= \sum_{j=i}^{n} \frac{(n-i)!(j-1)!}{n!(j-i)!} = \frac{(n-i)!(i-1)!}{n!} \sum_{j=i}^{n} \binom{j-1}{i-1} \qquad (1) \\ &= \frac{(n-i)!(i-1)!}{n!} \sum_{j=i-1}^{n-1} \binom{j}{i-1}. \end{aligned}$$

   Recall that

$$\sum_{j=i-1}^{n-1} \binom{j}{i-1} = \binom{n}{i}. \qquad (2)$$

   Now, substituting (2) into (1), we obtain:

$$P(A_i) = \frac{(n-i)!(i-1)!}{n!} \binom{n}{i} = \frac{1}{i}, \quad 1 \leq i \leq n.$$

   Thus, (iii) is true.

(b) Obviously $X_1 \sim U[1,n]$, and therefore $V(X_1) = (n^2-1)/12$. Thus, (iv) is true.

(c) By the law of total probability we obtain:

$$\begin{aligned} P(X_2 = k) &= \sum_{i=1}^{n} P(X_2 = k | X_1 = i) P(X_1 = i) \\ &= \frac{1}{n} \sum_{i=1}^{n-k} P(X_2 = k | X_1 = i). \end{aligned}$$

Since, $X_2|_{X_1=i} \sim U[1, n-i]$ we have:

$$P(X_2 = k) = \frac{1}{n} \sum_{i=1}^{n-k} \frac{1}{n-i} = \frac{1}{n} \sum_{r=k}^{n-1} \frac{1}{r}.$$

Thus, (ii) is true.

(d) The event $X_1 = n-2$ means that only two balls remain un the urn after stage 1. The event $T = 2$ occurs in this case if the $(n-1)$-st drawing is of the ball marked by the larger of the two remaining numbers, and the event $T = 3$ – if the $(n-1)$-st drawing is of the ball marked by the smaller of the numbers.

Hence $T|_{X_1=n-2} \sim U[2,3]$, and in particular $E(T|_{X_1=n-2}) = 5/2$. Thus, (ii) is true.

(e) For each i, $1 \le i \le n$, define a random variable Y_i by:

$$Y_i = \begin{cases} 1, & A_i \text{ will take place,} \\ 0, & \text{otherwise.} \end{cases}$$

In these terms,

$$T = Y_1 + Y_2 + \ldots + Y_n.$$

Therefore

$$E(T) = \sum_{i=1}^{n} E(Y_i) = \sum_{i=1}^{n} P(A_i) = \sum_{i=1}^{n} \frac{1}{i} \approx \ln n,$$

which converges to ∞ for $n \to \infty$. However, $\dfrac{E(T)}{n} \approx \dfrac{\ln n}{n} \underset{n\to\infty}{\longrightarrow} 0$. Thus, (iii) is true.

2. (a) Obviously $Z \sim B(1000, 0.75)$ and $Y = 1000 - Z$. Therefore, the required probability is

$$\begin{aligned} P(Z - Y = 200) &= P(2Z - 1000 = 200) = P(Z = 600) \\ &= \binom{1000}{600}\left(\frac{3}{4}\right)^{600}\left(\frac{1}{4}\right)^{400} = \binom{1000}{600}\cdot\frac{3^{600}}{4^{1000}}. \end{aligned}$$

Thus, (iii) is true.

(b) First, note that since $T \sim U[1, 199]$, we have $E(T) = 100$ and $V(T) = \frac{199^2-1}{12}$. Moreover, obviously $X|_T \sim B(T, 1/4)$ and in particular $E(X|T) = T/4$ and $E(X^2|T) = V(X|T) + E^2(X|T) = 3/16 \cdot T + 1/16 \cdot T^2$.
Therefore,

$$E(X) = E(E(X|T)) = E(T/4) = 1/4 \cdot E(T) = 25,$$

and

$$\begin{aligned} V(X) &= E(X^2) - E^2(X) = E(E(X^2|T)) - E^2(X) \\ &= 3/16 \cdot E(T) + 1/16 \cdot E(T^2) - E^2(X), \\ &= 3/16 \cdot E(T) + 1/16 \cdot E(T^2) - 1/16 \cdot E^2(T) \\ &= 3/16 \cdot E(T) + 1/16 \cdot V(T). \end{aligned}$$

Thus:

$$\begin{aligned} \mathrm{Cov}(X, T) &= E(X \cdot T) - E(X)E(T) \\ &= E(E(X \cdot T|T)) - E(X)E(T) \\ &= E(T \cdot T/4) - E(X)E(T) \\ &= 1/4 \cdot E(T^2) - E(X)E(T) \\ &= 1/4 \cdot (V(T) + E^2(T)) - 1/4 \cdot E(T)E(T) = V(T)/4. \end{aligned}$$

Therefore

$$\begin{aligned} \rho(X, T) &= \frac{\mathrm{Cov}(X, T)}{\sqrt{V(T) \cdot V(X)}} \\ &= \frac{1/4 \cdot V(T)}{\sqrt{V(T) \cdot (3/16 \cdot E(T) + 1/16 \cdot V(T))}} \\ &= \frac{V(T)}{\sqrt{V^2(T) + 300V(T)}} \\ &= \frac{1}{\sqrt{1 + \frac{300}{V(T)}}} = \sqrt{\frac{11}{12}}. \end{aligned}$$

Thus, (v) is true.

(c) The problem is equivalent to that of the drunkard walk, where the probability of moving right is $p = 3/4$, and the question is about the probability of the drunkard to never get to the left of his home. Hence the required probability is $1 - \frac{1-p}{p} = 1 - 1/3 = 2/3$.
Thus, (iii) is true.