Final #2

Mark the correct answer in each part of the following questions.

1. In the dwarf planet Pluto, each birth event follows a distribution where the probability of a single boy being born, that of twin boys being born, and that of triplet girls being born are $\frac{1}{3}$ each. Yesterday, there were 10 births recorded in Soroka of Pluto. Let X and Y denote the numbers of boys and girls, respectively, born in Soroka on that day.

   (a) The number of pairs (x, y), for which $P_{X,Y}(x, y) > 0$, is:

   (i) $\binom{9}{2}$.
   (ii) $\binom{10}{2}$.
   (iii) $\binom{11}{2}$.
   (iv) $\binom{12}{2}$.
   (v) None of the above.

   (b) $P(Y = 21) =$

   (i) $\frac{8\cdot\binom{10}{7}}{3^{10}}$.
   (ii) $\frac{16\cdot\binom{10}{7}}{3^{10}}$.
   (iii) $\frac{8\cdot\binom{10}{7}}{3^{9}}$.
   (iv) $\frac{16\cdot\binom{10}{7}}{3^{9}}$.
   (v) None of the above.

   (c) $P(Y = 3|X = 18) =$

   (i) $\frac{2}{7}$.
   (ii) $\frac{2}{9}$.
   (iii) $\frac{2}{11}$.

(iv) $\frac{2}{13}$.

(v) None of the above.

(d) $\rho(X, Y) =$

(i) -1.

(ii) $-\frac{\sqrt{3}}{2}$.

(iii) $-\frac{\sqrt{2}}{2}$.

(iv) $-\frac{\sqrt{3}}{3}$.

(v) None of the above.

(e) The Megiddo family plans to have children until they get a girls' triplet. Let Z denote the number of boys they will have. Then $E(Z) =$

(i) 1.

(ii) 2.

(iii) 3.

(iv) 4.

(v) None of the above.

2. We toss $2n$ coins and roll n dice. Let X be the number of heads the coins show and Y the number of 6-s the dice show. Put $Z = X - Y$. (For example, if $n = 2$, the coins show T, H, H, T, and the dice show 2, 6, then $X = 2, Y = 1, Z = 1$.)

(a) $P(X = 2n | Z = 2n - 1) =$

(i) 1/11.

(ii) 1/7.

(iii) 1/5.

(iv) 1/3.

(v) None of the above.

(b) $V(Z) =$

(i) $\frac{19}{36} \cdot n$.

(ii) $\frac{21}{36} \cdot n$.

(iii) $\frac{23}{36} \cdot n$.

(iv) $\frac{25}{36} \cdot n$.

(v) None of the above.

3. The Levy family gives their son Yossi every year for his birthday a money gift, distributed $U[1001, 2500]$ (in shekels). Out of this amount, Yossi gives his little sister Ruti a third and an additional 100 shekels. (For example, if Yossi gets 1500 shekels, he gives Ruti 600 shekels out of it.) Let X be the amount of money Yossi will keep for himself within 10 years and Y the amount Ruti will get. (For example, if Yossi gets every year 1500 shekels, then $X = 9000, Y = 6000$.)

   (a) The probability that, in nine out of the next 10 years, Yossi will get between 2001 and 2500 shekels each year, and in one of the years between 1001 and 2000, is:

      (i) $\frac{20}{3^{10}}$.

      (ii) $\frac{40}{3^{10}}$.

      (iii) $\frac{80}{3^{10}}$.

      (iv) $\frac{160}{3^{10}}$.

      (v) None of the above.

   (b) $\rho(X, Y)$ is

      (i) Strictly negative.

      (ii) 0.

      (iii) Between $1/e$ and $1/2$.

      (iv) 1.

      (v) None of the above.

4. An urn contains initially n white balls and n black balls, where n is some fixed positive integer. In an infinite-stage experiment, at each stage we draw a random ball from the urn and put there a black ball instead. For every $m \geq 0$, let X_m denote the number of white balls in the urn after m stages.

   (a) $P(X_n = 0) =$

(i) $\frac{1}{2^n}$.

(ii) $\frac{n!}{(2n)^n}$.

(iii) $\frac{n!}{(2n)!}$.

(iv) $\frac{1}{\binom{2n}{n}}$.

(v) None of the above.

(b)

(i) X_{2n} is not distributed according to any of the distributions presented in class. We have

$$E(X_{2n}) = n\left(1 - \frac{1}{2n}\right)^{2n}.$$

(ii) $X_{2n} \sim B\left(n, \left(1 - \frac{1}{2n}\right)^{2n}\right)$.

(iii) $X_{2n} \sim U[0, n]$.

(iv) If n is even, then $X_{n/2}$ is distributed as $Y + n/2$, where $Y \sim H(n/2, n, n)$.

(v) None of the above.

(c) Let Z be the smallest integer k such that $X_k = n-1$. (For example, if the first three balls drawn from the urn are black and the fourth is white, then $Z = 4$.

(i) $Z \sim G(1/2)$.

(ii) $Z \sim B(2n, 1/2)$.

(iii) $Z \sim \bar{B}(2n, 1/2)$.

(iv) $Z \sim H(n, n, n)$.

(v) None of the above.

Solutions

1. (a) The number k of births of triplets of girls may be any integer between 0 and 10, and the corresponding number of girls is $3k$. The number of boys in this case is anywhere between $10-k$ and $2(10-k)$, which is $11-k$ possibilities. Hence the required number of pairs is

$$\begin{aligned}(11-0)+(11-1)+\cdots+(11-10) &= 1+2+\cdots+11\\ &= \binom{12}{2}.\end{aligned}$$

Thus, (iv) is true.

(b) We have $Y=21$ when the number of births of girl triplets is 7. Now the number of such births is clearly distributed $B(10,1/3)$, and hence

$$P(Y=21)=\binom{10}{7}\left(\frac{1}{3}\right)^{7}\left(1-\frac{1}{3}\right)^{10-7}=\frac{8\cdot\binom{10}{7}}{3^{10}}.$$

Thus, (i) is true.

(c) The event $\{X=18\}$ occurs when there are either nine births of twin boys and one of a girls triplet, or eight births of twin boys and two of single boys. It follows that:

$$\begin{aligned}P(Y=3|X=18) &= \tfrac{P(Y=3,X=18)}{P(X=18)}\\ &= \tfrac{\binom{10}{1}(1/3)^{10}}{\binom{10}{1}(1/3)^{10}+\binom{10}{2}(1/3)^{10}}\\ &= \tfrac{2}{11}.\end{aligned}$$

Thus, (iii) is true.

(d) Let X_i and Y_i denote the number of the boys and of the girls, respectively, born in the i-th birth. Obviously, $X_i\sim U[0,2]$ and $Y_i=3Y_i'$, where $Y_i'\sim B(1,1/3)$, for $1\le i\le 10$. The only dependencies between these variables are of X_i and Y_i (with the same index i). Since $X = \sum_{i=1}^{10} X_i$ and $Y = \sum_{i=1}^{10} Y_i$, we have:

$$\begin{aligned} E(X) &= 10 \cdot \tfrac{0+2}{2} = 10, \\ V(X) &= 10 \cdot \tfrac{(2-0+1)^2-1}{12} = \tfrac{20}{3}, \\ E(Y) &= 10 \cdot 3 \cdot 1 \cdot \tfrac{1}{3} = 10, \\ V(Y) &= 10 \cdot 3^2 \cdot \tfrac{1}{3} \cdot \left(1 - \tfrac{1}{3}\right) = 20. \end{aligned}$$

Now:

$$\begin{aligned} E(XY) &= E\left(\textstyle\sum_{i=1}^{10} \sum_{j=1}^{10} X_i Y_j\right) \\ &= \textstyle\sum_{i=1}^{10} \sum_{j=1}^{10} E\left(X_i Y_j\right) \\ &= \textstyle\sum_{i \neq j} E(X_i)E(Y_j) + \sum_{i=1}^{10} E(X_i Y_i) \\ &= 10 \cdot 9 \cdot 1 \cdot 3 \cdot 1/3 + 10 \cdot 0 \\ &= 90. \end{aligned}$$

Consequently

$$\mathrm{Cov}(X, Y) = 90 - 10 \cdot 10 = -10,$$

and finally:

$$\rho(X, Y) = \frac{-10}{\sqrt{20/3} \cdot \sqrt{20}} = -\frac{\sqrt{3}}{2}.$$

Thus, (ii) is true.

(e) Let Z_i denote the number of boys the Megidos will have in the i-th birth, $1 \leq i < \infty$. This birth will take place if in all $i-1$ preceding births there will be only boys. If it takes place, there is a probability of 1/3 for each of 0, 1, and 2 boys. Hence:

$$E(Z_i) = \left(\frac{2}{3}\right)^{i-1} \cdot \frac{0+1+2}{3} = \left(\frac{2}{3}\right)^{i-1}, \qquad 1 \leq i < \infty.$$

Now $Z = \sum_{i=1}^{\infty} Z_i$, and therefore

$$E(Z) = \sum_{i=1}^{\infty} \left(\frac{2}{3}\right)^{i-1} = 3.$$

Thus, (iii) is true.

2. (a) Since $X \leq 2n$, we have $Z = 2n - 1$ when either $(X, Y) = (2n, 1)$ or $(X, Y) = (2n - 1, 0)$. Hence

$$\begin{aligned} P(X = 2n | Z = 2n - 1) &= \tfrac{P(X=2n, Z=2n-1)}{P(Z=2n-1)} \\ &= \tfrac{P(X=2n, Y=1)}{P(X=2n, Y=1) + P(X=2n-1, Y=0)} \\ &= \tfrac{1/2^{2n} \cdot \binom{n}{1}(1/6)^1(5/6)^{n-1}}{1/2^{2n} \cdot \binom{n}{1}(1/6)^1(5/6)^{n-1} + \binom{2n}{1}/2^{2n} \cdot (5/6)^{2n}} \\ &= \tfrac{n}{n+10n} \\ &= \tfrac{1}{11}. \end{aligned}$$

Thus, (ii) is true.

(b) We have:

$$V(X - Y) = V(X) + V(Y) = 2n \cdot 1/2 \cdot (1 - 1/2) + n \cdot 1/6 \cdot 5/6 = 23n/36.$$

Thus, (iii) is true.

3. (a) The probability that Yossi gets between 2001 and 2500 shekels at any specific year is

$$\frac{2500 - 2001 + 1}{2500 - 1001 + 1} = \frac{1}{3}.$$

Hence the number of years, out of ten, he gets such an amount is $B(10, 1/3)$-distributed. In particular, the probability that this will happen in nine of the years is

$$\binom{10}{1} \left(\frac{1}{3}\right)^9 \left(1 - \frac{1}{3}\right)^{10-1} = \frac{20}{3^{10}}.$$

Thus, (i) is true.

(b) Let S denote the total amount Yossi gets during 10 years. Clearly,

$$X = \frac{2S}{3} - 10 \cdot 100 = \frac{2S}{3} - 1000,$$

while

$$Y = \frac{S}{3} + 1000.$$

It follows that:

$$Y = X/2 + 1500.$$

Since Y is a linear function of X, and the first coefficient is positive, we have $\rho(X, Y) = 1$.

Thus, (iv) is true.

4. (a) The event $\{X_n = 0\}$ occurs if, at all the first n stages, a white ball is drawn. Denoting by A_i the event whereby the i-th ball drawn from the urn is white, $1 \le i \le n$, we have $\{X_n = 0\} = A_1 \cap A_2 \cap \cdots \cap A_n$, and therefore

$$\begin{aligned} P(X_n = 0) &= P(A_1)P(A_2|A_1)\cdots P(A_n|A_1 \cap A_2 \cap \cdots \cap A_{n-1}) \\ &= \tfrac{n}{2n}\tfrac{n-1}{2n}\cdots\tfrac{1}{2n} \\ &= \tfrac{n!}{(2n)^n}. \end{aligned}$$

Thus, (ii) is true.

(b) Marking the white balls in some way by $1, 2, \ldots, n$, and letting $Z_i = 1$ if the i-th ball is still in the urn after $2n$ stages and $Z_i = 0$ otherwise, $1 \le i \le n$, we have $X_{2n} = Z_1 + Z_2 + \cdots + Z_n$. Consequently,

$$E(X_{2n}) = \sum_{i=1}^{2n} E(Z_i) = nE(Z_1) = n\left(1 - \frac{1}{2n}\right)^{2n}.$$

Therefore, and since X_{2n} assumes the values $0, 1, \ldots, n$ with positive probabilities, if it were binomially distributed, the parameters would be n and $(1 - 1/2n)^n$. But then we would have

$$P(X_{2n} = n) = (1 - 1/2n)^{n^2},$$

whereas in fact

$$P(X_{2n} = n) = 1/2^{2n}.$$

If it were uniform, it would have to be $U[0, n]$-distributed, which the last formula (or the one for the expected value) shows to be false.

As to $X_{n/2}$, we have similarly

$$E(X_{n/2}) = n\left(1 - \frac{1}{2n}\right)^{n/2}.$$

If $X_{n/2}$ were distributed as $Y + n/2$, where $Y \sim H(n/2, n, n)$, its expectation would be

$$\frac{n}{2} \cdot \frac{n}{n+n} + \frac{n}{2} = \frac{3n}{4}.$$

Thus, (i) is true.

(c) The event $\{X_k = n-1\}$ occurs if, in the course of the first k stages, exactly one white ball has been drawn. Thus, Z is the number of drawings until the first white ball has been drawn. Since, until this happens, the probability of drawing a white ball is $1/2$, we have $Z \sim G(1/2)$.

Thus, (i) is true.