Chapter 1
Heidelberg lectures on Coleman integration

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1.1 Introduction

In the first half of February 2010 I spent 2 weeks at the Mathematics Center Heidelberg (MATCH) at the university of Heidelberg, as part of the activity PIA 2010 - The arithmetic of fundamental groups. In the first week I gave 3 introductory lectures on Coleman integration theory and in the second week I gave a research lecture on new work, which was (and still is) in progress, concerning Coleman integration in families. I later gave a similar sequence of lectures at the Hebrew University in Jerusalem.

This article gives an account of the 3 instructional lectures as well as the lecture I gave at the conference in Heidelberg with some (minimal) additions. I largely left things as they were presented in the lectures and I therefore apologize for the sometimes informal language used and the occasional proof which is only sketched. As in the lectures I made an effort to make things as self contained as possible.

The main goal of these lectures is to introduce Coleman integration theory. The goal of this theory is (in very vague terms) to associate with a closed one form $\omega \in \Omega(X)$, where X is a "space" over a p-adic field K, A locally analytic primitive F_ω, i.e., such that $dF_\omega = \omega$, in such a way that it is unique up to a constant.

In Section 1.4 we introduce Coleman theory. The presentation roughly follows Coleman's original approach [Col82, CdS88]. One essential difference is that we emphasize the semi-linear point of view. This turns out to be very useful in numerical computations of Coleman integrals. The presentation we give here, which does not derive the semi-linear properties from Coleman's work, is new.

In section 1.5 we give an account of the Tannakian approach to Coleman integration developed in [Bes02]. The main novelty is a more self contained and somewhat simplified proof from the one given in loc. cit. Rather than rely on the work of Chiarellotto [Chi98], relying ultimately on the thesis of Wildeshaus [Wil97] we unfold the argument and obtain some simplification by using the Lie algebra rather than its enveloping algebra.

In the final Section we discuss the new approach to Coleman integration in families. We discuss two complementary formulations, one in terms of the Gauss Manin connection and one in terms of differential Tannakian categories. As we find this new theory [Ovc08, Ovc09a, Ovc09b, Kam10] interesting and probably not familiar in the world of Arithmetic Geometry, we have given a rather lengthy account in Subsection 1.6.2, which might be interesting in its own right.

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1.2 Overview of Coleman theory

To appreciate the difficulty of integrating a closed form on a p-adic space, let us consider a simple example: We consider the space

$$X = \{z \in K, |z| = 1\}\ ,\ \omega = \frac{dz}{z}$$

Morally, then, the primitive F_ω should just be the logarithm function $\log(z)$. To try to find a primitive, we could pick $\alpha \in X$ and expand ω in a power series around α as follows:

$$\omega = \frac{d(\alpha+x)}{\alpha+x} = \frac{dx}{\alpha+x} = \frac{1}{\alpha}\frac{dx}{1+x/\alpha} = \frac{1}{\alpha}\sum\left(\frac{-x}{\alpha}\right)^n$$

and integrating term by term we obtain

$$F_\omega(\alpha+x) = -\sum\frac{1}{n+1}\left(\frac{-x}{\alpha}\right)^{n+1} + C$$

where these expansions converge on the disc for which $|x| < 1$.

So far, we have done nothing that could not be done in the complex world. However, in the complex world we could continue as follows: fix the constant of integration C on one of the discs. Then do analytic continuation: For each intersecting disc it is possible to fix the constant of integration on that disc uniquely so that the two expansions agree on the intersection. Going around the circle gives a multivalued function, which is the log function.

In the p-adic world, we immediately realize that such a strategy will not work because two open discs of radius 1 are either identical or completely disjoint. Thus, there is no obvious way of fixing simultaneously the constants of integration.

Starting with [Col82], Robert Coleman devises a strategy for coping with this difficulty using what he called “analytic continuation along Frobenius”. To explain this in our example, we take the map $\phi : X \to X$ given by $\phi(x) = x^p$. which is a lift of the p-power map. One notices immediately that $\phi^*\omega = p\omega$. Coleman's idea is that this relation should imply a corresponding relation on the integrals

$$\phi^* F_\omega = pF_\omega + C$$

where C is a constant function. It is easy to see that by changing F_ω by a constant, which we are allowed to do, we can assume that $C = 0$. The equation above now reads

$$F_\omega(x^p) = pF_\omega(x).$$

Suppose now that α satisfies the relation $\alpha^{p^k} = \alpha$. Then we immediately obtain

$$F_\omega(\alpha) = F_\omega(\alpha^{p^k}) = p^k F_\omega(\alpha) \Rightarrow F_\omega(\alpha) = 0.$$

This condition, together with the assumption that $dF_\omega = \omega$ fixes F_ω on the disc $|z - \alpha| < 1$. But it is well known that every $z \in X$ resides in such a disc, hence F_ω is completely determined.

In [Col82] Coleman also introduces iterated integrals (only on appropriate subsets of $\mathbb{P}^1$) which have the form

$$\int (\omega_n \cdot \int (\omega_{n-1} \cdots \cdot \omega_2 \cdot \int \omega_1) \cdots)$$

and in particular defines p-adic polylogarithms $\mathrm{Li}_n(z)$ by the conditions

$$\begin{aligned} d\,\mathrm{Li}_1(z) &= \frac{dz}{1-z} \\ d\,\mathrm{Li}_n(z) &= \mathrm{Li}_{n-1}(z)\frac{dz}{z} \\ \mathrm{Li}_n(0) &= 0 \end{aligned}$$

So that locally one finds

$$\mathrm{Li}_n(z) = \sum_{k=1}^{\infty} \frac{z^k}{k^n}$$

Then, in the paper [Col85] he extends the theory to arbitrary dimensions but without computing iterated integrals. In [CdS88] Coleman and de Shalit extend the iterated integrals to appropriate subsets of curves with good reduction.

In [Bes02] the author gave a Tannakian point of view to Coleman integration and extended the iterated theory to arbitrary dimensions. Other approaches exist. Colmez and independently Zarhin use functoriality with respect to algebraic morphisms. This approach does not need good reduction but can not handle iterated integrals. Vologodsky has a theory for algebraic varieties, which is similar in many respects to the theory in [Bes02], but that using alterations and monodromy operators in a very sophisticated way works also in the bad reductions case.

1.3 Background

Let K be a complete discrete valuation field with ring of integers R, residue field κ of prime characteristic p, uniformizer π and algebraic closure $\bar{K}$. We also fix an automorphism σ of K which reduces to the p-power map on κ, and when needed extend it to $\bar{K}$.

1.3.1 Rigid analysis

Let us recall first a few basic facts about rigid analysis, an excellent survey can be found in [Sch98]

The Tate algebra T_n is by definition

$$T_n = K < t_1,\dots,t_n >= \{\sum a_I t^I, a_I \in K, \lim_{I\to\infty} |a_I| = 0\}$$

which is the same as the algebra of power series with coefficients in K converging on the unit polydisc

$$B_n = \{(z_1,\dots,z_n) \in \bar{K}^n, |z_i| \le 1\}$$

An affinoid algebra A is a K-algebra with a surjective map $T_n \to A$ (for some n). One associates with A its maximal spectrum

$$\begin{aligned} X = \operatorname{spm}(A) &= \{m \subset A \text{ maximal ideal }\} \\ &= \{\psi : A \to \bar{K} \text{ a } K\text{-homomorphism}\}^{\operatorname{Gal}(\bar{K}/K)} \end{aligned}$$

(the latter equality is a consequence of the Noether normalization lemma for affinoid algebras from which it follows that a field which is a homomorphic image of such an algebra is a finite extension of K). Two easy examples are

$$\begin{aligned} \operatorname{spm}(T_n) &= B_n^{\operatorname{Gal}(\bar{K}/K)} \\ \operatorname{spm}(T_n/(t_1t_2-1)) &= \{(z_1,z_2) \in B_2, z_1z_2 = 1\}^{\operatorname{Gal}(\bar{K}/K)} \\ &= \{z \in \bar{K}, |z| = 1\}^{\operatorname{Gal}(\bar{K}/K)} \end{aligned}$$

(in what follows we will shorthand things so that the last space will simply be written $\{|z| = 1\}$ when there is no danger of confusion).

the maximal spectrum $X = \operatorname{spm}(A)$ of an affinoid algebra will be called an affinoid domain, and in a Grothendieckian style we associate with it its ring of functions $\mathscr{O}(X) = A$. Rigid geometry allows one to glue affinoid domains into more complicated spaces, and obtain the ring of functions on these spaces as well. We will say nothing about this except to mention that the space $B_n^\circ = \{|z_i| < 1\} \subset B_n$ can be obtained as the union of the spaces $\{z_i^k/\pi| \le 1\}$ and its ring of functions is not surprisingly

$$\mathscr{O}(B_n^\circ) = \left\{ \sum a_I t^I, \lim_{I\to\infty} |a_I| r^{|I|} = 0 \text{ for any } r < 1 \right\}$$

where $|(i_1,\ldots,i_n)| = i_1 + \cdots + i_n$.

1.3.2 Dagger algebras and Monsky-Washnitzer cohomology

The de Rham cohomology of rigid spaces is problematic in certain respects. To see an example of this, consider the first de Rham cohomology of T_1, which is the cokernel of the map

$$d : T_1 \to T_1 dt$$

This cokernel is infinite as one can write down a power series $\sum a_i t^i$ such that the a_i converge to 0 sufficiently slowly to make the coefficients of the integral $\sum a_i t^{i+1}/(i+1)$ not converge to 0. On the other hand, as B_1 can be considered a lift of the affine line, one should expect its cohomology to be trivial.

To remedy this, Monsky and Washnitzer considered so called weakly complete finitely generated algebras. An excellent reference is the paper [vdP86].

We consider the algebra

$$\mathscr{T}_n^\dagger = \left\{ \sum a_I t^I, a_I \in R, \exists r > 1 \text{ such that } \lim_{I\to\infty} |a_I| r^I = 0 \right\}$$

In other words, these are the power series converging on something slightly bigger than the unit polydisc. Integration reduces the radius of convergence, but only slightly - if the original power series converges to radius r the integral will no longer converge to radius r but will converge to any smaller radius, hence still overconverges.

Remark 1. The algebra of power series converging on the open polydisc of radius 1 also has trivial de Rham cohomology, but should not be considered a lifting of the affine line but rather of a point.

An R-algebra $A^\dagger$ is called a *weakly complete finitely generated* (wcfg) algebra if there is a surjective homomorphism $\mathscr{T}_n^\dagger \to A^\dagger$.

We will need to consider the module of differentials $\Omega^1_{A^\dagger}$ [vdP86, (2.3)], its higher analogues and the obvious de Rham complex $\Omega^\bullet_{A^\dagger}$.

One observes that $\mathscr{T}_n^\dagger/\pi$ is isomorphic to the polynomial algebra $\kappa[t_1,\ldots,t_n]$. Thus, if $A^\dagger$ is a wcfg algebra then $\bar{A} := A^\dagger/\pi$ is a finitely generated κ-algebra. Conversely, any finitely generated smooth κ-algebra can be obtained as an $\bar{A}$ for an appropriate $A^\dagger$ (a result of Elkik [Elk73]). In addition.

Proposition 1 ([vdP86, Theorem 2.4.4]). *We have:*

- *Any two such lifts are isomorphic.*
- *Any morphism $\bar{f} : \bar{A} \to \bar{B}$ can be lifted to a morphism $f^\dagger : A^\dagger \to B^\dagger$.*
- *Any two maps $A^\dagger \to B^\dagger$ with the same reduction induce homotopic maps $\Omega^\bullet_{A^\dagger} \otimes K \to \Omega^\bullet_{B^\dagger} \otimes K$*

Definition 1. The Monsky Washnitzer cohomology of $\bar{A}$ is the cohomology of the de Rham complex $\Omega^{\bullet}_{A^\dagger} \otimes K$,

$$H^i_{\mathrm{MW}}(\bar{A}/K) = H^i(\Omega^{\bullet}_{A^\dagger} \otimes K)$$

It is a consequence of the work of Berthelot that $H^i_{\mathrm{MW}}(\bar{A})$ is a finite dimensional K-vector space.

The Frobenius morphism $\varphi(x) = x^p$ of $\bar{A}$ can be lifted, by Proposition 1, to a σ linear morphism $\phi : A^\dagger \to A^\dagger$ (indeed, $A^\dagger$ with the homomorphism $R \xrightarrow{\sigma} R \to A^\dagger$ is a lift of $\bar{A}$ with the map $\kappa \xrightarrow{x^p} \kappa \to \bar{A}$ and the φ induces a homomorphism between $\bar{A}$ and this new twisted κ-algebra) and induces a well defined σ-linear endomorphism φ of $H^i_{\mathrm{MW}}(\bar{A})$. On the other hand, if κ is a finite field with $q = p^r$ elements, then φ^r is already κ-linear and therefore induces an endomorphism φ of $H^i_{\mathrm{MW}}(\bar{A})$. by [Chi98] one knows the possible eigenvalues of ϕ^r on Monsky-Washnitzer cohomology. in particular, we have

Theorem 1. *The eigenvalues of the κ-linear Frobenius ϕ^r on $H^1_{\mathrm{MW}}(\bar{A})$ are Weil numbers of weights* 1 *and* 2*.*

1.3.3 Specialization and locally analytic functions

One associates with $A^\dagger$ the K-algebra A, which is the completion of $A^\dagger \otimes K$ by the quotient norm induces from $\mathscr{T}_n^\dagger$. Assuming that $A^\dagger = \mathscr{T}_n^\dagger / I$ we have $A = T_n/I$. We further associate with A the space $X = \mathrm{spm}(A)$. Letting $X_\kappa = \mathrm{spec}(\bar{A})$ we have a specialization map $\mathrm{sp} : X \to X_\kappa$ which is defined as follows: Take a homomorphism $\psi : A \to L$, with L a finite extension of K. Then one checks by continuity that A maps to $\mathscr{O}_L$ and one associates with the kernel of ψ the kernel of its reduction mod π.

For our purposes, it will be convenient to consider the space X^{geo} of geometric points of X, which means K-linear homomorphisms $\psi : A \to \bar{K}$. This has a reduction map to the set of geometric points of X_κ obtained in the same way as above.

Definition 2. The inverse image under reduction of a geometric point $x : \mathrm{spec}\, \bar{\kappa} \to X_\kappa$. will be called the residue disc of x, denoted $U_x \subset X^{\mathrm{geo}}$.

By Hensel's Lemma and the smoothness assumption on $\bar{A}$ it is easy to see that U_x is naturally isomorphic to the space of geometric points of a unit polydisc.

Definition 3. The K-algebra of locally analytic functions on X, A_{loc}, is defined as the space of all functions $f : X^{\mathrm{geo}} \to \bar{K}$ which satisfy the following two conditions

- They are $\mathrm{Gal}(\bar{K}/K)$ equivariant in the sense that for any $\tau \in \mathrm{Gal}(\bar{K}/K)$ we have $f(\tau(x)) = \tau(f(x))$.
- restricted to each residue disc they are defined by a convergent power series.

There is an obvious injection $A \subset A_{\text{loc}}$. The algebra of our Coleman functions will lie in between these two K-algebras.

Another way of stating the equivariance condition for locally analytic functions, given the local expansion condition, is to say that given any $\tau \in \text{Gal}(\bar{K}/K)$ transforming the geometric point x of X_κ to the geometric point y, we have that τ translates the local expansion of f near x to the local expansion near y by acting on the coefficients. This way one can similarly define the A_{loc}-module Ω_{loc}^n of locally analytic n-forms on X and the obvious differential $d : \Omega_{\text{loc}}^{n-1} \to \Omega_{\text{loc}}^n$.

We define an action of the σ-semi-linear lift of Frobenius ϕ defined in the previous subsection on the spaces above. We first of all define an action on X^{geo} as follows: Suppose $\psi : A \to \bar{K} \in X^{\text{geo}}$ is a K-linear homomorphism. Then

$$\phi(\psi) = \sigma^{-1} \circ \psi \circ \phi \tag{1.1}$$

(recall that we have extended σ to $\bar{K}$). Note that this is indeed K-linear again. We can describe this action on points concretely as follows: Suppose $A = T_n/(f_1, \ldots, f_k)$ and let $g_i = \phi(t_i)$ so that ϕ is given by the formula

$$\phi(\sum a_I t^I) = \sum \sigma(a_I)(g_1, \ldots, g_n)^I \ .$$

Suppose that $\underline{z} := (z_1, \ldots, z_n) \in X^{\text{geo}}$, so that $f_i(\underline{z}) = 0$ for each i. Then we have

$$\phi(\underline{z}) = (\sigma^{-1} g_1(\underline{z}), \ldots, \sigma^{-1} g_1(\underline{z})) \ .$$

Having defined ϕ on points we now define it on functions by

$$\phi(f)(x) = \sigma f(\phi(x)) \tag{1.2}$$

From (1.1) it is quite easy to see that for $f \in A$ this is just the same as $\phi(f)$ as previously defined. We again have a compatible action on differential forms.

1.4 Coleman theory

We define Coleman integration is a somewhat different way then the one Coleman does, emphasizing a semi-linear condition and stressing the Frobenius equivariance.

Theorem 2. *Suppose that K is a finite extension of $\mathbb{Q}_p$. Then there exists a unique K-linear integration map*

$$\int : (\Omega_{A^\dagger}^1 \otimes K)^{d=0} \to A_{\text{loc}}/K$$

satisfying the following conditions:

1. *The map $d \circ \int$ is the canonical map $(\Omega_{A^\dagger}^1 \otimes K)^{d=0} \to \Omega_{\text{loc}}^1$.*
2. *The map $\int \circ d$ is the canonical map $A_K^\dagger \to A_{\text{loc}}/K$.*

3. One has $\phi \circ \int = \int \circ \phi$

In addition, the map is independent of the choice of ϕ.

Proof. Since $H^1_{\mathrm{MW}}(\bar{A})$ is finite dimensional, we may choose forms $\omega_1, \dots, \omega_n \in \Omega^1_{A^\dagger} \otimes K$ such that their images in $H^1(\Omega^\bullet_{A^\dagger} \otimes K)$ form a basis. If we are able to define the integrals $F_{\omega_i} := \int \omega_i$ for all the ω_i's, then the second condition immediately tells us how to integrate any other form. Put all the forms above into a column vector $\underline{\omega}$. Then we have a matrix $M \in M_{n\times n}(K)$ such that

$$\phi \underline{\omega} = M\underline{\omega} + d\underline{g}$$

where $\underline{g} \in (A_K^\dagger)^n$. Conditions 2 and 3 in the theorem tells us that this implies the relation

$$\phi F_{\underline{\omega}} = MF_{\underline{\omega}} + \underline{g} + \underline{c} \tag{1.3}$$

where $\underline{c} \in K^n$ is some vector of constants. We first would like to show that $\underline{c}$ may be assumed to vanish. For this we have the following key Lemma.

Lemma 1. *The map* $\sigma - M : K^n \to K^n$ *is bijective*

Proof. We need to show that for any $\underline{d} \in K^n$ there is a unique solution to the system of equations $\sigma(\underline{x}) = M\underline{x} + \underline{d}$. By repeatedly applying σ to this equation we can obtain an equation for $\sigma^i(\underline{x})$

$$\sigma^i(\underline{x}) = M_i\underline{x} + \underline{d}_i$$

where $M_i = \sigma^{i-1}(M) \cdot \sigma^{i-2}(M) \cdots \sigma(M) \cdot M$. Suppose now that κ has cardinality $q = p^r$. Then σ^r is the identity on K and so we obtain the equation $\underline{x} = M_r\underline{x} + \underline{d}_i$. The matrix M_r is exactly the matrix of the linear Frobenius ϕ^r on $H^1_{\mathrm{MW}}(\bar{A}/K)$, and by Theorem 1 the matrix $I - M_r$ is invertible. This shows that

$$\underline{x} = (I - M_r)^{-1}\underline{d}_i$$

is the unique possible solution to the equation. This shows that the map is injective, and since it is $\mathbb{Q}_p$-linear on a finite dimensional $\mathbb{Q}_p$-vector space it is also bijective.

Since ϕ acts as σ on constant functions we immediately get from the Lemma that by changing the constants in $F_{\underline{\omega}}$ we may assume that $\underline{c} = 0$ in (1.3).

We claim that now the vector of functions $F_{\underline{\omega}}$ is completely determined. Indeed, since $dF_{\underline{\omega}} = \underline{\omega}$ it is sufficient to determine it on a single point on each residue disc. So let x be such a point. Substituting x in (1.3) and recalling the action of ϕ on functions (1.2) we find

$$\sigma(F_{\underline{\omega}}(\phi(x))) = MF_{\underline{\omega}}(x) + \underline{g}(x)$$

Since $\phi(x)$ is in the same residue disc as x the difference

$$\underline{e} := F_{\underline{\omega}}(\phi(x)) - F_{\underline{\omega}}(x) = \int_x^{\phi(x)} \underline{\omega}$$

is computable from $\underline{\omega}$ alone. Substituting in the equation we find

$$\sigma(F_{\underline{\omega}}(x)+\sigma(\underline{e})) = MF_{\underline{\omega}}(x)+\underline{g}(x)$$

and rearranging we find an equation for $F_{\underline{\omega}}(x)$ that may be solved using Lemma 1.

It is fairly easy to see that the integral computed in this way indeed satisfies all the required properties. Since we have seen that the properties characterize the integral it follows that it is independent of the choice of basis $\underline{\omega}$. It remains to show that it is independent of the choice of ϕ. It is easy to see that it suffices to do this with respect to the equivariance property with respect to a linear Frobenius. So suppose we are given two linear Frobeni ϕ and ϕ' and that we have set up the theory for ϕ. we want to show that it also satisfies equivariance with respect to ϕ'. Let ω be a closed form and suppose we have chosen the constant in Coleman integration so that $F_{\phi(\omega)} = \phi F_\omega$. By Proposition 1 we have $h \in A_K^\dagger$ such that $\phi'(\omega)-\phi(\omega) = dh$. We now compute

$$\begin{aligned}\int \phi'(\omega) - \phi' \int \omega &= \int \phi'(\omega) - \phi' \int \omega - (\int \phi(\omega) - \phi \int \omega) \\ &= \int (\phi'(\omega)-\phi(\omega)) - (\phi' \int \omega - \phi \int \omega) \\ &= h - (\phi' \int \omega - \phi \int \omega)\end{aligned}$$

and substituting at a point x we get

$$h(x) - \int_{\phi(x)}^{\phi'(x)} \omega$$

We need to show that this is a constant independent of x. This follows because one can show that the function $H(x,y) = \int_x^y \omega$ is in fact an analytic function in two variables on the space of pairs (x,y) reducing to the same point, and that then the function h may be taken to be the pullback of H via the map (ϕ,ϕ') (we do not give full details here).

1.5 Coleman integration via isocristals

In this section we explain the approach to Coleman integration introduced in [Bes02]. We comment that the approach there works globally as well, but we only explain it in the affine (or, more precisely, affinoid) situation, in which we described Coleman's work.

The main idea is that the iterated integral

$$\int (\omega_n \int (\omega_{n-1} \int (\cdots \int \omega_1)\cdots))$$

is a solution of the system of differential equations

$$dy_0 = 0,\ dy_1 = \omega_1 y_0, \dots dy_n = \omega_n y_{n-1} \tag{1.4}$$

or, in vector notation

$$d\underline{y} = \Omega \underline{y}\ ,\ \Omega = \begin{pmatrix} 0 & 0 & 0 & \cdots & 0 \\ \omega_1 & 0 & 0 & \cdots & 0 \\ 0 & \omega_2 & 0 & \cdots & 0 \\ 0 & 0 & \omega_3 & \cdots & 0 \\ \dots & \dots & \dots & \dots & \dots \\ 0 & \cdots & 0 & \omega_n & 0 \end{pmatrix}.$$

This is just a unipotent differential equation. The Frobenius equivariance condition can now be interpreted as saying that we have a system $\underline{y}$ of "good" local solutions for this equation, in such a way that $\phi\underline{y}$ is a "good" system of solutions for the equation $d\underline{y} = \phi(\Omega)\underline{y}$. This, as well as the independence of the choice of the lift of Frobenius, turns out to be very nicely explained by the Tannakian formalism of unipotent isocristals.

1.5.1 The Tannakian theory of unipotent isocristals

We assume familiarity with the basic theory of neutral Tannakian categories. The standard reference is [DM82]

From here onward, we will write $A^\dagger$ instead of $A_K^\dagger$ to simplify the notation.

Definition 4. A unipotent isocristal on $\bar{A}$ is an $A^\dagger$-module M together with an integrable connection

$$\nabla : M \to M \otimes_{A^\dagger} \Omega^1_{A^\dagger}$$

which is an iterated extension of trivial connections (where trivial means the object $\mathbb{1} := (A^\dagger, d)$).

We first observe that the module M is in fact free, because it is an iterated extension of $A^\dagger$, which is obviously split.

A morphism of unipotent isocristals is just a horizontal (i.e., commuting with the connection) map of $A^\dagger$-modules.

We denote the category of unipotent isocristals on $\bar{A}$ by $\mathscr{U}n(\bar{A})$. It is a basic fact of the theory that, as the notation suggests, the category depends only on $\bar{A}$ and not on the particular choice of lift $A^\dagger$.

Example 1. Let $M \in \mathscr{U}n(\bar{A})$ have rank 2. Then it sits in a short exact sequence

$$0 \to \mathbb{1} \to M \to \mathbb{1} \to 0$$

which is (non-canonically) split. It is thus isomorphic to the object having underlying module ${A^\dagger}^2$ and connection

$$\nabla = d - \left(\begin{smallmatrix} 0 & 0 \\ \omega & 0 \end{smallmatrix}\right)$$

By associating with M the class of ω in $H^1_{\mathrm{dR}}(A^\dagger/K) = H^1_{\mathrm{MW}}(\bar{A}/K)$ it is easy to check that one obtains a bijection

$$\mathrm{Ext}^1_{\mathscr{U}n(\bar{A})}(\mathbb{1},\mathbb{1}) \cong H^1_{\mathrm{MW}}(\bar{A}/K)$$

Theorem 3. *The category $\mathscr{U}n(\bar{A})$ is a rigid abelian tensor category.*

For a proof of this fact see for example [CLS99, 2.3.3]. It is fairly standard and consists of checking that $\mathscr{U}n(\bar{A})$ is closed under sub and quotient objects, tensor products and duals.

To make $\mathscr{U}n(\bar{A})$ into a Neutral Tannakian category what is missing is a fiber functor, i.e., an exact functor into K-vector spaces preserving the tensor structure. We can associate such a functor to each κ-rational point as follows:

Definition 5. Let $x \in X_\kappa(\kappa)$ be a rational point. We associate with it the functor

$$\omega_x : \mathscr{U}n(\bar{A}) \to \mathrm{Vec}_K \ , \ \omega_x(M,\nabla) = \{v \in M(U_x), \nabla(v) = 0\}$$

where U_x is the residue disc of x and $M(U_x)$ consists of the sections of M on the rigid analytic space U_x.

The fact that ω_x is indeed a fiber functor is quite standard. The key point to observe is the following: a precondition for a functor such as ω_x to be a fiber functor is that the dimension of $\omega_x(M,\nabla)$ equals the rank of M. For a general differential equation there is no reason why this should be the case and one introduces a condition of overconvergence, which among other things guarantees this. A unipotent isocristal is always overconvergent. It is, however, easy to see without knowing this that indeed $\omega_x(M,\nabla)$ has the right dimension for a unipotent ∇ simply because finding horizontal sections amounts to iterated integration and one can integrate power series converging on the unit open polydisc to power series with the same property (Remark 1)

In the general theory of overconvergent isocristals one can realize the functor ω_x as simply the pullback x^* to an isocristal on $\mathrm{spec}(\kappa)$.

The general theory of Tannakian categories [DM82] tells us that the category $\mathscr{U}n(\bar{A})$ together with the fiber functor ω_x determine a fundamental group

$$G = G_x = \pi_1(\mathscr{U}n(\bar{A}), \omega_x)$$

which is an affine proalgebraic group, and an equivalence of categories between $\mathscr{U}n(\bar{A})$ and the category of finite dimensional K-algebraic representations of G. We begin by recalling that G represents the functor that sends a K-algebra F to the group

$$\begin{aligned}\mathrm{Aut}^{\otimes}(\omega_x \otimes F) := \{M \in \mathscr{U}n(\bar{A}) \to (&\alpha_M : \omega_x(M)\otimes F \to \omega_x(M)\otimes F)\ ,\\ &\alpha_M \text{ natural isomorphism and}\\ &\alpha_{M\otimes N} = \alpha_M \otimes \alpha_N\ ,\ \alpha_{\mathbb{1}} = \mathrm{id}\}\end{aligned} \tag{1.5}$$

The description of the Lie algebra $\mathfrak{g}$ of G is well known. Consider the algebra $K[\varepsilon]$ of Dual numbers where $\varepsilon^2 = 0$. Then $\mathfrak{g}$ is just the tangent space to G at the origin and is thus given by

$$\mathfrak{g} = \operatorname{Ker}(G(K[\varepsilon]) \to G(K))$$

In terms of the description (1.5) to G an element $\alpha \in \mathfrak{g}$ sends $M \in \mathscr{U}n(\bar{A})$ to

$$\alpha_M = \mathrm{id} + \varepsilon\beta_M \ , \ \beta_M \in \operatorname{End}(\omega_x(M))$$

(such an element is automatically invertible). The conditions on the α_M easily translate to conditions on the β_M and we obtain

$$\begin{aligned} \mathfrak{g} = \{(M \to & \beta_M \in \operatorname{End}(\omega_x(M)) \ , \\ & \beta_M \text{ natural } , \beta_{\mathbb{1}} = 0 \ , \\ & \beta_{(M\otimes N)} = \beta_M \otimes \mathrm{id}_{\omega_x(N)} + \mathrm{id}_{\omega_x(M)} \otimes \beta_N \} \end{aligned}$$

The Lie bracket is given in this representation by the commutator. We have

Lemma 2. *The elements of G are unipotent and the elements of $\mathfrak{g}$ are nilpotent in the sense that for every $M \in \mathscr{U}n(\bar{A})$ the corresponding α_M is unipotent and the corresponding β_M is nilpotent.*

Proof. Choose a flag $M = M_0 \supset M_1 \supset \cdots$ with trivial consecutive quotients. Then the naturality of α and β implies that with respect to a basis compatible with the associated flag on $\omega_x(M)$ the matrices of α_M and β_M are upper triangular, with 1 respectively 0 on the diagonal.

It follows that there is well defined algebraic exponential map $\exp : \mathfrak{g} \to G(K)$ sending β_M to $\exp(\beta_M)$ given by the usual power series. Tensoring with an arbitrary K-algebra we can easily see (using the fact that K has characteristic 0) that exp induces an isomorphism of affine schemes from the affine space associated with $\mathfrak{g}$ to G. The product structure on G translates in $\mathfrak{g}$ to the product given by the Baker-Campbell-Hausdorff formula. It is further clear that the following holds.

Proposition 2. *The reverse operations of differentiation and exponentiation give an equivalence between the categories of algebraic representations of G and continuous Lie algebra representations of $\mathfrak{g}$.*

Here, continuous representation means with respect to the discrete topology on the representation space and with respect to the inverse limit topology on $\mathfrak{g}$.

1.5.2 The Frobenius invariant path

Consider now two κ-rational points $x, z \in X_\kappa$. When we have a similarly defined space of paths $P_{x,z} := \operatorname{Iso}^{\otimes}(\omega_x, \omega_z)$ (same functoriality and tensor conditions) which is clearly a left principal homogeneous space for G_x (and a right one for G_z). In concrete terms, the path space $P_{x,z}$ consists of rules for "analytic continuation" for each unipotent differential equation (M,∇), of a solutions (= horizontal section) $\underline{y}_x \in M(U_x)^{\nabla=0}$ to $\underline{y}_z \in M(U_z)^{\nabla=0}$ compatible with morphisms and tensor products. Composition of paths

$$P_{x,z} \times P_{z,w} \to P_{x,w} \tag{1.6}$$

is derived from composition of isomorphisms.

Suppose now that $\bar{f} : \bar{B} \to \bar{A}$ is a morphism. The pullback $\bar{f}^*$ (pullback is in the geometric sense) is a tensor functor from $\mathscr{U}n(\bar{B})$ to $\mathscr{U}n(\bar{A})$. We have a natural isomorphism of functors

$$\omega_{\bar{f}(x)} \to \omega_x \circ \bar{f}^* , \tag{1.7}$$

which is compatible with the tensor structure. This is obvious from the general theory since, as you may recall, we interpreted ω_x as the pullback x^* to spec(κ). To translate into concrete terms chose a lifting $f : B^\dagger \to A^\dagger$ of $\bar{f}$. Then the assumptions imply that f maps U_x to $U_{\bar{f}(x)}$ and the isomorphisms is obtained by composition with f of the horizontal sections on $U_{\bar{f}(x)}$.

Suppose that z is an additional rational point on X_κ. Then it is easy to see that $\bar{f}$ induces a map $\bar{f} : P_{x,z} \to P_{\bar{f}(x),\bar{f}(z)}$. In concrete terms, suppose that $\alpha \in P_{x,z}$ (over some extension algebra) is a rule for analytic continuation of solutions from U_x to U_z, then $\bar{f}(\alpha)$ is a rule for analytic continuation from $U_{f(x)}$ to $U_{f(z)}$ given as follows: Start from a horizontal section in $M(U_{\bar{f}(x)})$. Pullback by f to obtain a horizontal section of $\bar{f}^*(M)$ on U_x. Apply the rule α to obtain a horizontal section on U_z and finally apply the inverse of pullback by f. It is formally checked that $\bar{f}$ is compatible with composition of paths (1.6). In particular, when $x = z$, $\bar{f} : G_x \to G_{\bar{f}(x)}$ is a group homomorphism and in general it is compatible with the structure of $P_{x,z}$ as a principal homogeneous space for G_x.

Suppose now that $\bar{f} : \bar{A} \to \bar{A}$ and $\bar{f}$ fixes both x and z. Then we can check what it means for a path $\alpha \in P_{x,z}$ to be fixed by $\bar{f}$: The analytic continuation α has the property that the following diagram commutes.

$$\begin{array}{ccc}
\omega_x(M) & \xrightarrow{\alpha_M} & \omega_z(M) \\
\downarrow & & \downarrow \\
\omega_x(\bar{f}^*M) & \xrightarrow{\alpha_{\bar{f}^*M}} & \omega_z(\bar{f}^*M)
\end{array}$$

where the vertical maps are the isomorphisms of (1.7) Even more concretely, restricting to the differential equation (1.4), α translates a solution $\underline{y}_x$ on U_x to a solution $\underline{y}_z$ on U_z in such a way that it now translates the local solution $f^*\underline{y}_x$ to the system

$$dy_0 = 0,\ dy_1 = f^*\omega_1 y_0, \dots dy_n = f^*\omega_n y_{n-1}$$

on U_x to the solution $f^*\underline{y}_z$ on U_z. In particular, if we think of a collection of solutions to $dy_0 = 0,\ dy_1 = \omega y_0$ compatible under α as an integral of ω, then the path α provides such an integral for each closed one-form ω in such a way that $\int f^*\omega =$

$f^*\omega$ (plus a constant arising from the choice of which solutions to extend). When $\bar{f}$ is a (κ-linear) Frobenius this is exactly what we want our Coleman integration to do. Thus, it is clear that the following Theorem provides the sought after generalization of Coleman integration.

Theorem 4. *Suppose that ϕ is a κ-linear Frobenius fixing the two κ-rational points x and z. Then there exists a unique $\gamma_{x,z} \in P_{x,z}(K)$ fixed by ϕ. Furthermore, these path are compatible under raising ϕ to some power and under composition.*

The proof of Theorem 4 is more or less an immediate consequence of the following Theorem.

Theorem 5. *Let ϕ be as above, fixing the rational point x. Then the map $g \mapsto \phi(g)^{-1}g$ from G_x to itself is an isomorphism of schemes.*

We first prove that Theorem 5 implies Theorem 4. Since G_x is unipotent, there exists a K-rational point $\gamma' \in P_{x,z}(K)$. Let $g' \in G_x(K)$ be such that $\phi(\gamma') = g'\gamma'$ and let $g \in G_x(K)$ be the element, whose uniqueness and existence is guaranteed by Theorem 5, such that $g' = \phi(g)^{-1}g$. Let $\gamma = g\gamma'$. Then

$$\phi(\gamma) = \phi(g)\phi(\gamma') = \phi(g)g'\gamma' = g\gamma' = \gamma$$

proving existence. On the other hand. If both γ and γ' are fixed by ϕ and if $g\gamma = \gamma'$, then $\phi(g) = g$ and by the uniqueness in Theorem 5 we have that g is the identity element and $\gamma' = \gamma$.

For the proof of Theorem 5 we need to study in more detail the Lie algebra $\mathfrak{g}$. As the group G is pro-algebraic, it can be written as an inverse limit of algebraic groups $\varprojlim_\alpha G_\alpha$. Its Lie algebra can thus be written as an inverse limit of finite dimensional Lie algebras

$$\mathfrak{g} = \varprojlim_{\alpha} \mathfrak{g}/\mathfrak{g}_\alpha$$

with some indexing set of α's. We consider the lower central series of $\mathfrak{g}$ obtained as follows:

$$\mathfrak{g}_1 = [\mathfrak{g},\mathfrak{g}] \ , \ \mathfrak{g}_{n+1} = [\mathfrak{g},\mathfrak{g}_n]$$

Here, the commutators should be taken in the topological sense, i.e., completed.

Proposition 3 (Wildeshaus [Wil97, p. 32]). *There is a canonical isomorphism*

$$\mathfrak{g}/\mathfrak{g}_1 \to \operatorname{Ext}^1_{\mathscr{U}n(\bar{A})}(\mathbb{1},\mathbb{1})^* \ .$$

Proof. We exhibit a natural pairing $\mathfrak{g} \times \operatorname{Ext}^1_{\mathscr{U}n(\bar{A})}(\mathbb{1},\mathbb{1}) \to K$ as follows: consider $\ell \in \mathfrak{g}$ and an extension

$$0 \to \mathbb{1} \to M \to \mathbb{1} \to 0$$

When applying ω_x we can use a compatible basis to write the matrix of ℓ on $\omega_x(M)$ as $\left(\begin{smallmatrix} 0 & \alpha \\ 0 & 0 \end{smallmatrix}\right)$. Then the pairing will send (ℓ, M) to α. Since the commutator of two matrices of the form $\left(\begin{smallmatrix} 0 & * \\ 0 & 0 \end{smallmatrix}\right)$ is 0, and since the representation of $\mathfrak{g}$ on $\omega_x(M)$ is continuous by Proposition 2, it is clear that the pairing factors via $(\mathfrak{g}/\mathfrak{g}_1) \times \operatorname{Ext}^1_{\mathscr{U}n(\bar{A})}(\mathbb{1},\mathbb{1})$.

To establish the isomorphism of the Proposition we need to use the full force of Tannakian duality, that is the part of theory implying that the category $\mathscr{U}n(\bar{A})$ is equivalent to the category of continuous Lie algebra representations of $\mathfrak{g}$. Thus, if the extension M is in the Kernel of the pairing, it corresponds to a trivial Lie algebra representation and is therefore trivial. In the reverse direction, suppose that $a : \mathfrak{g}/\mathfrak{g}_1 \to K$ is a continuous functional. It thus extends to a functional $a : \mathfrak{g} \to K$ which is continuous and which vanishes on all commutators. It follows easily that $\ell \to \begin{pmatrix} 0 & a(\ell) \\ 0 & 0 \end{pmatrix}$ is a continuous Lie algebra representation of $\mathfrak{g}$, which is an extension of the required type and gives back a when pairing with it. It follows that $\mathrm{Ext}^1(\mathbb{1},\mathbb{1})$ is isomorphic to the continuous dual of $\mathfrak{g}/\mathfrak{g}_1$. As $\mathrm{Ext}^1(\mathbb{1},\mathbb{1})$ is finite dimensional, it follows that so is $\mathfrak{g}/\mathfrak{g}_1$ and they are dual as discrete vector spaces.

Proposition 4. *The quotients $\mathfrak{g}_n/\mathfrak{g}_{n+1}$ are finite dimensional and the commutator induces a surjective map*

$$[\,] : \mathfrak{g}/\mathfrak{g}_1 \otimes \mathfrak{g}_{n-1}/\mathfrak{g}_n \to \mathfrak{g}_n/\mathfrak{g}_{n+1} \tag{1.8}$$

Proof. We prove this by induction. The case $n = 0$ for the finiteness follows from the previous Proposition. The Jacobi identity immediately implies that $[\mathfrak{g}_1, \mathfrak{g}_{n-1}] \subset \mathfrak{g}_{n+1}$ and by definition $[\mathfrak{g}, \mathfrak{g}_n] = \mathfrak{g}_{n+1}$. Thus, The map (1.8) is defined. To show surjectivity (which is not obvious because we are taking completed brackets) we can chose complementary subspaces V and W for $\mathfrak{g}_1$ in $\mathfrak{g}$ and for $\mathfrak{g}_n$ in $\mathfrak{g}_{n-1}$ respectively. Surjectivity follows if we show that the inclusion $[V,W] + \mathfrak{g}_{n+1} \subset \mathfrak{g}_n$ is an equality. But this is clearly the case after completion and so we are done because the sum of a finite dimensional subspace and a closed subspace is closed (prove this!). Finally, the surjectivity immediately proves that $\mathfrak{g}_n/\mathfrak{g}_{n+1}$ is finite dimensional again.

Corollary 1. *For every n the quotient $\mathfrak{g}/\mathfrak{g}_n$ is finite dimensional.*

Proposition 5. *The topology induced by the $\mathfrak{g}_n$ is stronger than the $\mathfrak{g}_\alpha$ topology on $\mathfrak{g}$.*

Proof. For each α the Lie algebra $\mathfrak{g}/\mathfrak{g}_\alpha$ is a finite dimensional nilpotent Lie algebra, implying that for a sufficiently large n its lower central series vanishes, from which it follows that $\mathfrak{g}_n \subset \mathfrak{g}_\alpha$.

Now we use again the action of a (κ-linear) Frobenius ϕ. By functoriality it induces a continuous endomorphism of $\mathfrak{g}$. It therefore clearly preserves the filtration $\mathfrak{g}_n$ and induces an endomorphism on the quotients $\mathfrak{g}/\mathfrak{g}_n$ and $\mathfrak{g}_n/\mathfrak{g}_{n+1}$.

Proposition 6. *The eigenvalues of ϕ on $\mathfrak{g}/\mathfrak{g}_n$ and $\mathfrak{g}_n/\mathfrak{g}_{n+1}$ have strictly negative weights.*

Proof. This follows immediately because ϕ has positive weights on $\mathrm{Ext}^1(\mathbb{1},\mathbb{1}) = H^1_{\mathrm{MW}}(\bar{A}/K)$ hence negative weights on its dual $\mathfrak{g}/\mathfrak{g}_1$, and by Proposition 4 we have a surjective map, compatible with ϕ, $(\mathfrak{g}/\mathfrak{g}_1)^{\otimes n+1} \to \mathfrak{g}_n/\mathfrak{g}_{n+1}$

Corollary 2. *The map $\phi - \mathrm{id}$ is invertible on $\mathfrak{g}/\mathfrak{g}_n$ and $\mathfrak{g}_n/\mathfrak{g}_{n+1}$.*

Proof (Proof of Theorem 5). For simplicity we prove bijectivity on K rational points. Since the proof relies on the Lie algebra it will work for any extension.

We begin with injectivity. Suppose that $\phi(g)=g$ for some $g\neq 1$. Then $g=\exp(\ell)$ for some $0\neq \ell\in\mathfrak{g}$ and since exp is an isomorphism compatible with ϕ we have $\phi(\ell)=\ell$. But for some sufficiently large n the image of ℓ in $\mathfrak{g}/\mathfrak{g}_n$ is non-zero and is therefore an eigenvector for ϕ with eigenvalue 1 contradicting Corollary 2.

To prove surjectivity, let $g'=\exp(\ell')\in G(K)$. Define a sequence $\ell_n\in\mathfrak{g}_n$ as follows: $\ell_0=\ell'$. Suppose we have defined ℓ_n. Consider the function

$$f(k)=\exp^{-1}\left(\exp(\phi k)^{-1}\exp(\ell_n)\exp(k)\right)=\ell_n+k-\phi(k)+\text{ commutators}$$

Since $1-\phi$ is invertible on $\mathfrak{g}_n/\mathfrak{g}_{n+1}$ by Corollary 2 we can find $k_n\in\mathfrak{g}_n$ such that $\ell_{n+1}:=f(k_n)\in\mathfrak{g}_{n+1}$. Now let

$$g_n=\exp(k_0)\exp(k_1)\cdots\exp(k_n)\ .$$

Then

$$(\phi(g_n))^{-1}g'g_n=\exp(\ell_{n+1})\ .$$

It follows from Proposition 5 that the limit $g=\lim_{n\to\infty}g_n$ exists and that $(\phi(g))^{-1}g'g=1$ or $g'=\phi(g)g^{-1}$ as required.

1.5.3 Coleman functions

The work of the previous subsection explains how to analytically continue solutions of differential equations to get Coleman functions. The functions themselves are obtained as components of the solutions - The iterated integral

$$\int(\omega_n\int(\omega_{n-1}\int(\cdots\int\omega_1)\cdots))$$

is going to be the component y_n in a system of local horizontal solutions of the system (1.4), compatible with respect to Frobenius invariant paths. One can do this in a more streamlined way, which extends also to the non-affine case, by considering arbitrary functionals on the underlying vector bundle for a connection instead of just the projection on the last component. This gives rise to the following definition.

Definition 6. An abstract Coleman function on $A^\dagger$ is a fourtuple, which we write $(M,\nabla,\underline{y}_x,s)$ in which ∇ is a unipotent integrable connection on an $A^\dagger$-module M, $\underline{y}_x$ refers to a system of horizontal sections for each U_x, compatible with the Frobenius invariant paths, and $s\in\operatorname{Hom}(M,A^\dagger)$.

We note that specifying for which points x one has the $\underline{y}_x$ does not matter. They are all derived from one of them by doing analytic continuation so one could instead just specify $\underline{y}_x$ for one x and this formulation is only done for symmetry. We further note that s is usually not horizontal (horizontal s's produce constant functions). In fact, one can define a notion of Coleman functions with values in any sheaf by changing the target of s.

A Coleman function is made into an actual locally analytic function by evaluating the s on the $\underline{y}_x$'s. Many abstract Coleman functions may produce the same function. One way in which this can happen is the following:

Definition 7. Two Coleman functions, $(M,\nabla,\underline{y}_x,s)$ and $(M',\nabla',\underline{y}'_x,s')$ are called equivalent if there exists a horizontal morphism $f : M \to M'$ carrying the $\underline{y}_x$'s to the $\underline{y}'_x$'s (by the properties of the invariant paths it suffices to check this for one x) and such that $s = s' \circ f$. More generally they are called equivalent if they are related by the equivalence relation generated by the above relation. An equivalence class of abstract Coleman functions is called a Coleman function.

It is trivial to check that equivalent abstract Coleman functions give rise to the same locally analytic function, which is therefore associated to the Coleman function as just defined. It is not immediately clear, but turns out to be true, that A Coleman function inducing the 0 function is indeed equivalent to 0. This is a consequence of the identity principle, to be discussed below. There are some advantages to defining Coleman functions without reliance on a "physical" representation as a locally analytic function. One example is integration of meromorphic differentials on curves.

We denote the K-algebra of all Coleman functions by A_{Col}. Coleman functions with values in a sheaf $\mathscr{F}$ will be denoted $A_{\mathrm{Col}}(\mathscr{F})$. In particular, we have degree n Coleman differential forms defined by $\Omega^n_{\mathrm{Col}} = A_{\mathrm{Col}}(\Omega^n)$.

Many properties of Coleman functions can easily be derived from the description above. It is easy to define sums and products of Coleman functions, compatible with the same operations on locally analytic functions. It is also easy to define pullbacks of Coleman functions by morphisms, compatible with the corresponding operation on locally analytic functions.

To give an example of the properties of Coleman functions we discuss the identity principle. This was proved by Coleman for $\mathbb{P}^1$ in [Col82] and for curves by Coleman and de Shalit [CdS88]. It says the following:

Proposition 7. *Suppose that the Coleman function F is* 0 *on one residue disc. Then it is identically* 0*.*

The proof of this result is based on the following construction: We recall that part of the data for a Coleman function is a section $s : M \to A^\dagger$. One can construct M_s which is the maximal subconnection contained in $\mathrm{Ker}(s)$. The point is to construct it concretely as the intersection of the sections s and its derivatives of all orders with respect to the dual connection. The consequence of this is that, with $\underline{y}_x$ a local horizontal section showing up in the definition of F, which is by assumption in $\mathrm{Ker}(s)$, the fact that $\nabla \underline{y}_x = 0$ actually implies that $\underline{y}_x \in M_s(U_x)$. We find F to be equivalent with $(M_s,\nabla,\underline{y}_x,0)$, and this is clearly equivalent to 0.

Corollary 3. *If $dF = 0$ then F is a constant function.*

Proof. The function F is a constant on some residue disc. Subtracting that constant we find a function which is 0 on one residue disc, hence 0 by the identity principle.

The main result about Coleman functions is the following Theorem.

Theorem 6. *The sequence*

$$0 \to K \to A_{\mathrm{Col}} \xrightarrow{d} \Omega^1_{\mathrm{Col}} \xrightarrow{d} \Omega^2_{\mathrm{Col}}$$

is exact.

Everything is already proved except for the fact that we may integrate a closed Coleman form. The idea is roughly that having a closed Coleman form ω, the condition $dF = \omega$ can be written as a new unipotent differential equation. The closeness of ω is used to find a subconnection which is integrable in addition to being unipotent, from which F can be constructed.

1.5.4 Tangential base points

One of the advantages of the Tannakian approach to Coleman integration is that new fiber functors are integrated in the theory with no extra cost. The prime example of this so far are fiber functors coming from Deligne's *tangential base points* [Del89]. In this subsection we sketch this extension and the application to polylogarithms and to multiple zeta values, in particular towards proving the *series shuffle product formula* for p-adic multiplze zeta values. Full details may be found in the paper [BF06].

The de Rham version of Deligne's tangential base point is defined as follows [Del89, 15.28-15.30]: Suppose C be a curve over a field K of characteristic 0, smooth at a point P, with a local parameter t at P, and suppose $\nabla : M \to M \otimes \Omega^1_C(\log P)$ is a connection with logarithmic singularities at P, so that locally $\nabla = d + \Gamma$ with Γ is a section of $\mathrm{End}(M) \otimes \Omega^1_C(\log P)$. One defines the residue connection on the constant bundle, with fiber the fiber of M at P, on the complement of 0 in the tangent line $T_P(C)$, with log singularities at $0, \infty$, by

$$\mathrm{Res}_P(\nabla) := d + (\mathrm{Res}_P \Gamma) d \log(\bar{t})$$

where $\bar{t}$ is the indued coordinate on the tangent space. While this looks like it depends on the parameter it is in fact not the case, up to a canonical isomorphism, and Deligne gives a coordinate free description.

There is no difficulty in replacing the algebraic curve by a p-adic analytic one. Since the action of a lift of Frobenius, assumed to fix P, extends to an action on the tangent space, one can analytically continue horizontal sections of ∇ along Frobenius to horizontal section of $\mathrm{Res}_P \nabla$ on residue discs in $T_P(C) - \{0\}$. One can set up a theory of Coleman functions "of algebraic origin" where the underlying bundle and connection are algebraic with logarithmic singularities at P, in such a way that these functions now have values at the points of $T_P(C) - \{0\}$.

This turns out to be far less mysterious than one might expect. Consider a unipotent differential equation with logarithmic singularities near P. It terms of the parameter t one easily sees that it has a full set of solutions in the ring $K[[t]][\log(t)]$.

Define the constant term (with respect to t) of an element in $K[[t]][\log(t)]$ by formally setting $\log(t) = 0$ and then evaluating at 0. In [BF06, Proposition 4.5] we show that taking the constant term of a Coleman function corresponds to analytically continuing to the tangent space and evaluating at the tangent point $\bar{t} = 1$.

This is already useful for p-adic polylogarithms. Recall from the introduction that these were defined to be Coleman functions that satisfy the unipotent system of differential equations:

$$
\begin{aligned}
d\,\mathrm{Li}_1(z) &= \frac{dz}{1-z} \\
d\,\mathrm{Li}_n(z) &= \mathrm{Li}_{n-1}(z)\frac{dz}{z} \\
\mathrm{Li}_n(0) &= 0
\end{aligned}
$$

The problem with this definition is that the equations have singularities at 0 and 1 and the boundary conditions are made at the singular point 0. In practice there is no problem because things are aranged in such a way that the functions are nonsingular at 0. Deligne pointed out in the complex case that one should interpret the boundary conditions at the singular point 0 to mean analytic continuation from the tangent vector $\bar{t} = 1$ at 0, and this holds true in the p-adic case as well: One replaces the condition $\mathrm{Li}_n(0) = 0$ by the equivalent condition that the constant term there is 0. One can use the same method to assign values to p-adic polylogarithms (and multiple polylogarithms) at 1.

We now briefly recall some material from the theory of multiplze zeta values, including the p-adic theory developed by Furusho [Fur04]. For $\underline{k} = (k_1, \dots, k_m)$, $k_i > 0$, $k_m > 1$ the multiple zeta value $\zeta(\underline{k})$ is defined as the (convergent) series.

$$
\zeta(\underline{k}) = \sum_{0<n_1<\cdots<n_m} \frac{1}{n_1^{k_1}\cdots n_m^{k_m}} \,. \tag{1.9}
$$

for example, for $\underline{k} = (k)$, $\zeta(\underline{k}) = \zeta(k)$ is the usual zeta value.

These numbers, already known to Euler, are of interest because of their algebraic interrelations, which are expected to reflect deep arithmetic information. The simplest types of relations are the so called series (or harmonic) shuffle product formulae. The easiest example of these, which we will concentrate on in this subsection (see [BF06] for the general theory) is the formula

$$
\zeta(a)\zeta(b) = \zeta(a,b) + \zeta(b,a) + \zeta(a+b) \tag{1.10}
$$

which one gets by dividing the summation over the infinite square $n_1, n_2 > 0$ into the sum over the bottom and top triangles and over the diagonal.

There is another type of relations for multiple zeta values which one obtains from an integral representation. To derive it, define the $\underline{k}$-th multiple polylogarithm, where the index $\underline{k}$ can now have $k_m = 1$, by the series

$$\mathrm{Li}_{\underline{k}}(z) = \sum_{0<n_1<\cdots<n_m} \frac{z^{n_m}}{n_1^{k_1}\cdots n_m^{k_m}} . \tag{1.11}$$

and observe that $\zeta(\underline{k}) = \mathrm{Li}_{\underline{k}}(1)$. From the power series expansion one easily arrives at the following (unipotent) differential equation:

$$d\,\mathrm{Li}_{k_1,\dots,k_m}(z) = \begin{cases} \mathrm{Li}_{k_1,\dots,k_m-1}(z)\frac{dz}{z} & k_m \neq 1 \\ \mathrm{Li}_{k_1,\dots,k_{m-1}}(z)\frac{dz}{1-z} & k_m = 1 \end{cases} \tag{1.12}$$

In particular, multiple polylogarithms are iterated integrals and can be written as integrals over certain triangular domains. In fact, borrowing from the description of multiple polylogarithms in terms of the KZ differential equation, associate with $\underline{k}$ the word $w = BA^{k_1-1}B\cdots BA^{k_m-1}$, and consider the differential form

$$\omega_i^w := \begin{cases} \frac{dt_i}{t_i} & \text{if } i\text{'th place in w is } A \\ \frac{dt_i}{1-t_i} & \text{otherwise} \end{cases} .$$

Then one obtains the formula

$$\zeta(\underline{k}) = \int_{0\le t_1\le t_2\le\cdots\le 1} \omega_1^w(t_1)\omega_2^w(t_2)\cdots$$

This serves as a source for the integral shuffle product formulas. The simplest example is:

$$\begin{aligned}
&\zeta(2)\cdot\zeta(2) = \\
&\int_{0\le t_1\le t_2\le 1} \frac{dt_1}{1-t_1}\cdot\frac{dt_2}{t_2}\cdot\int_{0\le s_1\le s_2\le 1}\frac{ds_1}{1-s_1}\cdot\frac{ds_2}{s_2} \\
&=\int_{t_1\le t_2\le s_1\le s_2}\Omega + \int_{s_1\le s_2\le t_1\le t_2}\Omega \\
&+\int_{t_1\le s_1\le t_2\le s_2}\Omega + \int_{s_1\le t_1\le s_2\le t_2}\Omega \\
&+\int_{s_1\le t_1\le t_2\le s_2}\Omega + \int_{t_1\le s_1\le s_2\le t_2}\Omega
\end{aligned}$$

where $\Omega = \frac{dt_1}{1-t_1}\cdot\frac{dt_2}{t_2}\cdot\frac{ds_1}{1-s_1}\cdot\frac{ds_2}{s_2}$. The six terms are themselves iterated integrals and one finds the formula

$$\zeta(2)^2 = 2\zeta(2,2)+4\zeta(1,3) .$$

In [Fur04] Furusho set out to develop a p-adic theory of multiple zeta values. The immediate problem is that the series (1.9) does not converge p-adically. In order to overcome this he checked that multiple polylogarithms, defined as Coleman functions using the differential equation (1.12) have a limit when z approached 1 and this limit is then defined to be the corresponding multiple zeta value. As explained before, one could simplify things by using the constant term.

Since p-adic multiple zeta values where defined using the multiple polylogarithm, it is perhaps not surprising that Furusho was only able to prove the analogues of the integral shuffle product formulae. The series product formulae were established in [BF06] and further work concerning generalized multiple zeta values, covering the case $n_m = 1$ as well, was later done in [FJ07]. The strategy for proving the sereis shuffle relation is rather simple, but certain intricacies have to be overcome by using the tangential base points and their generalizations. Again, we only deal with the simplest case, namely, the p-adic analogue for (1.10).

The natural function to consider for proving this is the two variable p-adic multiple polylogarithm defined near $(0,0)$ by

$$\mathrm{Li}_{(a,b)}(x,y) = \sum_{0<n<m} \frac{x^n y^m}{n^a m^b} .$$

One checks easily the differential relations between these functions:

$$x\frac{d}{dx}\mathrm{Li}_{(a,b)}(x,y) = \begin{cases} \mathrm{Li}_{(a-1,b)}(x,y) & a>1 \\ \frac{1}{x-1}(\mathrm{Li}_b(xy)-\mathrm{Li}_b(y)) & a=1 \end{cases}$$
$$y\frac{d}{dy}\mathrm{Li}_{(a,b)}(x,y) = \begin{cases} \mathrm{Li}_{(a,b-1)}(x,y) & b>1 \\ \frac{y}{1-y}\mathrm{Li}_a(xy) & b=1 \end{cases}$$

Thus, one may analytically continue $\mathrm{Li}_{(a,b)}(x,y)$ to Coleman functions in two variables. Now, the relation

$$\mathrm{Li}_a(x)\mathrm{Li}_b(y) = \mathrm{Li}_{(a,b)}(x,y) + \mathrm{Li}_{(b,a)}(y,x) + \mathrm{Li}_{a+b}(xy)$$

is obvious, because on the power series defining these functions near 0 it is true by the same summation proving the series shuffle product formula. Thus, to get the required formula one only needs to substitute $x = y = 1$. This is where the main difficulty in the entire argument is: It is by no means clear that $\mathrm{Li}_{(a,b)}(1,1) = \mathrm{Li}_{(a,b)}(1)$. Of course, the difficulty is increased by the fact that both points are singular for the differential equations defining the two functions.

To treat this difficulty, one has to work with the generalization of the notion of a tangential base point, which is also due to Deligne [Del89, 15.1-15.2]. Given a smooth variety X and a divisor $D = \sum_{i\in I} D_i$ with normal crossings and smooth components, set, for $J \subset I$, $D_J = \cap_{j\in J} D_j$. Let N_J be the normal bundle to D_J and let N_J^0 be the complement in N_J of $N_{J'}|_{D_J}$ for $J' \subset J$, and let N_J^{00} be the restriction of N_J^0 to $D_J^0 := D_J - \cup_{j\notin J} D_j$. Note that when $|I| = \dim(X) = 1$ so D is just one point P, we have $N_J^{00} = T_P(X) - \{0\}$. Deligne associates to a connection on X with logarithmic singularities along D, residue connections on every N_J^{00} with logarithmic singularities "at infinity".

Thus we again obtain new fiber functors on the category of unipotent connections by taking the fiber of the residue at points of the spaces N_J^{00}.

Remark 2. An important observations is that some of these constructions provide naturally isomorphic fiber functors. A typical example which captures the essense of things [BF06, Prop. 3.6 and Rem. 3.7] is the following: Suppose $X = \mathbb{A}^2$ and D_i is defined by $x_i = 0$ where x_i, $i = 1,2$, are the coordinates. One can start with a residue with log singularities along $D_1 + D_2$, take the residue along D_1, which can be interpreted again as a connection on $\mathbb{A}^2$ with logarithmic singularities along $\bar{x}_1 = 0, x_2 = 0$, restrict to $\bar{x}_1 = 1$, take the residue at the point $x_2 = 0$ and restrict to $\bar{x}_2 = 1$. Then this is exactly the same as taking the fiber at $(1,1)$ after taking the residue to $N^{00}_{\{1,2\}}$. Consequently, it is also the same as doing the above procedure with the roles of 1 and 2 reversed.

In [BF06, Section 4] we prove that if we have a Coleman function of algebraic origin on the space X, then one can analytically continue it to the spaces N^{00}_J and furthermore one obtains Coleman functions on these spaces. One can further deduce, essentially from the definition of the residue connection, differential relations between the Coleman functions restricted to the spaces N^{00}_J from the original differential relations. Indeed. In Proposition 4.4 there we proved, for the special case of restricting to the normal bundle of one of the components E of D, that

$$df = \sum \omega_i g_i \Rightarrow df^{(E)} = \sum (\mathrm{Res}_E\, \omega_i) g_i^{(E)} \tag{1.13}$$

where $f^{(E)}$ is the restriction to the normal bundle to E of f and where, if ω is locally written as $\omega' + hd\log(t)$, with t the defining parameter for E, then $\mathrm{Res}_E(\omega) = \omega'|_E + h|_E d\log(\bar{t})$.

Let us now apply these considerations to the functions $\mathrm{Li}_{(a,b)}$. One first observes that the differential equations defining these functions ultimately have singularities along $x = 0,1,\infty$, $y = 0,1,\infty$ and $xy = 1$, where the last divisor comes from the appearance of functions like $\mathrm{Li}(xy)$ in the expressions. Consequently, one should blow up $\mathbb{P}^1 \times \mathbb{P}^1$ at the point $(1,1)$ to make the singular locus normal crossings (the resulting space, if one blows up further the irrelevant points $(0,\infty), (\infty,0)$ is also the Deligne-Mumford compactification for the moduli space of curves of genus 0 with 5 marked points). One gets the picture in Figure 1.1

We now try to compute $\mathrm{Li}_{(a,b)}(1,1)$. First we should interpret it as the value of $\mathrm{Li}_{(a,b)}$ on a tangent vector $(\bar{x},\bar{y}) = (1,1)$ at the point $(1,1)$. The first step is to restrict to the divisor $y = 0$. By this we mean that we analytically continue to the normal bundle of $y = 0$ minus the 0 section and then restrict to the section $\bar{y} = 1$. In this case, the recepy outlined in (1.13), together with the restriction to $\bar{y} = 1$ boils down to removing the part multiplying dy and then set $y = 0$ in the formulas. The equations are therefore going to become

$$x\frac{d}{dx}\mathrm{Li}_{(a,b)}(x,1) = \begin{cases} \mathrm{Li}_{(a-1,b)}(x,1) & a > 1 \\ 0 & a = 1 \end{cases}$$

Since the boundary conditions on these functions are always set so that the constant term at 0 is 0 it follows immediately that the function $\mathrm{Li}_{(a,b)}(x,1)$ is identically 0 (this is not surprizing based on its expansion and the identity principle.

We now repeat the same considerations but this time restricting to the divisor $x=1$. In the same way as before the differential equations are going to be

$$y\frac{d}{dy}\mathrm{Li}_{(a,b)}(1,y) = \begin{cases} \mathrm{Li}_{(a,b-1)}(1,y) & b>1 \\ \frac{y}{1-y}\mathrm{Li}_a(y) & b=1 \end{cases}$$

which are of course the same differential equations satisfied by the single variable $\mathrm{Li}_{(a,b)}$. Based on the computation at $y=0$ and on Remark 2 we see that the

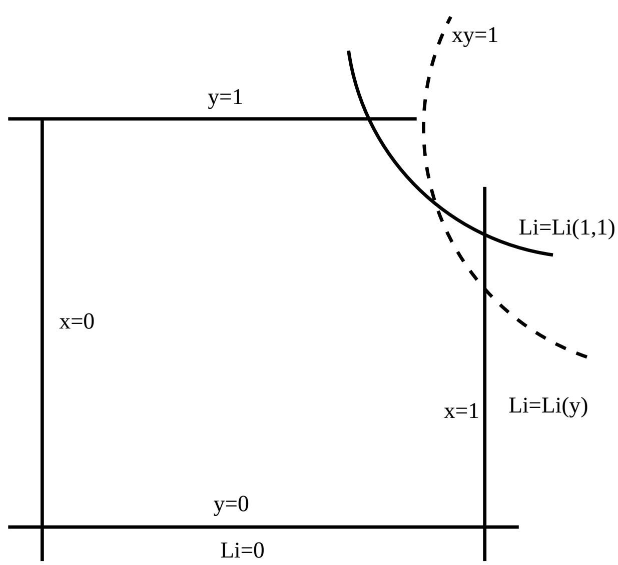

Fig. 1.1 Analytic continuation to $(1,1)$

boundary values at $y = 0$ for all of these functions are 0, which now gives the result $\mathrm{Li}_{(a,b)}(1,y) = \mathrm{Li}_{(a,b)}(y)$.

One can not now just substitute at the point $(1,1)$ since that point has been blown up. However, one checks that the differential equation for restricting to the exceptional divisor formed by the blowup, with an appropriately chosen coordinate, forces it to be constant. This makes the blowup banign (it is interesting to note that in the case $b = 1$ this is no longer the case and one therefore gets different normaliztions for the multiple zeta values, depending on a chosen location on the exceptional divisor. This is worked out in [FJ07]).

1.6 Coleman integration in families

An important observation to be made about Coleman integration is that it ultimately relies on the ground field to have residue field which is finite (by varying the field we can deal with residue fields which are algebraic over their prime fields) because we rely on the linear Frobenius. It is an interesting problem to try to remove this condition.

In this section we report on some recent work, which is still in progress. Full details will appear somewhere else. The problem that we want to address in this work is: given an algebraic family of closed forms, how can we integrate the family in a way better than just integrating each family member separately. More precisely, suppose that $X \xrightarrow{\pi} S$ is a smooth family of overconvergent rigid spaces (in this section we treat the nature of the spaces involved in a rather loose way). and $\omega \in \Omega^1_{X/S}$ is a relatively closed relative form, associate a Coleman integral F_ω which, when restricted to $s \in S$ is a Coleman integral of ω_s on $\pi^{-1}(s)$, but which is more canonical then just taking a choice of integral for each fiber, with the possibility of choosing a different constant of integration for each fiber. For affine S this can be thought of as doing Coleman integration "over $\mathscr{O}_S$", thus giving one solution to the problem posed in the previous paragraph.

Our motivation for treating this problem is computational and comes from the work of Lauder [Lau03, Lau04b, Lau04a]. In this work, one sees the possibility of computing the matrix of Frobenius on a variety by putting it inside a family, deforming to a fiber where this matrix can be computed easily, and then relying on the fact that the matrix of Frobenius satisfies a differential equation (derived from the Picard-Fuchs equation and computable) to recover the matrix by solving the equation with boundary terms provided by the simple fiber. One can speculate on the possibility of doing the same with Coleman integrals, given that the computation of the matrix of Frobenius is such an important part in the Computation of the Coleman integral, as we have seen in Section 1.4.

Following the approach to Coleman integration we presented, it makes sense to attempt to look for the answer by imposing additional constraints on the association of a Coleman integral to a form. Given the type of problem it makes sense to look for a differential condition. We would like to have a condition saying roughly that the formation of Coleman integrals commutes with differentiation in the direction of the base, i.e.,

$$\int \frac{\partial}{\partial s}\omega = \frac{\partial}{\partial s}F_{\omega} \tag{1.14}$$

where we assume for simplicity from now onward that S is one-dimensional, and the derivative refers to some vector field on the base.

There exists a well defined notion of differentiation of differential forms with respect to a vector field. However, there is no obvious way of lifting a vector field on S to a vector field on X (except when $X = Y \times S$, which is an interesting test case). Thus, Equation 1.14 does not quite make sense. Trying to get a meaningful statement out of it we are led to the following condition.

Lift the form ω to an absolute form $\tilde{\omega}$ on X (this can be done at least locally in the rigid topology. Since ω is relatively closed we may interpret $d\tilde{\omega}$ as an element of $\Gamma(X, \pi^*\Omega^1_S \otimes \Omega^1_{X/S})$. Note here that projecting from $\Omega^1_{X/S}$ to the first relative de Rham cohomology exactly yields the Gauss-Manin connection applied to the cohomology class of ω. Hypothesizing the existence of the theory of relative Coleman integration, we would like to integrate $d\tilde{\omega}$ to obtain a section

$$F_{d\tilde{\omega}} \in \Gamma(X, \pi^*\Omega^1_S \otimes \mathscr{O}_{\mathrm{Col}}(X/S))$$

with a hypothetical sheaf of relative Coleman functions $\mathscr{O}_{\mathrm{Col}}(X/S)$. Alternatively, we can integrate ω to get F_{ω}. We expect that $d_r F_{\omega} = \omega$, where d_r is the relative differential. Thus $dF_{\omega} - \tilde{\omega}$ is a one-form on X locally coming from the base. This suggest the following condition

$$dF_{\omega} - \tilde{\omega} = F_{d\tilde{\omega}} \tag{1.15}$$

Our goal in the rest of this section is to show that this is indeed a meaningful condition from two different points of view. The first involves The Gauss-Manin connection while the second comes from fairly recent work on differential Tannakian categories

1.6.1 Integration via the Gauss-Manin connection

A first attempt to use the condition (1.15) to get a relative Coleman integration theory follows roughly the same line as the approach in Section 1.4. We assume that X and S are affine (in the appropriate setting). We can chose a vector $\underline{\omega} \in (\Omega^1_{X/S})^n$ whose entries form a basis for $H^1_{\mathrm{dR}}(X/S)$ over $\mathscr{O}_S$ and chose a lifting $\underline{\tilde{\omega}} \in (\Omega^1_X)^n$. Since $\underline{\tilde{\omega}}$ consists of a basis, we find a relation of the form

$$d\underline{\tilde{\omega}} = \Theta(s) \otimes \underline{\omega} + d_r(\underline{g})$$

where $\Theta(s)$ in an n by n matrix with entries in Ω_S^1 and $\underline{g}$ has entries in $\Gamma(X, \pi^* \Omega_S^1)$. Applying (1.15) we get the following relation

$$dF_{\underline{\omega}} = \underline{\tilde{\omega}} + \Theta(s) F_{\underline{\omega}} + \underline{g} \,.$$

Rearranging terms we find

$$(d - \Theta(s)) F_{\underline{\omega}} = \underline{\tilde{\omega}} + \underline{g} \tag{1.16}$$

We now observe that on S the operator $d - \Theta(s)$ is nothing but the Gauss-Manin connection ∇_{GM} for the vector bundle $H^1_{\mathrm{dR}}(X/S)$. Consequently, the equation (1.16) describes $F_{\underline{\omega}}$ as a preimage, under $\pi^*\nabla_{\mathrm{GM}}$ of a certain one form $\underline{\tilde{\omega}} + \underline{g}$. Note that fiber by fiber $\pi^*\nabla_{\mathrm{GM}}$ restricts to just ordinary derivative while $\underline{\tilde{\omega}} + \underline{g}$ restricts to $\underline{\omega}$ so fiber by fiber we indeed obtain the required integrals of our forms.

In [Col89, Col94] Coleman extended his theory of integration to define integration (but not iterated integrals) of one forms with values in overconvergent Frobenius isocristals, that is, differential equations which overconverges in the appropriate sense, which have an action of Frobenius. Using this version of Coleman integration theory we obtain the required $F_{\underline{\omega}}$.

This method of integration can be extended to iterated integrals by using universal unipotent connections.

1.6.2 Differential Tannakian categories

We now give a fairly brief introduction to differential Tannakian categories. The Galois theory of differential equations is fairly well known. We recall that a differential ring is a ring R (we assume our rings are commutative) equipped with a derivation $\partial : R \to R$. Let K be a differential field. Starting from a linear differential equation

$$\nabla : \partial_x \underline{y} = A \cdot y \,, \tag{1.17}$$

where A has entries in K, the theory finds an extension differential field K^∇ over which all solutions of the equation are defined, and considers the group of automorphisms of K^∇ over K. The resulting Galois groups are algebraic groups. The Tannakian approach to differential Galois theory interprets these Galois groups as the groups associated with the Tannakian subcategory (of all linear differential equations) generated by the given one.

Differential algebraic groups and differential Tannakian categories start showing up when one considers parameterized systems of differential equations. Suppose K is equipped with two commuting derivations ∂_x and ∂_t (e.g., $\mathbb{C}(x,y)$). If we have a differential equation with respect to ∂_x, we can ignore the t-derivation completely and recover the same theory as before. Instead, we can look for a field extension which still carries two derivations, over which all solutions of the equation exist.

This, perhaps surprisingly, gives a fascinatingly different theory. It is best to consider an example (taken from [CS07, Example 3.1]).

Suppose $K = \mathbb{C}(x,t)$ and our differential equation is

$$\nabla : \partial_x y = \frac{t}{x} y$$

whose solution is $y = cx^t$. Thus, in standard differential Galois theory we would simply add x^t to K. However, since our field K^∇ should be closed with respect to both ∂_x and ∂_y, and since $\partial_t x^t = \log(x) x^t$ we have

$$K^\nabla = K(x^t, \log(x)) .$$

We now consider an automorphism σ of K^∇, commuting with the derivations and fixing K. It preserves solutions of the differential equation so

$$\sigma(x^t) = a(t) x^t .$$

On the other hand $\sigma(\log(x))$ should be constant with respect to t and differentiate to $1/x$ with respect to x. We therefore have

$$\sigma(\log(x)) = \log(x) + b$$

where $b \in \mathbb{C}$. We now have

$$\begin{aligned}(\partial_t a(t)) \cdot x^t + a(t) \log(x) x^t &= \partial_t (a(t) x^t) = \partial_t (\sigma(x^t)) \\ &= \sigma(\partial_t x^t) = \sigma(\log(x) x^t) = (\log(x) + b) a(t) x^t\end{aligned}$$

and it follows that $ba(t) = \partial_t a(t)$ hence $b = (\partial_t a(t))/a(t)$. We obtain our differential Galois group

$$\mathrm{Gal}(K^\nabla / K) = \{a(t) \neq 0 \ , \ \partial_t \frac{\partial_t a(t)}{a(t)} = 0\}$$

where the group structure is given by multiplication. This is an example of a *Linear differential algebraic group* (in this case over the field $\mathbb{C}(t)$).

We recall that a differential affine variety over a differential field K (with one derivation ∂_t for simplicity) is the subset of some K^n which is the set of solutions of some differential equation. In other words, if the coordinates are a_1 to a_n, it is defined by the vanishing of a polynomial in the a_i and their derivatives $(\partial_t)^j a_i$.

The ring of differential functions on the affine space $\mathbb{A}^n$ is the polynomial ring in an infinite number of (formal) variables

$$K\{\mathbb{A}^n\} := K[\{(\partial_t)^j a_i \ , \ j = 0, \dots, \infty \ , \ i = 1, \dots, n\}] .$$

It has a derivation, extending the one on K, given by

$$\partial_t((\partial_t)^j a_i) = (\partial_t)^{j+1} a_i$$

making it a *differential K-algebra*. Given a differential affine subvariety V of $\mathbb{A}^n$, we can associate with it the radical differential ideal I generated by the defining equations (differential ideal means closed under ∂_t) and its ring of functions $K\{V\} = K\{\mathbb{A}^n\}/I$. Just like in usual algebraic geometry we may now interpret V, or rather its set of K-rational points, as the set of differentiable K-algebra homomorphisms $K\{V\} \to K$ (K-algebra homomorphisms commuting with the derivation). Note that algebraic varieties are a special case of differential varieties but their rings of functions in the two cases are quite different. There are corresponding notions of morphisms between differential varieties $V \to W$, and these give rise to differential algebra morphisms $K\{W\} \to K\{V\}$.

A differential algebraic group is defined as a differential subvariety G of some GL_n which is closed under multiplication and inversion. By the usual procedure, this gives rise to a structure of a *differential Hopf algebra* on $K\{G\}$, i.e., a Hopf algebra together with a derivation extending the one on K and commuting with all structural morphisms (i.e., both multiplication and co-multiplication between $K\{G\}$ and $K\{G\} \otimes K\{G\}$ commute with the derivation, which on the latter object is defined by

$$\partial a \otimes b = (\partial a) \otimes b + a \otimes (\partial b)). \tag{1.18}$$

A homomorphism of differential algebraic groups, i.e., a differentiable algebraic morphism which is also a group homomorphism, induces a homomorphism of differentiable Hopf algebras. An interesting example is provided by the dlog homomorphism

$$d\log : \mathbb{G}_m \to \mathbb{G}_a \ , \ d\log(a) = \frac{\partial(a)}{a} \tag{1.19}$$

Definition 8. A representation of a Linear differentiable algebraic group G on a finite dimensional K-vector space V is a differentiable algebraic homomorphism $G \to \mathrm{GL}(V)$.

In this definition we make $\mathrm{GL}(V)$ into a differential algebraic group by identifying it with some GL_n by choosing a basis.

We first remark that even a standard algebraic group has some new representations when viewed as a differential algebraic group. For example, $\mathbb{G}_m$ has a two dimensional representation given by $\left(\begin{smallmatrix} 1 & d\log \\ 0 & 1 \end{smallmatrix}\right)$.

In terms of the Hopf algebra $K\{G\}$, a representation of G on V is given by a $K\{G\}$-comodule structure on V,

$$\rho : V \to V \otimes_K K\{G\}$$

in the same way as an algebraic G-representation would (no differentials here!). To see this, consider first the comodule corresponding to the standard representation of GL_n on K^n as an algebraic representation, then obtain the comodule structure for K^n with respect to $K\{G\}$ by simply composing with the embedding of the algebraic $K[G]$ in the differential $K\{G\}$. In concrete terms recalling that the comodule structure in the algebraic setting is given by sending v to the function (viewed as an element of $V \otimes K[G]$) $g \mapsto gv$), this gives the comodule structure

$$e_i \mapsto \sum e_j \otimes a_{ji} \tag{1.20}$$

Now, for a representation of G compose with the Hopf algebra homomorphism $K\{\mathrm{GL}_n\} \to K\{G\}$ to obtain the required $K\{G\}$-comodule structure, from which the representation is easily recoverable.

The fact that no differentials are involved in the Hopf-algebraic description of differential representation is perhaps confusing if G is algebraic, but it is not a contradiction to anything because $K[G]$ is quite different from $K\{G\}$. It nevertheless suggests that a Tannakian description of a differential algebraic group in terms of its category of representations needs to use something outside the structure of Tannakian category on this category of representations.

A Tannakian description of Linear differential algebraic groups was given quite recently by Ovchinikov [Ovc08, Ovc09a, Ovc09b, Kam10]. There is an alternative approach using model theory due to Kamensky [Kam10]. The papers by Kamensky do an excellent job of describing the categorical formalism. Unfortunately, for proofs they use model theory in a rather "black-box" approach (referring to deep work of Hrushovsky) which sheds little light on the algebraic point of view. We try to give here a minimal account, which we found useful in understanding the situation. We note however, that the above mentioned references due more, in the sense that they show, under some additional assumptions, that the Galois group is a pro-differential algebraic group rather than just a Hopf algebra (this is not equivalent in the differential algebraic setting see [Ovc08, p. 8]).

Suppose we are given a Linear differential algebraic group G, which we would like to recover from its category of representations Rep_G. If we take the Hopf-algebraic point of view, the usual Tannakian formalism already reconstructs for us the Hopf algebra $K\{G\}$ (since the category is just that of comodules for that algebra) and so we only need to recover the derivation. This is then not to be found in the category Rep_G itself, as this suffices exactly to recover the Hopf algebra structure by Tannakian duality. It must come from an additional structure on Rep_G. A so called *differential structure*.

For motivating this structure, consider again the differential equation (1.17) but over the field $\mathbb{C}(x,t)$. Using the fact that the two derivations commute we obtain

$$\partial_x(\partial_t \underline{y}) = \partial_t(A\underline{y}) = (\partial_t A)\underline{y} + A\partial_t \underline{y}\ .$$

This means that we obtain a new differential equation

$$\partial_x \begin{pmatrix} \underline{y} \\ \partial_t \underline{y} \end{pmatrix} = \begin{pmatrix} A & 0 \\ \partial_t A & A \end{pmatrix} \begin{pmatrix} \underline{y} \\ \partial_t \underline{y} \end{pmatrix} \tag{1.21}$$

which is an extension of the original equation by itself. This can be seen to be a functorial construction, and is the required differential structure.

Definition 9. A differential rigid abelian tensor category over the field K is a rigid abelian tensor category $\mathscr{T}$, and satisfying the condition that $\mathrm{End}(\mathbb{1}) = K$, together with a functor $D : \mathscr{T} \to \mathscr{T}$ sitting in a short exact sequence

$$0 \to \mathrm{id} \to D \to \mathrm{id} \to 0 \tag{1.22}$$

and which satisfies a certain list of axioms connecting D with the tensor structure (see [Ovc08, Ovc09a, Ovc09b, Kam10]).

Remark 3. In the description given in [Kam10] one defines a new category consisting of short exact sequences $0 \to M \to N \to M \to 0$ in the category $\mathscr{T}$. This is given a tensor structure as follows: The tensor product of $(0 \to M_1 \to N_1 \to M_1 \to 0)$ with $(0 \to M_2 \to N_2 \to M_2 \to 0)$ is the Baer sum of extensions of the first sequence tensored with M_2 with the second sequence tensore with M_1, both viewed as extensions of $M_1 \otimes M_2$ with itself. The requirements on D are now simply that it defines a tensor functor from $\mathscr{T}$ to this new category.

We note that we do not need to assume that K is a differential field because that will be forced from the axioms. To get a Tannakian theory we need to introduce a differential structure on Vec_K for a differential field K. To see this, recall first that for any ring R, R-module extensions of R by itself are equivalent to derivations of R - The R module structure associated with a derivation ∂ is given by

$$r \mapsto \left(\begin{smallmatrix} r & 0 \\ -\partial r & r \end{smallmatrix}\right)$$

By tensoring with an arbitrary R-module M we see that a derivation leads to a functor D from the category of R-modules to itself which sits in a short exact sequence as in (1.22). This holds of course for a differential field K providing the required structure on Vec_K. In concrete terms, for a K-vector space V, $D(V)$ is the abelian group $V \times V$ with the K-vector space structure given by

$$\alpha(v_1, v_2) = (\alpha v_1, \alpha v_2 - \partial(\alpha) v_1) \tag{1.23}$$

A useful convention is to identify the vector $v \in V$ with $(0, v) \in D(V)$ and to denote the map $v \mapsto (v, 0)$ by ∂. This way, the action of the field is given by the following equations

$$\alpha v = \alpha v$$
$$\partial(\alpha v) = (\partial \alpha) v + \alpha \partial v$$

The functoriality is given sending $T : V \to W$ to $(T, T) : D(V) \to D(W)$. Note however that in terms of standard bases this description is misleading: Suppose that B is a matrix with entries in K such that multiplying by B gives a linear map $B : K^n \to K^m$. Then, in terms of the standard bases provided, e.g. for K^n by $(e_1, 0), \ldots, (e_n, 0), (0, e_1), \ldots, (0, e_n)$ the matrix of $D(B)$ is going to be

$$\begin{pmatrix} B & 0 \\ \partial(B) & B \end{pmatrix} \tag{1.24}$$

The above description immediately suggests the extension of D to differential algebraic representations and to the category of Hopf-comodules. Indeed, applying

D to the standard representation of GL_n on K^n we get, by (1.24), the representation of GL_n on $K^n \oplus K^n$ given by $A \mapsto \begin{pmatrix} A & 0 \\ \partial(A) & A \end{pmatrix}$ whose associated comodule is given by

$$\begin{aligned} e_i &\mapsto \sum e_j \otimes a_{ji} \\ \partial e_i &\mapsto \sum (\partial e_j \otimes a_{ji} + e_j \otimes \partial a_{ji}) \end{aligned}$$

from which we get the extension of D to comodules:

$$D(\rho)(v) = \rho(v)\ ,\ D(\rho)\partial v = \partial(\rho(v))$$

where ∂ acts on a tensor product in the obvious way (1.18).

Definition 10. A differential tensor functor $\mathscr{T}_1 \xrightarrow{F} \mathscr{T}_2$ between two differential rigid abelian tensor categories is a tensor functor together with a natural isomorphism $D_2 \circ F \cong F \circ D_1$ compatible in the obvious way with the short exact sequence (1.22). A morphism of differential tensor functors $\alpha : F \to F'$ is a natural transformation of tensor functors which commutes with D in the sense that the diagram

$$\begin{array}{ccc} F \circ D_1 & \xrightarrow{\alpha} & F' \circ D_1 \\ \downarrow & & \downarrow \\ D_2 \circ F & \xrightarrow{D_2(\alpha)} & D_2 \circ F' \end{array}$$

commutes.

Example 2. Quite clearly the forgetful functor $\mathrm{Rep}_G \to \mathrm{Vec}_K$ is a differential tensor functor. Another example is solutions of differential equations. Consider the functor Sol that takes a differential equation ∇ as in (1.17) over the field $\mathbb{C}(x,t)$ to its space of solutions considered as a vector space over $\mathbb{C}(t)$. Then, according to (1.21), we can map $D(\mathrm{Sol}(\nabla))$ to $\mathrm{Sol}(D(\nabla))$ using the formula

$$(\underline{y}_1, \underline{y}_2) \mapsto (\underline{y}_1, \underline{y}_2 + \partial_t \underline{y}_1)$$

(note that to make this a $\mathbb{C}(t)$ linear map we exactly need to give $D(\mathrm{Sol}(\nabla))$ the vector space structure (1.23)).

Definition 11. A differential fiber functor on a differential rigid abelian tensor category is a differential tensor functor ω to Vec_K. If the category has a fiber functor it is called neutral Tannakian.

Theorem 7 ([Ovc09b, Theorem 1]). *Let $\mathscr{T}$ be a neutral differential Tannakian category with the differential fiber functor ω. Then $\mathscr{T}$ is equivalent to the category* Rep_G *of finite dimensional representations of an affine differential group scheme G. Furthermore, for a differential K-algebra F we have*

$$G(F) = \mathrm{Aut}(\omega \otimes F) \tag{1.25}$$

where Aut *here means automorphisms of the differential tensor functor in the sense of Definition 10.*

We sketch a proof of this result. Standard Tannakian theory tells us that $\mathscr{T}$ is equivalent to the category of representations of a certain affine group scheme, or equivalently to the category of comodules over some Hopf algebra H. The result will follow if we construct a derivation on H in such a way that the functor D on $\mathscr{T}$ corresponds to the functor D on the category of H-comodules.

We now recall [Del90] that the Hopf algebra H may be described concretely as an "algebra of matrix coefficients". An element in such an algebra is provided by a pair $(T,\mathscr{E})$ where $T\in\mathscr{T}$, $\mathscr{E}\in\omega(T)\otimes\omega(T)^*$, where $\omega(T)^*$ is the K-dual of $\omega(T)$. When $\mathscr{T}$ is the category of representations of an affine group scheme G and $\omega(T)$ is just the underlying vector space to $T\in\mathscr{T}$, then a pair $(T,v\otimes w^*)$ is to be thought of as corresponding to the function on G given by $g\mapsto w^*(gv)$. One identifies two pairs $(T_1,\mathscr{E}_1)$ and $(T_2,\mathscr{E}_2)$ if there exists a map $f:T_1\to T_2$ in such a way that both $\mathscr{E}_1$ and $\mathscr{E}_2$ are obtained from an element of $\omega(T_1)\otimes\omega(T_2)^*$ via the obvious maps. Clearly, two identified elements give rise to identical functions. One can easily derive from this formulas for the multiplication and comultiplication.

This description immediately suggests the way to define the derivation on H. Indeed, by (1.24) the derivatives of the matrix coefficients associated with the representation T are visible in $D(T)$. A bit of thought gives the following formula for the derivative of matrix coefficients

$$\partial(T,v\otimes w^*)=(D(T),(\partial v)\otimes(\partial w^*))$$

(if one is puzzled by the fact that ∂ is applied twice, note that the K-algebra structure of H can be obtained by multiplying the with elements of K either v or w^*). It now becomes a routine check to see that ∂ is indeed a derivation of the Hopf algebra H and it is quite obvious by the way we defined it that the D on H-comodules and on vector spaces correspond.

Finally, the description of the differential points of G is fairly easy. A point in the usual sense with values in F corresponds on the one hand the a morphism $H\to F$ and on the other hand to an automorphism of $\omega\otimes F$. Now, both the condition that the homomorphism preserve the derivation, and the condition that the automorphism is an automorphism of differential functors boil down to saying that "if α_T is represented by the matrix A, then $\alpha_{D(T)}$ is represented by the matrix $\left(\begin{smallmatrix}A&0\\ \partial A&0\end{smallmatrix}\right)$."

1.6.3 Applications to Coleman integration in families

A major difficulty with the above construction is that it is dependent on the choice of a derivation. If we want to get a theory which takes all derivations into account (like connections do) we are led, after some thought to make the following construction.

Recall that we are assuming a situation $\pi:X\to S$. Suppose M is a vector bundle on X equipped with a relative connection

$$\nabla : M \to M \otimes \Omega^1_{X/S}$$

which is integrable. Suppose we can lift ∇ to an absolute connection $\tilde{\nabla} : M \to M \otimes \Omega^1_X$. Because ∇ is integrable, the curvature of $\tilde{\nabla}$ has at least one form coming from S and can thus be projected on $\Omega^1_S \otimes \Omega^1_{X/S}$:

$$C = \tilde{\nabla}^2 \in \Omega^1_S \otimes \Omega^1_{X/S} \otimes \mathrm{End}(M)$$

We define a new module with connection $D = D_{\tilde{\nabla}} = D(M,\nabla)_{\tilde{\nabla}}$ where $D = M \oplus \Omega^1_S \otimes M$ and the connection is defined by

$$\nabla_D(m_1, \alpha \otimes m_2) = (\nabla m_1, \alpha \otimes \nabla m_2 - C \cdot M) \; .$$

This connection is integrable. It is independent of $\tilde{\nabla}$ up to a canonical isomorphism: Suppose $\tilde{\nabla}' = \tilde{\nabla} + A$ is another lift. Here $A \in \Gamma(X, \pi^{-1}\Omega^1_S \otimes \mathrm{End}(M))$ because it projects to 0 in relative forms. Then it is easy to compute that the corresponding curvature is $C' = C + \nabla(A)$ (where ∇ takes Ω^1_S as constants and acts in the induced way on $\mathrm{End}(M)$). Then we get a canonical horizontal isomorphism between $D_{\tilde{\nabla}}$ and $D_{\tilde{\nabla}'}$ given by

$$(m_1, \alpha \otimes m_2) \mapsto (m_1, \alpha \otimes m_2 + A m_1) \; .$$

Consequently we can glue these objects, coming from different local liftings of ∇, to obtain a global object $D(M,\nabla)$. Clearly, there is a short exact sequence of vector bundles with relative connections,

$$0 \to \Omega^1_S \otimes M \to D(M,\nabla) \to M \to 0$$

because all the horizontal isomorphisms constructed commute with these short exact sequences. Clearly, the construction D is functorial.

For vector bundles over S we can make an analogous functorial construction. For such a vector bundle M define $D(M) = M \oplus \Omega^1_S \otimes M$ with the $\mathscr{O}_S$-module structure

$$s \cdot (m_1, \alpha \otimes m_2) = (s m_1, s\alpha \otimes m_2 + ds \otimes m_1) \; .$$

Suppose now that X and S are residue discs. Then, mimicking the constructions in Example 2 we have a well behaved solutions functor

$$\mathrm{Sol} : \{ \text{ Relative connections } (M, \nabla : M \to \Omega^1_{X/S}) \} \to \{ \text{ Vector bundles on } S \}$$

given by taking horizontal sections, and a map

$$D \circ \mathrm{Sol} \to \mathrm{Sol} \circ D \; , \; (m_1, \alpha \otimes m_2) \mapsto (m_1, \alpha \otimes m_2 + \tilde{\nabla} m_1) \qquad (1.26)$$

where m_1 and m_2 are horizontal sections for ∇, implying that $\tilde{\nabla} m_1 \in \Omega^1_S \otimes M$

There is some way to go before we can incorporate these constructions into a functioning Tannakian differential Tannakian formalism. The main problem is of having a good Tannakian theory over rings (but see [Wed04, Sch09] for some progress on these matters). Assuming such a formalism it seems reasonable to prove the following.

Conjecture 1. For any two residue discs in X there is a unique differentiable path invariant under Frobenius between the two solution functors.

Note that a differentiable path, in the sense of differential Tannakian categories should really be thought of as a horizontal path, only that there is no connection on paths.

We expect the proof of this conjecture to roughly follow the method described in Subsection 1.5.2. The Lie algebra will inherit a connection, and its graded pieces are going to be dominated again by tensor powers of $H^1_{\mathrm{dR}}(X/S)^*$ with the connection induced by the Gauss-Manin connection. Differential paths are now going to be related with horizontal sections on the Lie algebra, over some residue disc in S. But these are now well behaved vector spaces over the ground field so we can apply Frobenius as before to complete the argument.

Let us close by observing the relation of this conjecture with the condition (1.15). In the notation introduced before stating this condition, suppose we have a closed $\omega \in \Omega^1_{X/S}$ and we lift it to a $\tilde{\omega} \in \Omega_X$. correspondingly we have the connection ∇ and its lift $\tilde{\nabla}$ given by

$$\nabla = d_r - \begin{pmatrix} 0 & 0 \\ \omega & 0 \end{pmatrix} , \ \tilde{\nabla} = d - \begin{pmatrix} 0 & 0 \\ \tilde{\omega} & 0 \end{pmatrix} .$$

We can now compute that the connection ∇_D is going to be given by the following formula

$$\nabla_D \left(\begin{pmatrix} y_1 \\ y_2 \end{pmatrix}, \alpha \otimes \begin{pmatrix} y_3 \\ y_4 \end{pmatrix} \right) = \left(\nabla \begin{pmatrix} y_1 \\ y_2 \end{pmatrix}, \alpha \otimes \nabla \begin{pmatrix} y_3 \\ y_4 \end{pmatrix} + \begin{pmatrix} 0 & 0 \\ d\tilde{\omega} & 0 \end{pmatrix} \begin{pmatrix} y_1 \\ y_2 \end{pmatrix} \right)$$

where $d\tilde{\omega}$ is to be thought of as residing inside $\Omega^1_S \otimes \Omega^1_{X/S}$. Now we check what it means for $\left(\begin{smallmatrix} 1 \\ F_\omega \end{smallmatrix}\right)$ to be a horizontal section at the residue discs of x and z, say, which is compatible with respect to translation by a differential path. Appropriately translating the condition in Definition 10 we find that it simply means that another horizontal section that translates under the same path is the image of $\left(\begin{smallmatrix} 1 \\ F_\omega \end{smallmatrix}\right)$ under (1.26) which is

$$\left(\begin{pmatrix} 1 \\ F_\omega \end{pmatrix}, \begin{pmatrix} 0 \\ dF_\omega - \tilde{\omega} \end{pmatrix} \right) .$$

In other words, $dF_\omega - \tilde{\omega}$ is a Coleman integral, which is just (1.15).

References

[Bes02] A. Besser. Coleman integration using the Tannakian formalism. *Math. Ann.*, 322(1):19–48, 2002.

[BF06] A. Besser and H. Furusho. The double shuffle relations for p-adic multiple zeta values. In *Primes and knots*, volume 416 of *Contemp. Math.*, pages 9–29. Amer. Math. Soc., Providence, RI, 2006.

[CdS88] R. Coleman and E. de Shalit. p-adic regulators on curves and special values of p-adic L-functions. *Invent. Math.*, 93(2):239–266, 1988.
[Chi98] B. Chiarellotto. Weights in rigid cohomology applications to unipotent F-isocrystals. *Ann. Sci. École Norm. Sup. (4)*, 31(5):683–715, 1998.
[CLS99] B. Chiarellotto and B. Le Stum. F-isocristaux unipotents. *Compositio Math.*, 116(1):81–110, 1999.
[Col82] R. Coleman. Dilogarithms, regulators, and p-adic L-functions. *Invent. math.*, 69:171–208, 1982.
[Col85] R. Coleman. Torsion points on curves and p-adic abelian integrals. *Annals of Math.*, 121:111–168, 1985.
[Col89] R. Coleman. p-adic integration. notes from lectures at the University of Minnesota, 1989.
[Col94] R. Coleman. A p-adic Shimura isomorphism and p-adic periods of modular forms. *Contemp. math.*, 165:21–51, 1994.
[CS07] P. J. Cassidy and M. F. Singer. Galois theory of parameterized differential equations and linear differential algebraic groups. In *Differential equations and quantum groups*, volume 9 of *IRMA Lect. Math. Theor. Phys.*, pages 113–155. Eur. Math. Soc., Zürich, 2007.
[Del89] P. Deligne. Le groupe fondamental de la droite projective moins trois points. In *Galois groups over* $\mathbf{Q}$ *(Berkeley, CA, 1987)*, volume 16 of *Math. Sci. Res. Inst. Publ.*, pages 79–297. Springer, New York, 1989.
[Del90] P. Deligne. Catégories tannakiennes. In *The Grothendieck Festschrift, Vol. II*, volume 87 of *Progr. Math.*, pages 111–195. Birkhäuser Boston, Boston, MA, 1990.
[DM82] P. Deligne and J.S. Milne. Tannakian categories. In *Hodge cycles, motives, and Shimura varieties*, volume 900 of *Lect. Notes in Math.*, pages 101–228. Springer, 1982.
[Elk73] R. Elkik. Solutions d'équations à coefficients dans un anneau hensélien. *Ann. Sci. École Norm. Sup. (4)*, 6:553–603 (1974), 1973.
[FJ07] H. Furusho and A. Jafari. Regularization and generalized double shuffle relations for p-adic multiple zeta values. *Compos. Math.*, 143(5):1089–1107, 2007.
[Fur04] H. Furusho. p-adic multiple zeta values. I. p-adic multiple polylogarithms and the p-adic KZ equation. *Invent. Math.*, 155(2):253–286, 2004. citing [8],[11],[16].
[Kam10] M. Kamensky. Model theory and the tannakian formalism. Preprint, 2010.
[Lau03] Alan G. B. Lauder. Homotopy methods for equations over finite fields. In *Applied algebra, algebraic algorithms and error-correcting codes (Toulouse, 2003)*, volume 2643 of *Lecture Notes in Comput. Sci.*, pages 18–23. Springer, Berlin, 2003.
[Lau04a] Alan G. B. Lauder. Counting solutions to equations in many variables over finite fields. *Found. Comput. Math.*, 4(3):221–267, 2004.
[Lau04b] Alan G. B. Lauder. Deformation theory and the computation of zeta functions. *Proc. London Math. Soc. (3)*, 88(3):565–602, 2004.
[Ovc08] A. Ovchinnikov. Tannakian approach to linear differential algebraic groups. *Transform. Groups*, 13(2):413–446, 2008.
[Ovc09a] A. Ovchinnikov. Differential Tannakian categories. *J. Algebra*, 321(10):3043–3062, 2009.
[Ovc09b] A. Ovchinnikov. Tannakian categories, linear differential algebraic groups, and parametrized linear differential equations. *Transform. Groups*, 14(1):195–223, 2009.
[Sch98] P. Schneider. Basic notions of rigid analytic geometry. In *Galois representations in arithmetic algebraic geometry (Durham, 1996)*, volume 254 of *London Math. Soc. Lecture Note Ser.*, pages 369–378. Cambridge Univ. Press, Cambridge, 1998.
[Sch09] D. Schappi. Tannaka duality for comonoids in cosmoi. Preprint, 2009.
[vdP86] M. van der Put. The cohomology of Monsky and Washnitzer. *Mém. Soc. Math. France (N.S.)*, 23:33–59, 1986. Introductions aux cohomologies p-adiques (Luminy, 1984).
[Wed04] Torsten Wedhorn. On Tannakian duality over valuation rings. *J. Algebra*, 282(2):575–609, 2004.
[Wil97] J. Wildeshaus. *Realizations of polylogarithms*. Springer-Verlag, Berlin, 1997.