THE SYNTOMIC REGULATOR FOR K_4 OF CURVES

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ABSTRACT. Let C be a curve defined over a discrete valuation field of characteristic zero where the residue field has positive characteristic. Assuming that C has good reduction over the residue field, we compute the syntomic regulator on a certain part of $K_4^{(3)}(C)$. The result can be expressed in terms of p-adic polylogarithms, and Coleman integration.

1. INTRODUCTION

Let K be a complete discrete valuation field of characteristic zero, R its valuation ring, and κ its residue field. Assume κ is of positive characteristic p. If $\mathcal{X}/R$ is a scheme, smooth and of finite type, then, after tensoring with $\mathbb{Q}$, one can decompose the K-theory of $\mathcal{X}$ according to the weights eigenspaces, i.e.,

$$K_n(\mathcal{X}) \otimes_{\mathbb{Z}} \mathbb{Q} = \sum_j K_n^{(j)}(\mathcal{X}) \ ,$$

where $K_n^{(j)}(\mathcal{X})$ consists of those α in $K_n(\mathcal{X}) \otimes_{\mathbb{Z}} \mathbb{Q}$ such that $\psi^k(\alpha) = k^j \alpha$ for all Adams operators ψ^k, see [Sou85, Proposition 5]. There is a regulator map

$$K_n^{(j)}(\mathcal{X}) \to H_{\mathrm{syn}}^{2j-n}(\mathcal{X}, j),$$

see [Bes00a]. In many interesting cases the target group of the regulator is isomorphic to the rigid cohomology group of the special fiber $\mathcal{X}_\kappa$, in the sense of Berthelot, $H_{\mathrm{rig}}^{2j-n-1}(\mathcal{X}_\kappa/K)$. We will be interested in the situation where $\mathcal{X}$ is a proper smooth curve $\mathcal{C}$ over R with generic fiber C, and the K-group is $K_4^{(3)}(\mathcal{C})$. $K_4^{(3)}(C)$ is known to be isomorphic to $K_4^{(3)}(\mathcal{C})$ under localisation, see Section 2.2. The target group for the regulator in this case is $H_{\mathrm{rig}}^1(\mathcal{C}_\kappa/K) \cong H_{\mathrm{dR}}^1(C/K)$. Poincaré duality gives a pairing

$$H_{\mathrm{dR}}^1(C/K) \times H_{\mathrm{dR}}^1(C/K) \to H_{\mathrm{dR}}^2(C/K) \cong K \ .$$

If α is an element of $K_4^{(3)}(C)$ and ω is in $H^0(C, \Omega_{C/K}) \subset H_{\mathrm{dR}}^1(C/K)$, we want to compute the image of the pairing of $\mathrm{reg}(\alpha)$ and ω in K.

The complex $\mathcal{M}_{(3)}(F)$ was defined in [dJ95, Section 3] for any field F of characteristic zero. It consists of three terms in cohomological degrees 1, 2 and 3:

$$M_3(F) \to M_2(F) \otimes F_{\mathbb{Q}}^* \to F_{\mathbb{Q}}^* \otimes \bigwedge^2 F_{\mathbb{Q}}^* \ , \tag{1.1}$$

with $F_{\mathbb{Q}}^* = F^* \otimes_{\mathbb{Z}} \mathbb{Q}$, and $M_n(F)$ a $\mathbb{Q}$-vector space on symbols $[x]_n$ for x in $F \setminus \{0, 1\}$, modulo nonexplicit relations depending on n. The maps in the complex are given by

$$\mathrm{d}[x]_3 = [x]_2 \otimes x$$

and

$$\mathrm{d}[x]_2 \otimes y = (1-x) \otimes (x \wedge y).$$

There are maps

$$H^2(\mathcal{M}_{(3)}(F)) \to K_4^{(3)}(F)$$

and

$$H^3(\mathcal{M}_{(3)}(F)) \to K_3^{(3)}(F)\ .$$

The last of those two maps is in fact an isomorphism.

We can apply this with F the function field of C, but as the syntomic regulator needs some information over the residue field, we have to use an analogous complex $\mathcal{M}_{(3)}(\mathcal{O})$ for $\mathcal{O}$ the local ring of F consisting of functions on C that are generically defined on the special fiber $\mathcal{C}_\kappa$. The complex has the shape

$$M_3(\mathcal{O}) \to M_2(\mathcal{O}) \otimes \mathcal{O}^*_{\mathbb{Q}} \to \mathcal{O}^*_{\mathbb{Q}} \otimes \bigwedge^2 \mathcal{O}^*_{\mathbb{Q}}\ , \tag{1.2}$$

with in this case $M_n(\mathcal{O})$ a $\mathbb{Q}$-vector space on symbols $[u]_n$ for u in $\mathcal{O}^\flat$, the *special units* of $\mathcal{O}$, namely those that do not reduce to 1 in the residue field, again modulo nonexplicit relations depending on n. The maps in the complex are given by

$$\mathrm{d}[u]_3 = [u]_2 \otimes u$$

and

$$\mathrm{d}[u]_2 \otimes v = (1-u) \otimes (u \wedge v).$$

In fact, one may view $M_2(\mathcal{O}) \subseteq M_2(F)$, see Remark 2.41.

The complex comes with maps

$$H^p(\mathcal{M}_{(3)}(\mathcal{O})) \to K_{6-p}^{(3)}(\mathcal{O})$$

for $p = 2$ and 3.

To explain the terms in which the formula for the regulator will be expressed, we need to introduce Coleman integration theory (see Section 3). Coleman [Col82, CdS88] defined an integration theory on curves with good reduction over $\mathbb{C}_p$ and on certain rigid analytic subdomains of these, which he termed "Wide open spaces". One first needs to chose a branch of the p-adic logarithm, i.e., a group homomorphism $\log : {\mathbb{C}_p}^* \to \mathbb{C}_p$, such that around $z = 1$, it is given by the usual power series expansions for $\log(1+y)$ (this amounts to specifying $\log(p)$ in $\mathbb{C}_p$). Once this is done, the theory incluses single valued iterated integrals on the appropriate domain, called "Coleman functions". In particular, one can define $\mathrm{Li}_2(z) = \int_0^z -\log(1-x)\mathrm{d}\log x$, which extends to a Coleman function on $\mathbb{C}_p \setminus \{1\}$, see the beginning of Section VI in [Col82]. We also define $L_2(x) = \mathrm{Li}_2(x) + \log(x)\log(1-x)$ and $L_{\mathrm{mod},2}(x) = \mathrm{Li}_2(x)+\frac{1}{2}\log(x)\log(1-x)$. The latter function satisfies $L_{\mathrm{mod},2}(x)+L_{\mathrm{mod},2}(1/x) = 0$ for x in $\mathbb{C}_p \setminus \{0,1\}$ [Col82, Proposition 6.4(ii)].

It turns out that, on every residue disc U_x around a point x (points reducing to the same point) with local parameter $z = z_x$, Coleman functions can be expanded as $G(z) = \sum_i f_i(z)\log^i(z)$, where all $f_i(z)$ are in $K[[z, z^{-1}]]$. For such a function $G(z)$, we define the constant term $c_z(G)$ at x with respect to the parameter z as the constant term of f_0, see Definition 6.7. In general the constant term will depend on the choice of the local parameter z, but it is a remarkable fact that for any holomorphic ω on C and any g in $K(C)$, for all choices of $\int L_2(g)\omega$, $c_z(\int L_2(g)\omega)$ is in fact independent of the choice of the local parameter, see Lemma 9.4. Therefore, for any holomorphic ω on C, and any nonzero rational functions f and g on C with the divisor of f given by $\sum_j n_j(x_j)$, we define

$$\int_{(f)} L_2(g)\omega = \sum_j n_j c_{z_j}\left(\int L_2(g)\omega\right) ,$$

where z_j is a parameter around x_j.

Under the regulator, each element of $K_4^{(3)}(\mathcal{O})$ maps to $H^1_{\mathrm{dR}}(U/K)$ for some wide open space U in C in the terminology of Coleman. There exists a canonical projection $H^1_{\mathrm{dR}}(U/K) \to H^1_{\mathrm{dR}}(C/K)$, compatible under restriction to smaller U's, see [Bes00b, Proposition 4.8]. We still denote the composition $K_4^{(3)}(\mathcal{O}) \to H^1_{\mathrm{dR}}(U/K) \to H^1_{\mathrm{dR}}(C/k)$ by reg.

We are now ready to state the first theorem.

Theorem 1.3. *Let C be a smooth proper curve over K with good reduction over the maximal ideal of R. Let $\mathcal{C}$ be a smooth proper model of C over R, F the function field of C, $\mathcal{O}$ the local ring consisting of elements in F that are generically defined on the closed fiber of $\mathcal{C}$. If $\alpha = \sum_i [g_i]_2 \otimes f_i$ is an element of $H^2(\mathcal{M}_{(3)}(\mathcal{O}))$ and ω is a holomorphic form on C, then the composition of maps*

$$H^2(\mathcal{M}_{(3)}(\mathcal{O})) \longrightarrow K_4^{(3)}(\mathcal{O}) \xrightarrow{\mathrm{reg}} H^1_{\mathrm{dR}}(C/K) \longrightarrow H^2_{\mathrm{dR}}(C/K) \xrightarrow{\sim} K$$

(with the third map given by cupping $\eta \mapsto \eta \cup \omega$ and the fourth map by duality) maps α to

$$2\sum_i \int_{(f_i)} L_2(g)\omega - 2\sum_i \sum_{g_i(x)\neq 0,\infty} \mathrm{ord}_x(f_i) F_\omega(x) L_{\mathrm{mod},2}(g_i(x)) .$$

We would like to be able to take into account the boundary map

$$K_4^{(3)}(F) \xrightarrow{\partial} \textstyle\coprod_{x\in C^{(1)}} K_3^{(2)}(k(x)) .$$

With this in mind, in case F is the function field of a smooth, geometrically irreducible separated curve over a number field k, another complex, $\mathcal{M}_{(3)}(C)$, was defined on page 164 of [dJ00] (although all the necessariy calculations for this case can already be found in [dJ96, Proposition 5.1]) as the total complex associated to the double complex

$$\begin{tikzcd}
M_3(F) \arrow[r, "\mathrm{d}"] \arrow[d] & M_2(F)\otimes_{\mathbb{Q}} F^*_{\mathbb{Q}} \arrow[r, "\mathrm{d}"] \arrow[d, "\partial_1"'] & F^*_{\mathbb{Q}} \otimes \bigwedge^2 F^*_{\mathbb{Q}} \arrow[d, "\partial_2"'] \\
0 \arrow[r] & \coprod_x \widetilde{M}_2(k(x)) \arrow[r, "\mathrm{d}"] & \coprod_x \bigwedge^2 k(x)^*_{\mathbb{Q}} .
\end{tikzcd} \tag{1.4}$$

Here for any field L of characteristic zero, $\widetilde{M}_2(L)$ is a $\mathbb{Q}$-vector space on symbols $[z]_2$ with z in L^*, $z \neq 1$, with unknown relations, but which include the relation $[z]_2 + [z^{-1}]_2 = 0$ for all z in L^*, $z \neq 1$. In fact, it is constructed as the quotient of $M_2(L)$ by imposing those relations. This defines a cohomological complex in degrees 1 and 2,

$$\widetilde{M}_2(L) \to \bigwedge^2 L^*_{\mathbb{Q}} ,$$

where $\mathrm{d}[z]_2 = (1-z) \wedge z$. For a number field L, one knows that $H^1(\widetilde{\mathcal{M}}_{(2)}(L)) \cong K_3^{(2)}(L)$ (see [dJ96, Theorem 5.3]) and clearly $H^2(\widetilde{\mathcal{M}}_{(2)}(L)) \cong K_2^{(2)}(L) = K_2(L)_{\mathbb{Q}}$ as well. The maps are as follows. The d's in the top row are as in $\mathcal{M}_{(3)}(F)$. In the bottom row, $\mathrm{d}[z]_2 = (1-z) \wedge z$. For the vertical maps, $\partial_{1,x}([g]_2 \otimes f) = \mathrm{ord}_x(f) \cdot [g(x)]_2$, with the convention that $[0]_2 = [1]_2 = [\infty]_2 = 0$. Finally, $\partial_{2,x}$ can be described as follows. Let π be a uniformizer at x, u_j units at x. Then $\partial_{2,x}$ is determined by

$$\begin{aligned} \pi \wedge u_1 \wedge u_2 &\mapsto u_1(x) \wedge u_2(x) \\ u_1 \wedge u_2 \wedge u_3 &\mapsto 0 \ . \end{aligned}$$

Forgetting the bottom row gives an obvious map $\mathcal{M}_{(3)}(C) \to \mathcal{M}_{(3)}(F)$.

∂_1 induces a map $H^2(\mathcal{M}_{(3)}(F)) \to \coprod_{x \in C^{(1)}} H^1(\widetilde{\mathcal{M}}_{(2)}(k(x)))$, and the diagram

$$\begin{tikzcd}
H^2(\mathcal{M}_{(3)}(F)) \arrow[r] \arrow[d, "2\partial_1"] & K_4^{(3)}(F) \arrow[d, "\partial"'] \\
\coprod_{x \in C^{(1)}} H^1(\widetilde{\mathcal{M}}_{(2)}(k(x))) \arrow[r] & \coprod_{x \in C^{(1)}} K_3^{(2)}(k(x))
\end{tikzcd}$$

commutes, up to sign, and up to $\partial\left(K_3^{(2)}(k) \cup F_{\mathbb{Q}}^*\right)$ in the lower right corner, see [dJ96, Corollary 5.4]. This implies that the image of the composition

$$H^2(\mathcal{M}_{(3)}(C)) \to H^2(\mathcal{M}_{(3)}(F)) \to K_4^{(3)}(F)$$

is contained in $K_4^{(3)}(C) \oplus K_3^{(2)}(k) \cup F_{\mathbb{Q}}^* \subseteq K_4^{(3)}(F)$, see [dJ96, Theorem 5.2] or [dJ00, Corollary 4.3].

In order to give the syntomic regulator the information it needs over the residue field, we construct a complex similar to $\mathcal{M}_{(3)}(C)$ in the situation that $\mathcal{C}$ is a smooth proper geometrically connected curve over $R \bigcap k \subset k \subset K$. It is the complex $\mathcal{M}_{(3)}(\mathcal{C})$, defined as the total complex associated to the double complex

$$\begin{tikzcd}
M_3(\mathcal{O}) \arrow[r, "\mathrm{d}"] \arrow[d] & M_2(\mathcal{O}) \otimes_{\mathbb{Q}} F_{\mathbb{Q}}^* \arrow[r, "\mathrm{d}"] \arrow[d, "\partial_1"'] & \mathcal{O}_{\mathbb{Q}}^* \otimes \bigwedge^2 \mathcal{O}_{\mathbb{Q}}^* \arrow[d, "\partial_2"'] & . \\
0 \arrow[r] & \coprod_x \widetilde{M}_2(k(x)) \arrow[r, "\mathrm{d}"] & \coprod_x \bigwedge^2 k(x)_{\mathbb{Q}}^* &
\end{tikzcd}$$

Here $\mathcal{O}$ is the local ring of rational functions on $\mathcal{C}$ that are generically defined on the special fiber. The top row is the complex $\mathcal{M}_{(3)}(\mathcal{O})$ and the vertical maps are induced from the ones in (1.4), as there are natural maps $\mathcal{M}_{(n)}(\mathcal{O}) \to \mathcal{M}_{(n)}(F)$. Note that even though we start out with elements in $\mathcal{O}^*$ in the top row, after taking the vertical maps we have little control over the elements we get, and therefore we have to work with elements in the residue fields rather than with elements in the corresponding semilocal rings.

Again, there is an obvious map

$$\mathcal{M}_{(3)}(\mathcal{C}) \to \mathcal{M}_{(3)}(\mathcal{O})$$

corresponding to forgetting the bottom row. This gives us a map $H^2(\mathcal{M}_{(3)}(\mathcal{C})) \to H^2(\mathcal{M}_{(3)}(\mathcal{O}))$, and composing this with the map $H^2(\mathcal{M}_{(3)}(\mathcal{O})) \to K_4^{(3)}(\mathcal{O})$ gives us a map $H^2(\mathcal{M}_{(3)}(\mathscr{C})) \to K_4^{(3)}(\mathcal{O})$. Its image is contained in $K_4^{(3)}(\mathscr{C}) \oplus K_3^{(2)}(k) \cup \mathcal{O}_{\mathbb{Q}}^*$ inside $K_4^{(3)}(\mathcal{O}.)$

We are now ready to state the results corresponding to $\mathcal{M}_{(3)}(\mathscr{C})$.

Theorem 1.5. *Let C be a smooth, proper, geometrically irreducible curve over the number field $k \subset K$ with good reduction over the maximal ideal of $k \bigcap R$, and let $\mathcal{O}$ be the local ring consisting of rational functions on C that are generically defined on the special fiber. Let $\alpha = \sum_i [g_i]_2 \otimes f_i$ be an element of $H^2(\mathcal{M}_{(3)}(\mathscr{C}))$ and let ω be a holomorphic form on C. Then the composition of maps*

$$H^2(\mathcal{M}_{(3)}(\mathscr{C})) \longrightarrow K_4^{(3)}(\mathcal{O}) \xrightarrow{\text{reg}} H^1_{\text{dR}}(C/K) \xrightarrow{\cdot \cup \omega} H^2_{\text{dR}}(C/K) \xrightarrow{\sim} K$$

where the fourth map is given by duality, maps α to

$$2 \sum_i \int_{(f_i)} L_2(g_i)\omega \ .$$

Remark 1.6. One can also use the projection $K_4^{(3)}(\mathscr{C}) \oplus K_3^{(2)}(k) \cup \mathcal{O}_{\mathbb{Q}}^* \to K_4^{(3)}(\mathscr{C})$ in Theorem 1.5, and get a composition of maps

$$H^2(\mathcal{M}_{(3)}(\mathscr{C})) \longrightarrow K_4^{(3)}(\mathscr{C}) \xrightarrow{\text{reg}} H^1_{\text{dR}}(C/K) \longrightarrow H^2_{\text{dR}}(C/K) \xrightarrow{\sim} K \ ,$$

which is given by the same formula, see Theorem 1.10.

Theorem 1.5 will be deduced in Section 9 from Theorem 1.10 below. To state it we need to introduce the notion of the triple index. It is a generalization of the "local index" which was introduced in [Bes00b, Section 4]. Informally speaking, working on an annulus e over $\mathbb{C}_p$, $e \cong \{r < |z| < 1\}$, the triple index associates to the integrals F, G and H of three rigid analytic one-forms on e (in this case these forms are simply Laurent series converging on e multiplied by dz) together with choices of integrals for $F\mathrm{d}G$, $F\mathrm{d}H$ and $G\mathrm{d}F$, a number $\langle F, G; H\rangle_e$ in $\mathbb{C}_p$ which is supposed to be a generalization of $\operatorname{Res}_e FGdH$. Note that the integrals appearing in the data for the triple index make perfect sense once one admits a log function to correspond to the integral of dz/z, and are determined up to a constant by the form they integrate. Suppose now that $C/\mathbb{C}_p$ is a curve with good reduction and that C contains discs $D_i \cong \{|z| < 1\}$. The rigid analytic domain $U = C \setminus \cup_i (D_i - e_i)$ where $e_i \subset D_i$ is the annulus corresponding to $\{r < |z| < 1\}$ is called a wide open space by Coleman. The $e_i \subset U$ are called the annuli ends of U. Suppose that F, G and H are Coleman functions defined on U such that restricted to the e_i's they are of the type allowing us to compute the triple indices $\langle F|_{e_i}, G|_{e_i}; H|_{e_i}\rangle_{e_i}$. We may use auxiliary data composed of Coleman integrals restricted to e_i for computing these. It sometimes turns out that the sum of triple indices over all the e_i depends only on F, G, and H and not on the auxiliary data. This applies in particular to the sum of triple indices in the theorem below. It is further known that this sum of triple indices behaves well with respect to shrinking the wide open space U. Finally, if everything is defined over a subfield K of $\mathbb{C}_p$ then this sum of triple indices is in K.

The main complex for doing calculations is yet another complex, the complex

$$\mathcal{C}^\bullet(\mathcal{O}) : \mathcal{C}^1(\mathcal{O}) \to \mathcal{C}^2(\mathcal{O}) \tag{1.7}$$

in cohomological degrees 1 and 2, which we will construct in Section 2.5.4. It comes with an isomorphism

$$K_4^{(3)}(\mathcal{O})/K_3^{(2)}(\mathcal{O}) \cup \mathcal{O}_{\mathbb{Q}}^* \cong H^1(\mathcal{C}^\bullet(\mathcal{O}))$$

and a surjection

$$K_3^{(3)}(\mathcal{O}) \to H^2(\mathcal{C}^\bullet(\mathcal{O}))\,,$$

see (2.63) and Remark 2.34. $\mathcal{C}^2(\mathcal{O}) = K_2^{(2)}(\mathcal{O}) \otimes \mathcal{O}_{\mathbb{Q}}^*$, and if f is in $\mathcal{O}^*$ and g is in $\mathcal{O}^\flat$, then $\mathcal{C}^1(\mathcal{O})$ contains a symbol of the form $[g]_2 \cup (f)$, with $\mathrm{d}[g]_2 \cup (f) = \{(1-f)^{-1}, g\} \otimes f$ in $\mathcal{C}^2(\mathcal{O})$. One can define a map of (shifted) complexes given by

$$\begin{tikzcd} M_3(\mathcal{O}) \arrow[r] \arrow[d] & M_2(\mathcal{O}) \otimes \mathcal{O}_{\mathbb{Q}}^* \arrow[r] \arrow[d] & F_{\mathbb{Q}}^* \otimes \bigwedge^2 \mathcal{O}_{\mathbb{Q}}^* \arrow[d] \\ 0 \arrow[r] & \mathcal{C}^1(\mathcal{O}) \arrow[r] & \mathcal{C}^2(\mathcal{O}) \end{tikzcd} \tag{1.8}$$

as follows. We map $[g]_2 \otimes f$ to $[g]_2 \cup f$, and $f \otimes (g \wedge h)$ to $\{f, g\} \otimes h - \{f, h\} \otimes g$. This gives a map on cohomology groups, of which the interesting one for us is the one in the commutative diagram

$$\begin{tikzcd} H^2(\mathcal{M}_{(3)}(\mathcal{O})) \arrow[r] \arrow[d] & K_4^{(3)}(\mathcal{O}) \arrow[d] \\ H^1(\mathcal{C}^\bullet(\mathcal{O})) \arrow[r, "\sim"] & K_4^{(3)}(\mathcal{O})/K_3^{(2)}(\mathcal{O}) \cup \mathcal{O}_{\mathbb{Q}}^*\,, \end{tikzcd} \tag{1.9}$$

see (2.66), where the map on the right is the natural projection.

We can use triple indices to compute the cup product of the regulator with arbitrary cohomology classes on C, and not just with those of holomorphic forms. It is well known that such a cohomology class can be represented by a form of the second kind on C, i.e., a meromorphic form all of whose residues are 0.

Theorem 1.10. *Let ω be a form of second kind on C. Then the composition of maps*

$$K_4^{(3)}(C) \xrightarrow{\sim} K_4^{(3)}(\mathfrak{C}) \xrightarrow{\mathrm{reg}} H^1_{\mathrm{dR}}(C/K) \xrightarrow{\cdot\cup\omega} H^2_{\mathrm{dR}}(C/K) \xrightarrow{\sim} K \tag{1.11}$$

factors through the composition of maps

$$K_4^{(3)}(C) \cong K_4^{(3)}(\mathfrak{C}) \to K_4^{(3)}(\mathcal{O}) \to K_4^{(3)}(\mathcal{O})/K_3^{(2)}(\mathcal{O}) \cup \mathcal{O}_{\mathbb{Q}}^* \cong H^1(\mathcal{C}^\bullet(\mathcal{O}))\,. \tag{1.12}$$

If α in $K_4^{(3)}(C)$ maps to $\sum_i [g_i]_2 \cup (f_i)$ in $H^1(\mathcal{C}^\bullet(\mathcal{O}))$ under (1.12)*, then under* (1.11) *α is mapped to*

$$\sum_i \sum_e \left\langle \log(f_i), \log(g_i); \int F_\omega \,\mathrm{dlog}(1-g_i) \right\rangle_e\,,$$

where F_ω is any Coleman integrals of ω and the sum of triple indices is over all annuli ends e of a wide open space U on which all f_i, g_i and $1-g_i$ are invertible, and ω is holomorphic.

The proof of this theorem is given at the end of section 8. It is somewhat harder to state, since it involves the notion of the triple index. However, it is very useful in applications since it is easier to find explicit examples to which Theorem 1.10 applies, with F the function field of certain elliptic curves, see [dJ96, Section 6].

It is also important to note that, unlike the situation over the complex numbers, where it is known that computing the cup product of the regulator with holomorphic forms suffices to describe it completely in the case we are considering because those linear maps surject onto the dual of the target space of the regulator (see the beginning of Section 4 of [dJ96], especially Proposition 4.1), the situation over the p-adics is not clear.

We end the introduction with some conjectures. The regulator formulae that we obtain do not depend on any integrality assumptions. This is only required because the syntomic regulator is a map from the K-theory of an integral model. One has inclusions and identifications

(1.13)

$$\begin{tikzcd}
K_4^{(3)}(\mathscr{C}) \arrow[r, hook] \arrow[d, equal] & K_4^{(3)}(\mathscr{C}) \oplus K_3^{(2)}(k) \cup \mathcal{O}^*_{\mathbb{Q}} \arrow[r, hook] \arrow[d, equal] & K_4^{(3)}(\mathcal{O}) \arrow[d, equal] \\
K_4^{(3)}(C) \arrow[r, hook] & K_4^{(3)}(C) \oplus K_3^{(2)}(k) \cup F^*_{\mathbb{Q}} \arrow[r, hook] & K_4^{(3)}(F)
\end{tikzcd}$$

in K-theory, see Section 2.2. Therefore we can make the following Conjecture. The third part of this Conjecture refers to the complex $\mathcal{C}^\bullet(F)$,which is similar to $\mathcal{C}^\bullet(\mathcal{O})$, see Section 2.4.4. $[g]_2 \cup f$ exists in $\mathcal{C}^1(F)$ for g in $F^* \setminus \{1\}$ and f in F^*.

Conjecture 1.14.

(1) *The composition*

$$H^2(\mathcal{M}_{(3)}(F)) \longrightarrow K_4^{(3)}(F) \cong K_4^{(3)}(\mathcal{O}) \xrightarrow{\text{reg}} H^1_{\text{dR}}(C/K) \longrightarrow H^2_{\text{dR}}(C/K) \xrightarrow{\sim} K$$

(with the third map given by cupping $\eta \mapsto \eta \cup \omega$ and the fourth map by duality) is given by the formula in Theorem 1.3.

(2) *The composition*

$$H^2(\mathcal{M}_{(3)}(C)) \longrightarrow K_4^{(3)}(F) \cong K_4^{(3)}(\mathcal{O}) \xrightarrow{\text{reg}} H^1_{\text{dR}}(C/K) \xrightarrow{\cdot\cup\omega} H^2_{\text{dR}}(C/K) \xrightarrow{\sim} K$$

is given by exactly the same formulae as in Theorem 1.5.

(3) *The composition of maps in* (1.11) *factors through the composition of maps*

$$K_4^{(3)}(C) \to K_4^{(3)}(F) \to K_4^{(3)}(F)/K_3^{(2)}(F) \cup F^*_{\mathbb{Q}} \cong H^1(\mathcal{C}^\bullet(F)) ,$$

and if α in $K_4^{(3)}(C)$ maps to $\sum_i [g_i]_2 \cup (f_i)$ in $H^1(\mathcal{C}^\bullet(F))$, then the result of applying (1.11) *to α is given by the same formula as in Theorem 1.10.*

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Notation

Unless stated otherwise, throughout the paper, we will be working with the following notation.

K will be a discrete valuation field of characteristic zero, with valuation ring R, and residue field κ of positive characteristic p. We shall assume that κ is a subfield of $\overline{\mathbb{F}}_p$. In various places, k will be a number field inside K. In that case we denote by $\mathbb{F}$ the residue field of $k \bigcap R$ inside κ.

$\mathscr{C}$ will be a smooth, proper, geometrically irreducible curve over R or over $k \bigcap R$ in K.

The generic fiber is denoted C, the special fiber is denoted $\mathcal{C}_{\mathbb{F}}$ in the first case and by $\mathcal{C}_{\mathbb{F}}$ in the second. F will be either $k(C)$ or $K(C)$, $\mathcal{O}$ will be the ring of integers in F for the valuation on F corresponding to the generic point of $\mathcal{C}_{\mathbb{F}}$ or $\mathcal{C}_{\mathbb{F}}$. It therefore consists of those elements in F that are generically defined on the special fiber.

If S is a subset of a group, then we denote by $<S>$ the subgroup generated by S, and if S is a subset of a $\mathbb{Q}$-vector space, we denote by $<S>_{\mathbb{Q}}$ the $\mathbb{Q}$-vector subspace generated by S.

All tensor products will be over $\mathbb{Q}$, unles specified otherwise.

For the convenience of the reader, we give a commutative diagram, which plays the role of "*Leitteppich*" for the proofs in this paper.

(1.15)

$$\begin{tikzcd}
& H^2(\mathcal{M}_{(3)}(\mathcal{C})) \arrow[r] \arrow[d] & H^2(\mathcal{M}_{(3)}(\mathcal{O})) \arrow[r] \arrow[d] & H^1(\mathcal{C}^\bullet(\mathcal{O})) \arrow[d] \\
& K_4^{(3)}(\mathcal{C}) \oplus K_3^{(2)}(k) \cup \mathcal{O}_{\mathbb{Q}}^* \arrow[r] & K_4^{(3)}(\mathcal{O}) \arrow[r] \arrow[d, "\mathrm{reg}"] & K_4^{(3)}(\mathcal{O})/K_3^{(2)}(\mathcal{O}) \cup \mathcal{O}_{\mathbb{Q}}^* \arrow[d, dotted] \\
& & H^1_{\mathrm{dR}}(C) \arrow[r, "\int \cdot \wedge \omega"] & K
\end{tikzcd}$$

The top left square comes from the natural map $\mathcal{M}_{(3)}(\mathcal{C}) \to \mathcal{M}_{(3)}(\mathcal{O})$ corresponding to leaving out the bottom row in the first; the top right square is the map in (1.9), induced from (1.8). Those results will be established in Section 2. The composition of maps

$$K_4^{(3)}(\mathcal{O}) \xrightarrow{\mathrm{reg}} H^1_{\mathrm{dR}}(C) \xrightarrow{\int \cdot \wedge \omega} K$$

will be shown to factor through the quotient map $K_4^{(3)}(\mathcal{O}) \to K_4^{(3)}(\mathcal{O})/K_3^{(2)}(\mathcal{O}) \cup \mathcal{O}_{\mathbb{Q}}^*$ in the proof of Proposition 4.18 and Proposition 8.4. Its value is then computed in Section 8, using the techniques developed in the preceeding sections, which gives Theorem 1.10. Finally, Theorems 1.3 and 1.5 are deduced from this in Section 9.

2. K-THEORY

2.1. **Introduction.** The main idea is the same as in [dJ96], but the fact that we will be working with a discrete valuation ring rather than a field gives rise to some complications. In order to highlight the idea we start with a rather gentle exposition. For the proofs of the statements that are used in the construction, we refer the reader to [dJ95], especially Sections 2.1 through 2.3, and 3. There most of the work was done over $\mathbb{Q}$, but in fact the proofs hold over our base $\mathcal{O}$, a discrete valuation ring of characteristic zero, without any change.

If B is a Noetherian scheme of finite Krull dimension d, then according to [Sou85, Proposition 5], one can write

$$K_n(B) \otimes_{\mathbb{Z}} \mathbb{Q} = \oplus_{j=\min 2,n}^{n+d} K_n^{(j)}(B) \tag{2.1}$$

where $K_n^{(j)}(B)$ consists of all α in $K_n(B) \otimes_{\mathbb{Z}} \mathbb{Q}$ such that $\psi^k(\alpha) = k^j \alpha$ for all Adams operators ψ^k. (The regularity assumption at the beginning of Section 4 of loc. cit. is not necessary, see [GS99, Proposition 8].) If in addition B is separated and regular, then the pullback $K_*(B) \to K_*(\mathbb{A}^1_B)$ is an isomorphism, see [Qui73, §7]. The weight behaves naturally with respect to pullback, also giving us $K_m^{(j)}(B) \cong K_m^{(j)}(\mathbb{A}^1_B)$ under pullback. And under suitable hypotheses for a closed embedding, there is a pushforward Gysin map with a shift in weights corresponding to the codimension (see, e.g., [dJ95, Proposition 2.3]).

Let $X_B = \mathbb{P}^1_B \setminus \{t = 1\}$ with t the standard affine coordinate on $\mathbb{P}^1_B$. Write $\square^1_B$ for the closed subset $\{t = 0, \infty\}$ in $\mathbb{P}^1_B$. Then the relative exact sequence for the couple $(X_B; \square^1_B)$ gives us

$$\cdots \to K_{n+1}(X_B) \to K_{n+1}(\square^1_B) \to K_n(X_B; \square^1_B) \to K_n(X_B) \to K_n(\square^1_B) \to \cdots$$

for $n \geq 0$. Because the map pullback $K_{n+1}(B) \to K_{n+1}(X_B)$ is an isomorphism, combining it with the pullback $K_{n+1}(X_B) \to K_{n+1}(\square^1_B) = K_{n+1}(B)^2$ shows that the map $K_{n+1}(X_B) \to K_{n+1}(\square^1_B)$ corresponds to the diagonal embedding $K_{n+1}(B) \to K_{n+1}(B)^2$. As this holds for all $n \geq 0$, we get that we have an isomorphism $K_n(X_B; \square^1_B) \cong K_{n+1}(B)$ for $n \geq 0$. Note that we have a choice of sign here in the isomorphism of the cokernel of $K_n(B) \to K_n(B)^2$ with $K_n(B)$. This results in similar choices of signs in the maps $H^p(\mathcal{M}_{(n)}(\mathcal{O})) \to K^{(n)}_{2n-p}(\mathcal{O})$ (resp. $H^p(\widetilde{\mathcal{M}}_{(n)}(\mathcal{O})) \to K^{(n)}_{2n-p}(\mathcal{O})$) later on in this section.

We will have to go up one level in the relativity. If we let $\square^2_B$ be shorthand for

$$\{t_1 = 0, \infty\}; \{t_2 = 0, \infty\}\ ,$$

then we can get a long exact sequence

$$\begin{aligned}\cdots \to K_{n+1}(X^2_B; \{t_1 = 0, \infty\}) \to K_{n+1}(\{t_2 = 0, \infty\}; \{t_1 = 0, \infty\}) \to \\ \to K_n(X^2_B; \square^2_B) \to K_n(X^2_B; \{t_1 = 0, \infty\}) \to K_n(\{t_2 = 0, \infty\}; \{t_1 = 0, \infty\}) \to \cdots .\end{aligned}$$

The composition

$$\begin{aligned}K_{n+1}(X_B; \{t_1 = 0, \infty\}) \xrightarrow{\sim} K_{n+1}(X^2_B; \{t_1 = 0, \infty\}) \to \\ \to K_{n+1}(\{t_2 = 0, \infty\}; \{t_1 = 0, \infty\}) \cong K_{n+1}(X_B; \{t_1 = 0, \infty\})^2\end{aligned}$$

(with the first the pullback along the projection $(t_1, t_2) \mapsto t_2$) is the diagonal embedding, hence we obtain an isomorphism $K_n(X^2_B; \square^2_B) \cong K_{n+1}(X_B; \square^1_B)$ for $n \geq 0$. Therefore we get $K_n(X^2_B; \square^2_B) \cong K_{n+1}(X_B; \square_B) \cong K_{n+2}(B)$ for $n \geq 0$. A similar argument with weights gives us an isomorphism $K_n^{(j)}(X^2_B; \square^2_B) \cong K^{(j)}_{n+2}(B)$ for $n \geq 0$.

In order to get elements in $K_{n+2}(X^2_B; \square^2_B)$, we use localization sequences. We first explain the idea for $K_{n+1}(X_B; \square_B)$, because for $K_{n+2}(X^2_B; \square^2_B)$ the process involves a spectral sequence. If u is an element in our discrete valuation ring $\mathcal{O}$ such that both u and $1 - u$ are units, then we get an exact localization sequence

$$\cdots \to K_m(\mathcal{O}) \to K_m(X_{\mathcal{O}}; \square^1_{\mathcal{O}}) \to K_m(X_{\mathcal{O},\mathrm{loc}}; \square^1_{\mathcal{O}}) \to K_{m-1}(\mathcal{O}) \to \cdots$$

where $X_{\mathcal{O},\mathrm{loc}} = X_{\mathcal{O}} \setminus \{t = u\}$ and we identified $\{t = u\} \subset X_{\mathcal{O}}$ with $\mathcal{O}$ (or rather $\mathrm{Spec}(\mathcal{O})$). We used here that u and $1 - u$ are units in $\mathcal{O}$ so that $\{t = u\}$ does not meet $\square^1_{\mathcal{O}}$ or $\{t = 1\}$, and that $\mathcal{O}$ is regular in order to identify $K_m(\mathcal{O})$ with $K'_m(\mathcal{O})$. (If we want to leave out $\{t = u\}$ and $\{t = v\}$ simultaneously for two distinct elements u and v in $\mathcal{O}$ such that all of u, v, $1 - u$ and $1 - v$ are units, which we shall do below, this already becomes far more complicated and one is force to use a spectral sequence.) The image of $K_2(\mathcal{O}) \to K_2(X_{\mathcal{O}}; \square^1_{\mathcal{O}})$ can be controlled by looking at the weights, which for the bit that we are interested in gives us

$$\cdots \to K_2^{(1)}(\mathcal{O}) \to K_2^{(2)}(X_{\mathcal{O}}; \square^1_{\mathcal{O}}) \to K_2^{(2)}(X_{\mathcal{O},\mathrm{loc}}; \square^1_{\mathcal{O}}) \to K_1^{(1)}(\mathcal{O}) \to \cdots ,$$

so that $K_3^{(2)}(\mathcal{O}) \cong \operatorname{Ker}\left(K_2^{(2)}(X_{\mathcal{O},\mathrm{loc}}; \square^1_{\mathcal{O}}) \to K_1^{(1)}(\mathcal{O})\right)$. Because of weights in K-theory, one knows that $K_2^{(1)}(\mathcal{O}) = 0$, so we can analyze $K_2^{(2)}(X_{\mathcal{O}}; \square^1_{\mathcal{O}})$ as subgroup of $K_2^{(2)}(X_{\mathcal{O},\mathrm{loc}}; \square^1_{\mathcal{O}})$. In [dJ95, Section 3.2] universal elements $[S]_n$ were constructed, of which we want to use $[S]_2$ here. It gives rise to an element $[u]_2$ in $K_2^{(2)}(X_{\mathcal{O},\mathrm{loc}}; \square^1_{\mathcal{O}})$ with boundary $(1-u)^{-1}$ in $K_1^{(1)}(\mathcal{O})$. If we use this for various u (suitably modifiying the localization sequence above into a spectral sequence) and also consider elements coming from the cup product

$$K_1^{(1)}(X_{\mathcal{O},\mathrm{loc}}; \square^1_{\mathcal{O}}) \times K_1^{(1)}(\mathcal{O}) \to K_2^{(2)}(X_{\mathcal{O},\mathrm{loc}}; \square^1_{\mathcal{O}})$$

we can get part of $K_2^{(2)}(X_{\mathcal{O}}; \square^1_{\mathcal{O}}) \cong K_3^{(2)}(\mathcal{O})$ by intersecting the kernel of the map corresponding to $K_2^{(2)}(X_{\mathcal{O},\mathrm{loc}}; \square^1_{\mathcal{O}}) \to K_1^{(1)}(\mathcal{O})$ with the space generated by the symbols $[u]_2$ and the image $K_1^{(1)}(X_{\mathcal{O},\mathrm{loc}}; \square^1_{\mathcal{O}}) \cup K_1^{(1)}(\mathcal{O})$ of the cup product.

2.2. **Preliminary material.** The goal of this subsection is to describe some basic facts about the various K-groups of F, $\mathcal{O}$, C and $\mathcal{C}$, including the identifications and inclusions in (1.13).

There are two cases to consider. The first is the case were $F = k(C)$ for a smooth, projective curve $\mathcal{C}$ over $R \bigcap k$ with generic fiber C, the other is where $F = K(C)$ for a smooth projective curve $\mathcal{C}$ over R, again with generic fibre C. Both cases are very similar, so we will treat them together.

For the case that $F = k(C)$, we let $\mathcal{C}_{\mathbb{F}}$ be the special fibre of $\mathcal{C}$, which is a smooth projective curve over a finite field $\mathbb{F}$. Because $\mathcal{C}_{\mathbb{F}}$ is regular, there is an exact localisation sequence

(2.2)
$$\cdots \longrightarrow K_4^{(2)}(\mathbb{F}(\mathcal{C}_{\mathbb{F}})) \longrightarrow K_4^{(3)}(\mathcal{O}) \longrightarrow K_4^{(3)}(F) \longrightarrow K_3^{(2)}(\mathbb{F}(\mathcal{C}_{\mathbb{F}})) \longrightarrow \cdots .$$

By [Har77, Korollar 2.3.2]), $K_n(L)$ is torsion for $n \geq 2$ for all function fields L of curves over finite fields, so in particular, $K_4^{(3)}(\mathcal{O}) \overset{\sim}{\to} K_4^{(3)}(F)$. If $F = K(C)$, we get

$$\cdots \longrightarrow K_4^{(2)}(\kappa(\mathcal{C}_\kappa)) \longrightarrow K_4^{(3)}(\mathcal{O}) \longrightarrow K_3^{(2)}(\kappa\mathcal{C}_\kappa) \longrightarrow \cdots$$

By our assumptions (see the beginning of the Introduction), $\kappa \subseteq \overline{\mathbb{F}}_p$. This means that we can compute $K_n(\kappa(\mathcal{C}_\kappa))$ as a direct limit of K_n of function fields of curves over finite fields, see [Qui73, Proposition 2.2]. But we have just seen that this is a direct limit over torsion, which is torsion, so $K_4^{(2)}(\kappa(\mathcal{C}_\kappa))$ and $K_3^{(2)}(\kappa(\mathcal{C}_\kappa))$ are zero as they are part of $K_4(\kappa(\mathcal{C}_\kappa))$ and $K_3(\kappa(\mathcal{C}_\kappa))$ respectively. Therefore we have an isomorphism $K_4^{(3)}(\mathcal{O}) \cong K_4^{(3)}(F)$ also in the case that $F = K(C)$. This enables us to identify $K_4^{(3)}(\mathcal{O})$ and $K_4^{(3)}(F)$ in both cases, which we shall do from now on.

If $F = k(C)$, there is an exact localisation sequence

$$\cdots \longrightarrow K_4^{(2)}(\mathcal{C}_{\mathbb{F}}) \longrightarrow K_4^{(3)}(\mathcal{C}) \longrightarrow K_4^{(3)}(C) \longrightarrow K_3^{(2)}(\mathcal{C}_{\mathbb{F}}) \longrightarrow \cdots ,$$

and $K_n^{(2)}(\mathcal{C}_{\mathbb{F}})$ is trivial for $n \geq 2$. One can see this by writing down the localisation sequence

$$\begin{tikzcd} \cdots \arrow[r] & \coprod_{x \in \mathcal{C}_{\mathbb{F}}^{(1)}} K_n^{(1)}(k(x)) \arrow[r] & K_n^{(2)}(\mathcal{C}_{\mathbb{F}}) \arrow[r] & K_n^{(2)}(\mathbb{F}(\mathcal{C}_{\mathbb{F}})) \arrow[r] & \cdots \end{tikzcd}$$

and using the fact that $K_n^{(1)}(L)$ is zero for any field L for $n \geq 2$. A similar localisation sequence, together with a direct limit argument as before, tells us that $K_4^{(3)}(\mathcal{C}) \cong K_4^{(3)}(C)$ also if we start working with $F = K(C)$.

Remark 2.3. We now have two identifications fitting into a commutative diagram

$$\begin{tikzcd} K_4^{(3)}(\mathcal{C}) \arrow[r] \arrow[d, equal] & K_4^{(3)}(\mathcal{O}) \arrow[d, equal] \\ K_4^{(3)}(C) \arrow[r] & K_4^{(3)}(F) \end{tikzcd} ,$$

either working with $F = k(C)$ or $F = K(C)$. The bottom arrow (and hence the top arrow) is an inclusion if C is a curve over a number field k, and $F = k(C)$. This follows from the localisation sequence

$$\begin{tikzcd} \cdots \arrow[r] & \coprod_{x \in C^{(1)}} K_4^{(2)}(k(x)) \arrow[r] & K_4^{(3)}(C) \arrow[r] & K_4^{(3)}(F) \arrow[r, "\partial"] & \coprod_{x \in C^{(1)}} K_3^{(2)}(k(x)) \arrow[r] & \cdots \end{tikzcd} .$$

Because all $k(x)$ are number fields, $K_4^{(2)}(k(x)) = 0$.

Remark 2.4. Under the assumption that C is geometrically irreducible over a number field k, let us show that we also have a direct sum $K_4^{(3)}(C) \oplus K_3^{(2)}(k) \cup F_{\mathbb{Q}}^*$ inside $K_4^{(3)}(F)$. Note that $F_{\mathbb{Q}}^* = K_1^{(1)}(F)$, so this makes sense. But $K_4^{(3)}(C) = \mathrm{Ker}(\partial) \subset K_4^{(3)}(F)$ in the localisation sequence in Remark 2.3, and for f in $F_{\mathbb{Q}}^*$, $\partial K_3^{(2)}(k) \cup f = K_3^{(2)}(k) \cup \partial f$ in $\coprod_{x \in C^{(1)}} k(x)_{\mathbb{Q}}^*$. Because ∂f is trivial if and only if f is in $k_{\mathbb{Q}}^*$, and $K_3^{(2)}(k) \cup k_{\mathbb{Q}}^*$ is contained in $K_4^{(3)}(k)$, which is zero as k is a number field, this shows that $K_3^{(2)}(F) \cup F_{\mathbb{Q}}^*$ injects into $\coprod_{x \in C^{(1)}} k(x)_{\mathbb{Q}}^*$ under ∂.

Remark 2.5. If k is a number field contained in F, let us observe that $K_3^{(2)}(k) \cup \mathcal{O}_{\mathbb{Q}}^* = K_3^{(2)}(k) \cup F_{\mathbb{Q}}^*$. F^* is up to torsion generated by $\mathcal{O}^*$ and p as p is a power of a uniformizing element of $\mathcal{O}$. But $K_3^{(2)}(k) \cup p$ is contained in $K_4^{(3)}(k)$, which is zero as k is a number field.

Proposition 2.6. *At several points below we will want to know that for a discrete valuation ring $\mathcal{O}$, with residue field κ and field of fractions F, for all $n \geq 1$, the sequence*

$$\begin{tikzcd} K_n^{(n)}(\mathcal{O}) \arrow[r] & K_n^{(n)}(F) \arrow[r] & K_{n-1}^{(n-1)}(\kappa) \arrow[r] & 0 \end{tikzcd}$$

is exact, and that the map $\mathcal{O}_{\mathbb{Q}}^{* \otimes n} \to \mathrm{Ker}\left(K_n^{(n)}(F) \to K_{n-1}^{(n-1)}(\kappa)\right)$ is surjective.

Proof. For $n = 1$ this follows from the exact sequence $1 \to \mathcal{O}^* \to F^* \to \mathbb{Z} \to 0$ corresponding to the valuation, tensored with $\mathbb{Q}$.

For any field L, $K_n^{(n)}(L) \cong K_n^M(L)_{\mathbb{Q}}$, where $K_n^M(L)$ is the Milnor K-theory of L, which is $\mathbb{Z}$ if $n = 0$, L^* if $n = 1$, and $L^{* \otimes_{\mathbb{Z}} n} / < \ldots \otimes x \otimes \ldots \otimes (1 - x) \otimes \ldots >$ if $n \geq 2$, see [Sou85, Théorème 2].

The class of $a_1 \otimes \ldots \otimes a_n$ in $K_n^M(L)$ is denoted by $\{a_1, \ldots, a_n\}$. There is a group homomorphism $K_n^M(F) \to K_n^M(\kappa)$, the Tame symbol, which is uniquely determined by the properties that, if the u_j are in $\mathcal{O}^*$, and π is a uniformizer of $\mathcal{O}$, then $\{u_1, \ldots, u_n\}$ is in its kernel, and $\{u_1, \ldots, u_{n-1}, \pi\}$ is mapped to $\{\bar{u}_1, \ldots, \bar{u}_{n-1}\}$.

Let us show that the map

$$K_n^M(F) \to K_{n-1}^M(\kappa)$$

given by the Tame symbol is surjective, with the kernel consisting of the subgroup of $K_n^M(F)$ generated by the image of $\mathcal{O}^{*\otimes_{\mathbb{Z}} n}$. For $n = 1$ this is clear, and the surjectivity in general follows from the fact that $\{u_1, \ldots, u_{n-1}, \pi\}$ with the u_i in $\mathcal{O}^*$ maps to $\{\bar{u}_1, \ldots, \bar{u}_{n-1}\}$ in $K_{n-1}^M(\kappa)$, where π is a uniformizer of $\mathcal{O}$. As for the kernel, notice that, by antisymmetry in Milnor K-theory, $K_n^M(F)$ is generated by elements $\{u_1, \ldots, u_n\}$ and $\{u_1, \ldots, u_{n-1}, \pi\}$, with all u_i in $\mathcal{O}^*$. Clearly, the elements of the form $\{u_1, \ldots, u_n\}$ are in the image of $\mathcal{O}^{*\otimes_{\mathbb{Z}} n}$, so we only have to show it for the elements of the form $\{u_1, \ldots, u_{n-1}, \pi\}$. Noticing that the Steinberg symbols $\ldots \otimes x \otimes \ldots \otimes (1-x) \otimes \ldots$ in $\mathcal{O}^{*\otimes_{\mathbb{Z}}^n}$ surject onto those in $\kappa^{*\otimes_{\mathbb{Z}}^n}$, we see that we only have to show that an element that is a sum of elements of the form $\{u_1, \ldots, u_{n-1}, \pi\}$ that lies in the kernel of the Tame symbol, and has the property that the corresponding sum of $u_1 \otimes \ldots \otimes u_{n-1}$ becomes trivial in $\kappa^* \otimes \mathbb{Z} \ldots \otimes_{\mathbb{Z}} \kappa^*$ lies in the image of $\mathcal{O}^{*\otimes_{\mathbb{Z}} n}$. From the exact sequence

$$1 \to 1 + \mathcal{O}\pi \to \mathcal{O}^* \to \kappa \to 1$$

and the fact that, if we have exact sequences $0 \to A_i \to B_i \to C_i \to 0$ $(i = 1, \ldots, n)$ of Abelian groups, then the kernel of $B_1 \otimes_{\mathbb{Z}} \ldots \otimes_{\mathbb{Z}} B_n \to C_1 \otimes_{\mathbb{Z}} \ldots \otimes_{\mathbb{Z}} C_n$ is the image of $A_1 \otimes_{\mathbb{Z}} B_2 \otimes_{\mathbb{Z}} \ldots \otimes_{\mathbb{Z}} B_n + B_1 \otimes_{\mathbb{Z}} A_2 \otimes_{\mathbb{Z}} B_3 \otimes \ldots \otimes_{\mathbb{Z}} B_n + \ldots$, we see that the sum of $u_1 \otimes \ldots \otimes u_{n-1}$ must lie in the image of $(1+\mathcal{O}\pi) \otimes_{\mathbb{Z}} \mathcal{O}^* \otimes_{\mathbb{Z}} \ldots \otimes_{\mathbb{Z}} \mathcal{O}^* + \mathcal{O}^* \otimes_{\mathbb{Z}} (1 + \mathcal{O}\pi) \otimes_{\mathbb{Z}} \ldots \otimes_{\mathbb{Z}} \mathcal{O}^* + \ldots$. But each element $\{u_1, \ldots, u_{n-1}, \pi\}$ with all u_i in $\mathcal{O}^*$ and at least one of them in $1 + \mathcal{O}\pi$ lies in the image of $\mathcal{O}^{*\otimes_{\mathbb{Z}} n}$. Namely, an element in $1 + \mathcal{O}\pi$ is of the form $1 - \pi^d u$ for some u in $\mathcal{O}^*$, $d > 0$. If $d = 1$ we can rewrite $\{\ldots, 1 - \pi u, \ldots, \pi\} = -\{\ldots, 1 - \pi u, \ldots, u\}$. If $d > 1$, then using that $\frac{1-\pi^d u}{1-\pi} = 1 - \pi \frac{\pi^{d-1}u - 1}{1-\pi}$, we find that $\{\ldots, 1 - \pi^d u, \ldots, \pi\} = \{\ldots, 1 - \pi \frac{\pi^{d-1}u-1}{1-\pi}, \ldots, u\}$, which reduces to the case $d = 1$ as $\frac{\pi^{d-1}u-1}{1-\pi}$ is in $\mathcal{O}^*$. $\square$

Assumption 2.7. *Throughout the construction of the complexes in the various subsections below, we let F be a field of characteristic zero. In the constructions for complexes for $\mathcal{O}$, $\mathcal{O}$ will be a discrete valuation ring $\mathcal{O}$, with residue field κ and field of fractions F, which we assume to be of characteristic zero. We shall always assume that $|\kappa| > 2$, so that $\mathcal{O}^\flat$ is nonempty and $<\mathcal{O}^\flat> = \mathcal{O}^*$.*

2.3. **A few more preliminaries.** It will be convenient to introduce the notation $F^\flat = F^* \backslash \{1\}$, as well as $\mathcal{O}^\flat = \{u \text{ in } \mathcal{O}^* \text{ such that } 1-u \text{ is in } \mathcal{O}^*\}$, and $\kappa^\flat = \kappa^* \backslash \{1\}$.

Throughout the remainder of Section 2, we will let X_F^{loc} be the scheme obtained from $X_F = \mathbb{P}_F^1 \setminus \{t = 1\}$ by removing all points $t = u$ with u in $F^\flat$. We write $X_F^{2,\mathrm{loc}}$ for $(X_F^{\mathrm{loc}})^2$. Similarly, we let $X_{\mathcal{O}} = \mathbb{P}_{\mathcal{O}}^1 \setminus \{t = 1\}$, we write $X_{\mathcal{O}}^{\mathrm{loc}}$ for the scheme obtained from $X_{\mathcal{O}}$ by removing all subschemes $t = u$ with u in $\mathcal{O}^\flat$, and we write $X_{\mathcal{O}}^{2,\mathrm{loc}}$ for $(X_{\mathcal{O}}^{\mathrm{loc}})^2$. Finally, for κ, we let $K_\kappa = \mathbb{P}_\kappa^1 \setminus \{t = 1\}$, we write X_κ^{loc} for the scheme obtained from X_κ by removing all subschemes $t = u$ with u in $\kappa^\flat$, and we write $X_\kappa^{2,\mathrm{loc}}$ for $(X_\kappa^{\mathrm{loc}})^2$. (Of course, we would have to remove such a closed subscheme for only a finite set of u's first, and then take a direct limit. But by [Qui73, roposition 2.4] and some exact sequences in relative K-theory this will give us the K-theory of $X_{\kappa}^{\mathrm{loc}}$ anyway. Moreover, as such a direct limit over finite subsets of $\mathcal{O}^\flat$ or $F^\flat$ is clearly filtered, hence exact, this procedure will commute with taking spectral sequences etc. below, so that we work directly in the direct limit.)

Because writing $\{t = 0, \infty\}$ or $\{t_1 = 0, \infty\}; \{t_2 = 0, \infty\}$ can be rather too long in places, we often abbreviate the first by simply writing $\square$, and the second by writing $\square^2$.

Let $(1 + I)^* = K_1^{(1)}(X_F^{\mathrm{loc}}; \square)$. From the exact sequence

$$\cdots \to K_2^{(1)}(\square) \to K_1^{(1)}(X_F^{\mathrm{loc}}; \square) \to K_1^{(1)}(X_F^{\mathrm{loc}}) \to K_1^{(1)}(\square) \to \dots$$

we see that $(1 + I)^* \subset K_1^{(1)}(X_F^{\mathrm{loc}})$ as $K_2^{(1)}(\square) \cong K_2^{(1)}(F)^{\oplus 2} = 0$. So we can describe $(1 + I)^*$ explicitly as those elements in $K_1^{(1)}(X_F^{\mathrm{loc}})$ that restrict to 1 at $t = 0$ and $t = \infty$. Because $K_1(X_F^{\mathrm{loc}})$ is given by the units in the ring corresponding to a localisation of the affine line, we find that

(2.8)

$$(1 + I)^* = \left\{ \prod_j \left(\frac{t - u_j}{t - 1} \right)^{n_j} \text{ with } u_j \text{ in } F^\flat \text{ , } n_j \text{ in } \mathbb{Z}, \text{ such that } \prod_j u_j^{n_j} = 1 \right\} \otimes_{\mathbb{Z}} \mathbb{Q} \,.$$

Note that in particular the divisor map

$$(1 + I)^* \to \coprod_{t \in F^\flat} K_0^{(0)}(F) \tag{2.9}$$

is an injection.

Note that, if A is any $\mathbb{Q}$-subspace of $K_n^{(l)}(X_F^{\mathrm{loc}}; \square)$, and we use the cup product $(1+I)^* \cup A \to K_{n+1}^{(l+1)}(X_F^{2,\mathrm{loc}}; \square^2)$ by pulling $(1 + I)^*$ back along the first projection, and A along the second, then $\mathrm{d}((1 + I)^* \cup A) = (\mathrm{d}(1 + I)^*) \cup A - (1 + I)^* \cup (\mathrm{d}A)$, and $\coprod_{t_1 \in F^\flat} A/(\mathrm{d}(1 + I)^*) \cup A \cong A \otimes F_{\mathbb{Q}}^*$ because $F^\flat$ generates F^*, and the functions in $(1 + I)^*$ (without $\dots \otimes_{\mathbb{Z}} \mathbb{Q}$) give exactly the multiplicative relations among the elements in $F^\flat$. Of course, by reversing the role of the projections we can do this with t_2 instead of t_1 instead. This will be used in order to change $\coprod_{t \in F^\flat} \dots$ into $\dots \otimes_{\mathbb{Q}} F_{\mathbb{Q}}^*$ in localisation sequences or spectral sequences below.

Under Assumption 2.7, we can do the same for $\mathcal{O}$. Namely, define $(1 + I)_{\mathcal{O}}^* = K_1^{(1)}(X_{\mathcal{O}}^{\mathrm{loc}}; \square)$. Because $K_2^{(1)}(\mathcal{O}) = 0$, and $K_1^{(1)}(\mathcal{O}) = \mathcal{O}_{\mathbb{Q}}^*$, one see by exactly the same argument as for $(1 + I)^*$ that

(2.10)

$$(1+I)_{\mathcal{O}}^* = \left\{ \prod_j \left(\frac{t - u_j}{t - 1} \right)^{n_j} \text{ with } u_j \text{ in } \mathcal{O}^\flat \text{ , } n_j \text{ in } \mathbb{Z}, \text{ such that } \prod_j u_j^{n_j} = 1 \right\} \otimes_{\mathbb{Z}} \mathbb{Q} \,.$$

In particular, we have $(1 + I)_{\mathcal{O}}^* \subseteq (1 + I)^*$ under localisation of the base from $\mathcal{O}$ to F. Note that we used here that $(1 + I)_{\mathcal{O}}^*$ gives us exactly the relations needed to turn $\coprod_{t \in \mathcal{O}^\flat} \dots$ into $\dots \otimes \mathcal{O}_{\mathbb{Q}}^*$, as $(1+I)_{\mathcal{O}}^*$ (without $\dots \otimes_{\mathbb{Z}} \mathbb{Q}$) gives the multiplicative relations among elements in $\mathcal{O}^\flat$, and $\mathcal{O}^\flat$ generates $\mathcal{O}^*$.

Finally, we like to mention that for x in F, under the map $K_0^{(0)}(F)_{|t=x} \to K_0^{(1)}((X_F; \square)) \cong F_{\mathbb{Q}}^*$, 1 is mapped to $x^{\pm 1}$, see [dJ95, Lemma 3.14]. The same holds for $\mathcal{O}$ instead of F, and this is compatible with products.

2.4. **Construction of the complexes for F.** Most of the constructions of the complexes below were already in this subsection and in Subsection 2.5 below were carried out in earlier papers ([dJ95], [dJ96] and [BdJ03]), but we include them for the sake of convenience. Also, in various cases the constructions were carried out more generally, in which case they tend to become dependent on assumptions on weights in K-theory. Rather than letting the reader trying to find his way through the various papers, we'll quickly redo the construction in the cases that we need and extend it where needed, so it will be clear what we get without making assumptions about weights.

2.4.1. *Construction of the complexes $\mathcal{M}_{(2)}(F)$ and $\widetilde{\mathcal{M}}_{(2)}(F)$.* The principle of the construction of the complex $\mathcal{M}_{(2)}(F)$ was first used in Bloch's Irvine notes (finally published as [Blo00]). The construction of $\mathcal{M}_{(2)}(F)$ and $\widetilde{\mathcal{M}}_{(2)}(F)$ can be found in [dJ95, Section 3].

We start with the localisation sequence

$$\begin{aligned} \cdots \longrightarrow \textstyle\coprod_{t\in F^\flat} K_2^{(1)}(F) \longrightarrow K_2^{(2)}(X_F;\square) \longrightarrow K_2^{(2)}(X_F^{\mathrm{loc}};\square) \longrightarrow \\ \textstyle\coprod_{t\in F^\flat} K_1^{(1)}(F) \longrightarrow K_1^{(2)}(X_F;\square) \longrightarrow \cdots . \end{aligned} \tag{2.11}$$

Because $K_2^{(1)}(F) = 0$ for any field F by (2.1), this means that the cohomological complex (in degrees 1 and 2)

$$RC_{(2)}(F) : K_2^{(2)}(X_F^{\mathrm{loc}};\square) \to \coprod_{t\in F^\flat} K_1^{(1)}(F) \tag{2.12}$$

has cohomology groups $H^1(RC_{(2)}(F)) \cong K_3^{(2)}(F)$ and $H^2(RC_{(2)}(F)) \cong K_2^{(2)}(F)$.

In [dJ95, Section 3.2], see also [Blo90], for every x in $F^\flat$ an element $[x]_2$ was constructed in $K_2^{(2)}(X_F^{\mathrm{loc}};\square)$ with the property that its boundary in $\coprod K_1^{(1)}(F)$ is $(1-x)^{-1}_{|t=x}$. Let

$$\mathrm{Symb}_1(F) = K_1^{(1)}(F) = F_{\mathbb{Q}}^* ,$$

and

$$\mathrm{Symb}_2(F) = <[x]_2 \text{ with } x \text{ in } F^\flat>_{\mathbb{Q}} + (1+I)^* \cup \mathrm{Symb}_1(F) .$$

Then we get a subcomplex of (2.12)

$$Symb_2(F) : \mathrm{Symb}_2(F) \to \coprod_{t\in F^\flat} \mathrm{Symb}_1(F) . \tag{2.13}$$

Letting $F_{\mathbb{Q}}^*$ act on the right in (2.9) gives the acyclic subcomplex

$$(1+I)^* \cup F_{\mathbb{Q}}^* \to \mathrm{d}(\dots) . \tag{2.14}$$

We take the quotient of (2.13) by (2.14), to obtain the complex

$$\mathcal{M}_{(2)}(F) : M_2(F) \to F_{\mathbb{Q}}^* \otimes F_{\mathbb{Q}}^* , \tag{2.15}$$

where we used that $\mathrm{d}(1+I)^*$ gives exactly the right relations to turn $\coprod_{t\in F^\flat} \dots$ into $\dots \otimes F_{\mathbb{Q}}^*$, as $F^\flat$ generates F^*, and $M_2(F) = \mathrm{Symb}_2(F)/(1+I)^* \cup \mathrm{Symb}_1(F) = \mathrm{Symb}_2(F)/(1+I)^* \cup F_{\mathbb{Q}}^*$. Then $M_2(F)$ is a $\mathbb{Q}$-vector space generated by the $[x]_2$, x in $F^\flat$, and thew boundary of $[x]_2$ is $(1-x)\otimes x$.

Note that from the maps

$$\mathcal{M}_{(2)}(F) \leftarrow Symb_2(F) \to RC_{(2)}(F)$$

with the left one a quasi isomorphism, we get that we have maps

$$H^p(\mathcal{M}_{(2)}(F)) \to K^{(2)}_{4-p}(F)$$

for $p = 1$ and 2. The map for $p = 1$ is an injection as the corresponding statement holds for $RC_{(2)}(F)$ and $Symb_2(F)$ is a subcomplex, and we are in the lowest degree. For $p = 2$ the map is an isomorphism because $K_2^{(2)}(F)$ is the quotient of $F^*_{\mathbb{Q}} \otimes F^*_{\mathbb{Q}}$ by $<x \otimes (1-x)$ with x in $F^\flat>$.

We will need to quotient out the complex $\mathcal{M}_{(2)}(F)$ even further, in order to end up with a second term $\bigwedge^2 F^*_{\mathbb{Q}}$ rather than $F^*_{\mathbb{Q}} \otimes F^*_{\mathbb{Q}}$. Namely, consider the subcomplex of $\mathcal{M}_{(2)}(F)$

$$<[x]_2 + [x^{-1}]_2>_{\mathbb{Q}} \to \mathrm{d}(\dots)\ .$$

Note that $\mathrm{d}([x]_2+[x^{-1}]_2) = x\otimes x$, so that the second term is in fact $\mathrm{Sym}^2(F^*_{\mathbb{Q}})$. It is known that this is an acyclic subcomplex, see [dJ95, Lemma 3.7 and Remark 3.10], and by taking the quotient complex we get

$$\widetilde{\mathcal{M}}_{(2)}(F) : \widetilde{M}_2(F) \to \bigwedge^2 F^*_{\mathbb{Q}}\ ,$$

with $\widetilde{M}_2(F) = M_2(F)/<[x]_2 + [x_2^{-1}]>_{\mathbb{Q}}$, and $\mathrm{d}[x]_2 = (1-x)\wedge x$.

Because $\widetilde{\mathcal{M}}_{(2)}(F)$ is quasi isomorphic to $\mathcal{M}_{(2)}(F)$ we have maps

$$H^p(\widetilde{\mathcal{M}}_{(2)}(F)) \to K^{(2)}_{4-p}(F)\ . \tag{2.16}$$

Again this maps is an injection for $p = 1$ and an isomorphism for $p = 2$.

There are essentially two ways of generalizing the complex $\mathcal{M}_{(2)}(F)$. The first one is to look at another part of the localisation sequence (2.11), the other to replace X_F by X_F^n for $n \geq 2$, and use localisation there, which will give a spectral sequence. The first will be used to construct the complex $\mathcal{C}^\bullet(F)$ in Section 2.4.4 below, the second (with $n = 2$) will be used for constructing the complex $\mathcal{M}_{(3)}(F)$ below.

2.4.2. *Construction of the complex* $\mathcal{M}_{(3)}(F)$. The complex $\mathcal{M}_{(3)}(F)$ was defined in [dJ95, Section 3]. It consists of three terms in cohomological degrees 1, 2 and 3:

$$M_3(F) \to M_2(F) \otimes F^*_{\mathbb{Q}} \to F^*_{\mathbb{Q}} \otimes \bigwedge^2 F^*_{\mathbb{Q}} \tag{2.17}$$

and comes equipped with maps $H^2(\mathcal{M}_{(3)}(F)) \to K_4^{(3)}(F)$ and $H^3(\mathcal{M}_{(3)}(F)) \to K_3^{(3)}(F)$. The last of those two maps is in fact an isomorphism.

Although we will need the complex $\mathcal{M}_{(3)}(\mathcal{O})$ in order to have information about the special fiber, we describe the complex $\mathcal{M}_{(3)}(F)$ first, as it is notationally easier. Moreover, in the part of the complex we are interested in, we can view $\mathcal{M}_{(3)}(\mathcal{O})$ as a subcomplex of $\mathcal{M}_{(3)}(F)$, see Remark 2.41.

Consider the divisors on X_F^2 defined by putting $t_i = u_j$ for some u_j in $F^\flat$ for $i = 1$ or 2. Then there is a spectral sequence (see [dJ96, page 257] or [dJ95, Page 221])

(2.18)

$$\begin{array}{ccc}
\vdots & \vdots & \vdots \\
K_2^{(3)}(X_F^{2,\mathrm{loc}};\square^2) & \coprod_{t_1\in F^\flat} K_1^{(2)}(X_F^{\mathrm{loc}};\square) \coprod \coprod_{t_2\in F^\flat} K_1^{(2)}(X_F^{\mathrm{loc}};\square) & \coprod_{t_1,t_2\in F^\flat} K_0^{(1)}(F) \\
K_3^{(3)}(X_F^{2,\mathrm{loc}};\square^2) & \coprod_{t_1\in F^\flat} K_2^{(2)}(X_F^{\mathrm{loc}};\square) \coprod \coprod_{t_2\in F^\flat} K_2^{(2)}(X_F^{\mathrm{loc}};\square) & \coprod_{t_1,t_2\in F^\flat} K_1^{(1)}(F) \\
K_4^{(3)}(X_F^{2,\mathrm{loc}};\square^2) & \coprod_{t_1\in F^\flat} K_3^{(2)}(X_F^{\mathrm{loc}};\square) \coprod \coprod_{t_2\in F^\flat} K_3^{(2)}(X_F^{\mathrm{loc}};\square) & \coprod_{t_1,t_2\in F^\flat} K_2^{(1)}(F) \\
\vdots & \vdots & \vdots
\end{array}$$

converging to $K_*^{(3)}(X_F^2;\square^2) \cong K_{*+2}^{(3)}(F)$. The only terms in it that contribute to $K_4^{(3)}(F)$ are $K_2^{(3)}(X_F^{2,\mathrm{loc}};\square^2)$ and $\coprod_{t_1\in F^\flat} K_2^{(2)}(X_F^{\mathrm{loc}};\square) \coprod \coprod_{t_2\in F^\flat} K_2^{(2)}(X_F^{\mathrm{loc}};\square)$ because $\coprod_{t_1,t_2\in F^\flat} K_1^{(2)}(F)$ is trivial. Let $RC_{(3)}(F)$ be the cohomological complex in degrees 1, 2 and 3, consisting of the row in (2.18) that begins with $K_3^{(3)}(X_F^{2,\mathrm{loc}};\square^2)$:

(2.19)

$$\begin{aligned} RC_{(3)}(F) : {} & K_3^{(3)}(X_F^{2,\mathrm{loc}};\square^2) \to \\ & \coprod_{t_1\in F^\flat} K_2^{(2)}(X_F^{\mathrm{loc}};\square) \coprod \coprod_{t_2\in F^\flat} K_2^{(2)}(X_F^{\mathrm{loc}};\square) \to \coprod_{t_1,t_2\in F^\flat} K_1^{(1)}(F)\ . \end{aligned}$$

This complex was denoted $C_{(3)}$ in [dJ95, Section 3.1], but considering the notational overload of the letter C in this paper, we prefer to think of it as a Row Complex rather than just a Complex.

Note that $K_1^{(2)}(F)$ equals zero, so for $p = 2$ and 3 there is a map

$$H^p(RC_{(3)}(F)) \to K_{6-p}^{(3)}(F)\ . \tag{2.20}$$

For x in $F^\flat$, in addition to the element $[x]_2$ in $K_2^{(2)}(X_F^{\mathrm{loc}};\square)$ of Section 2.4.1, there is also an element $[x]_3$ in $K_3^{(3)}(X_F^{2,\mathrm{loc}};\square^2)$ (see [dJ95, Section 3.2]) with boundary $-[x]_{2|t_1=x} + [x]_{2|t_2=x}$ in $\coprod_{t_1\in F^\flat} K_2^{(2)}(X_F^{\mathrm{loc}};\square) \coprod \coprod_{t_2\in F^\flat} K_2^{(2)}(X_F^{\mathrm{loc}};\square)$ in (2.18). Let us define $\mathrm{Symb}_n(F) \subseteq K_n^{(n)}(X_F^{n-1,\mathrm{loc}};\square^{n-1})$ for $n = 1$, 2 and 3 by setting

$$\mathrm{Symb}_1(F) = F_{\mathbb{Q}}^*\ ,$$

$$\mathrm{Symb}_2(F) = <[u]_2 \text{ with } u \text{ in } F^\flat>_{\mathbb{Q}} + (1+I)^* \cup \mathrm{Symb}_1(F)\ ,$$

and

$$\mathrm{Symb}_3(F) = <[u]_3 \text{ with } u \text{ in } F^\flat>_{\mathbb{Q}} + (1+I)^* \,\tilde{\cup}\, \mathrm{Symb}_2(F)\ .$$

For $n = 2$, those are the definitions given in Section 2.4.1, and for $n = 3$, by $\tilde{\cup}$ we mean the following. In the projection X_F^2 to X_F, we can use one of the factors to pull back $(1+I)^*$, the other to pull back $\mathrm{Symb}_2(F)$ and then take the product to land in $\mathrm{Symb}_3(F)$, giving us two cup products. The $\tilde{\cup}$ indicates that we take the sum of the images of both possibilities for those cup products.

Because, in (2.19), $\mathrm{d}[u]_2 = (1-u)^{-1}_{|t=u}$, and $\mathrm{d}[u]_3 = -[u]_{2|t_1=u} + [u]_{2|t_2=u}$, it follows that

(2.21)

$$\mathit{Symb}_{(3)}(F) : \mathrm{Symb}_3(F) \to \coprod_{t_1 \in F^\flat} \mathrm{Symb}_2(F) \amalg \coprod_{t_2 \in F^\flat} \mathrm{Symb}_2(F) \to \coprod_{t_1,t_2 \in F^\flat} \mathrm{Symb}_1(F)$$

is a subcomplex of (2.19). It is shown in [dJ95, Lemma 3.9 and Remark 3.10] that the subcomplex

(2.22)

$$(1+I)^* \,\tilde{\cup}\, \mathrm{Symb}_2(F) \to \coprod_{t_1 \in F^\flat} (1+I)^* \cup F^*_{\mathbb{Q}} \amalg \coprod_{t_2 \in F^\flat} (1+I)^* \cup F^*_{\mathbb{Q}} + \mathrm{d}(\dots) \to \mathrm{d}(...)$$

of (2.21) is acyclic.

S_2 acts on the spectral sequence (2.18) by swapping t_1 and t_2. It therefore also acts on the complex (2.19) above. Because the symbol $[x]_3$ is alternating by construction (see [dJ95, Section 3.2]), we can take the alternating parts of (2.21) and (2.22), and form the quotient complex

$$\mathcal{M}_{(3)}(F) : M_3(F) \to M_2(F) \otimes F^*_{\mathbb{Q}} \to F^*_{\mathbb{Q}} \otimes \bigwedge^2 F^*_{\mathbb{Q}} \,, \tag{2.23}$$

where

$$M_3(F) = \mathrm{Symb}_3(F) / \left((1+I)^* \,\tilde{\cup}\, \mathrm{Symb}_2(F) \right)^{\mathrm{alt}} \,,$$

and

$$M_2(F) = \mathrm{Symb}_2(F)/(1+I)^* \cup F^*_{\mathbb{Q}}$$

as before in Subsection 2.4.1. Note that, for $n = 2$ and 3, $M_n(F)$ is a $\mathbb{Q}$-vector space on symbols $[x]_n$ for x in $F^\flat$, modulo nonexplicit relations depending on n. The maps in the complex are given by

$$\mathrm{d}[x]_3 = [x]_2 \otimes x$$

and

$$\mathrm{d}[x]_2 \otimes y = (1-x) \otimes (x \wedge y).$$

As before, we used here that $\mathrm{d}(1+I)^*$ gives exactly the right relations to turn $\coprod_{t \in F^\flat} \dots$ into $\dots \otimes F^*_{\mathbb{Q}}$, as $F^\flat$ generates F^*. As $\mathit{Symb}_{(3)}(F)$ is a subcomplex of $\mathit{RC}_{(3)}(F)$, this gives us maps

$$\mathcal{M}_{(3)}(F) \leftarrow \mathit{Symb}_{(3)}(F)^{\mathrm{alt}} \to \mathit{RC}_{(3)}(F)^{\mathrm{alt}} \to \mathit{RC}_{(3)}(F)$$

with the left map a quasi isomorphism. Combining this with (2.20) gives us a map

$$H^p(\mathcal{M}_{(3)}(F)) \to K^{(3)}_{6-p}(F)$$

for $p = 2$ and 3.

Remark 2.24. For $p = 1$, we still get a map

$$H^1(\mathcal{M}_{(3)}(F)) \to K^{(3)}_5(F)/K^{(2)}_4(F) \cup F^*_{\mathbb{Q}} \,.$$

2.4.3. *Construction of the complex* $\mathcal{M}_{(3)}(C)$. We now return to the situation where we have a number field $k \subset K$, and a smooth, projective, geometrically irreducible curve C over k. with function field $F = k(C)$.

Because we are interested in finding elements in $K_4^{(3)}(C)$, we introduce yet another complex, $\mathcal{M}_{(3)}(C)$, which is the total complex associated to the double complex

$$\begin{tikzcd}
M_3(F) \arrow[r, "\mathrm{d}"] \arrow[d] & M_2(F) \otimes_{\mathbb{Q}} F^*_{\mathbb{Q}} \arrow[r, "\mathrm{d}"] \arrow[d, "\partial_1"'] & F^*_{\mathbb{Q}} \otimes \bigwedge^2 F^*_{\mathbb{Q}} \arrow[d, "\partial_2"'] \\
0 \arrow[r] & \coprod_x \widetilde{M}_2(k(x)) \arrow[r, "\mathrm{d}"] & \coprod_x \bigwedge^2 k(x)^*_{\mathbb{Q}} \, .
\end{tikzcd}$$

Here the coproduct is over all closed points x in C. The boundary maps are as follows. The d's in the top row are as in $\mathcal{M}_{(3)}(F)$. In the bottom row, $\mathrm{d}[z]_2 = (1-z) \wedge z$. For the vertical maps, $\partial_{1,x}([g]_2 \otimes f) = \mathrm{ord}_x(f) \cdot [g(x)]_2$, with the convention that $[0]_2 = [1]_2 = [\infty]_2 = 0$. Finally, $\partial_{2,x}$ described as follows. Let π be a uniformizer at x, u_j units at x. Then $\partial_{2,x}$ is determined by

$$\pi \wedge u_1 \wedge u_2 \mapsto u_1(x) \wedge u_2(x) \text{ and } u_1 \wedge u_2 \wedge u_3 \mapsto 0 \, .$$

We have an obvious map $\mathcal{M}_{(3)}(C) \to \mathcal{M}_{(3)}(F)$, corresponding to the localization map in (2.2). In [dJ96, Theorem 5.2], it is shown that this induces a commutative diagram

$$\begin{tikzcd}
H^2(\mathcal{M}_{(3)}(C)) \arrow[r] \arrow[d] & H^2(\mathcal{M}_{(3)}(F)) \arrow[d] \\
K_4^{(3)}(C) \oplus K_3^{(2)}(k) \cup F^*_{\mathbb{Q}} \arrow[r] & K_4^{(3)}(F) \, .
\end{tikzcd}$$

Note that it was shown in Remark 2.4 that $K_4^{(3)}(C) \oplus K_3^{(2)}(k) \cup F^*_{\mathbb{Q}}$ is indeed a direct sum, and that the lower vertical map is an injection.

Remark 2.25. If k is totally real then $K_3^{(2)}(k)$ is zero. But in general we can use the projection

$$K_4^{(3)}(C) \oplus K_3^{(2)}(k) \cup F^*_{\mathbb{Q}} \to K_4^{(3)}(C)$$

to get a map

$$H^2(\mathcal{M}_{(3)}(C)) \to K_4^{(3)}(C)$$

as the composition

$$H^2(\mathcal{M}_{(3)}(C)) \to K_4^{(3)}(C) \oplus K_3^{(2)}(k) \cup F^*_{\mathbb{Q}} \to K_4^{(3)}(C) \, .$$

Note that an element $\sum_i [g_i]_2 \otimes f_i$ in $H^2(\mathcal{M}_{(3)}(F))$ satisfies

$$\sum_i (1-g_i) \otimes (g_i \wedge f_i) = 0 \tag{2.26}$$

in $F^*_{\mathbb{Q}} \otimes \bigwedge^2 F^*_{\mathbb{Q}}$. The additional condition for it to lie in $H^2(\mathcal{M}_{(3)}(C))$ is that

$$\sum_i \mathrm{ord}_x(f_i)[g_i(x)]_2 = 0 \tag{2.27}$$

in $\widetilde{M}_2(k(x))$ for all closed points x in C, with the convention that $[0]_2 = [1]_2 = [\infty]_2 = 0$.

2.4.4. *Construction of the complex $\mathcal{C}^\bullet(F)$.* The complex $\mathcal{C}^\bullet(F)$ was introduced in [dJ96, Section 3], but was first constructed in [Blo90]. We briefly recall its construction in order to clarify the construction of the corresponding complex for $\mathcal{O}$ later in Section 2.5.4.

One starts with another part of the exact localisation sequence (2.11) in relative K-theory.

$$\begin{tikzcd}
\cdots \arrow[r] & \coprod_{t\in F^\flat} K_3^{(2)}(F) \arrow[r] & K_3^{(3)}(X_F;\square) \arrow[r] & K_3^{(3)}(X_F^{\mathrm{loc}};\square) \arrow[r] & {} \\
& & \coprod_{t\in F^\flat} K_2^{(2)}(F) \arrow[r] & K_2^{(3)}(X_F;\square) \arrow[r] & \cdots \; .
\end{tikzcd}\tag{2.28}$$

Because $K_2^{(3)}((X_F;\square)) \cong K_3^{(3)}(F) \cong K_3^M(F)_\mathbb{Q}$, so that the map $\coprod_{t\in F^\flat} K_2^{(2)}(F) \to K_2^{(3)}(X_F;\square)$ is surjective, this shows that the cohomological complex in degrees 1 and 2

$$AC_{(3)}(F) : K_3^{(3)}(X_F^{\mathrm{loc}};\square) \to \coprod_{t\in F^\flat} K_2^{(2)}(F)\tag{2.29}$$

has maps

$$H^1(AC_{(3)}(F)) \cong K_4^{(3)}(F)/K_3^{(2)}(F)\cup F_\mathbb{Q}^*$$

and

$$H^2(AC_{(3)}(F)) \cong K_3^{(3)}(F)\ .$$

(Here AC stands for Auxiliary Complex.)

Again we have an acyclic subcomplex

$$(1+I)^*\cup K_2^{(2)}(F) \to \mathrm{d}(\dots)\ ,\tag{2.30}$$

and therefore the quotient complex $\mathcal{C}^\bullet(F)$ is a cohomological complex in degree 1 and 2,

$$\mathcal{C}^\bullet(F) : \mathcal{C}^1(F) \to \mathcal{C}^2(F)\ ,$$

with

$$\mathcal{C}^1(F) = \frac{K_3^{(3)}(X_F^{\mathrm{loc}};\square)}{(1+I)^*\cup K_2^{(2)}(F)}$$

and

$$\mathcal{C}^2(F) = K_2^{(2)}(F)\otimes F_\mathbb{Q}^*\ .$$

It comes with maps

$$H^1(\mathcal{C}^\bullet(F)) \cong K_4^{(3)}(F)/K_3^{(2)}(F)\cup F_\mathbb{Q}^*$$

and

$$H^2(\mathcal{C}^\bullet(F)) \cong K_3^{(3)}(F)\ .$$

Note that if g is in $F^\flat$, and f is in F^*, then $[g]_2\cup f$ lies in $K_3^{(3)}(X_F^{\mathrm{loc}};\square)$. In fact, if we take the class of $[g]_2$ in $M_2(F)$ instead, then we do get a well defined class in $\mathcal{C}^1(F)$, as $(1+I)^*\cup F_\mathbb{Q}^*\cup f$ goes to zero in $\mathcal{C}^1(F)$ by definition. Under the differential in the complex, $[g]_2\cup(f)$ is mapped to $\{(1-g)^{-1}, f\}\otimes g = -\{1-g, f\}\otimes g$, so the condition for an element $\sum_i [g_i]_2\cup(f_i)$ to be in $H^1(\mathcal{C}^\bullet(F))$ is that

$$\sum_i \{1-g_i, f_i\}\otimes g_i = 0\tag{2.31}$$

in $K_2^{(2)}(F)\otimes F_\mathbb{Q}^*$.

The map

$$M_{(2)}(F) \otimes F_{\mathbb{Q}}^{*} \to \mathcal{C}^1(F)$$

given by

$$[g]_2 \otimes f \mapsto [g]_2 \cup f$$

fits into a commutative diagram

$$\begin{tikzcd} M_3(F) \arrow[r] \arrow[d] & M_2(F) \otimes F_{\mathbb{Q}}^{*} \arrow[r] \arrow[d] & F_{\mathbb{Q}}^{*} \otimes \bigwedge^2 F_{\mathbb{Q}}^{*} \arrow[d] \\ 0 \arrow[r] & \mathcal{C}^1(F) \arrow[r] & \mathcal{C}^2(F) \end{tikzcd} \tag{2.32}$$

where we map $f \otimes g \wedge h$ to $\{f, g\} \otimes h - \{f, h\} \otimes g$. Multiplying the map $H^2(\mathcal{M}_{(3)}(F)) \to K_4^{(3)}(F)$ by -1 if necessary, we obtain a commutative diagram

$$\begin{tikzcd} H^2(\mathcal{M}_{(3)}(F)) \arrow[r] \arrow[d] & K_4^{(3)}(F) \arrow[d] \\ H^1(\mathcal{C}^{\bullet}(F)) \arrow[r] & K_4^{(3)}(F)/K_3^{(2)}(F) \cup F_{\mathbb{Q}}^{*} \end{tikzcd} \tag{2.33}$$

see [dJ96, Proposition 3.2].

2.5. Construction of the complexes for $\mathcal{O}$ and $\mathbb{C}$.

Remark 2.34. At various stages there will be some properties of the complexes for $\mathcal{O}$ that depend on $K_3^{(2)}(\kappa)$ of $K_5^{(2)}(\kappa)$ being trivial. Clearly, this applies to $\mathcal{O}$ as in the Introduction by our remarks about the K-groups of $\mathbb{F}(\mathbb{C}_{\mathbb{F}})$ and $\kappa(\mathbb{C}_{\kappa})$ in Section 2.2.

2.5.1. *Construction of the complex $\mathcal{M}_{(2)}(\mathcal{O})$.* When we try to imitate the localisation sequence (2.11) for $\mathcal{O}$ rather than F, we are dealing with the two dimensional scheme $X_{\mathcal{O}}$, and we end up with a spectral sequence instead,

$$\begin{array}{ccc} \vdots & \vdots & \\ K_1^{(2)}(X_{\mathcal{O}}^{\mathrm{loc}}; \square) & \coprod_{t \in \mathcal{O}^{\flat}} K_0^{(1)}(F) & \\ K_2^{(2)}(X_{\mathcal{O}}^{\mathrm{loc}}; \square) & \coprod_{t \in \mathcal{O}^{\flat}} K_1^{(1)}(F) & \coprod_{t \in \kappa^{\flat}} K_0^{(0)}(\kappa) \\ K_3^{(2)}(X_{\mathcal{O}}^{\mathrm{loc}}; \square) & \coprod_{t \in \mathcal{O}^{\flat}} K_2^{(1)}(F) & \coprod_{t \in \kappa^{\flat}} K_1^{(0)}(\kappa) \\ \vdots & \vdots & \vdots \end{array} \tag{2.35}$$

which converges to $K_*^{(2)}(X_{\mathcal{O}}^{\mathrm{loc}}; \square) \cong K_{*+1}^{(2)}(\mathcal{O})$.

Because $K_2^{(1)}(F)$, $K_1^{(0)}(\kappa)$ and $K_2^{(0)}(\kappa)$ are all trivial, if we let $RC_{(2)}(\mathcal{O})$ be the cohomological complex in degrees 1, 2 and 3, given by

$$K_2^{(2)}(X_{\mathcal{O}}^{\mathrm{loc}}; \square) \to \coprod_{t \in \mathcal{O}^{\flat}} K_1^{(1)}(F) \to \coprod_{t \in \kappa^{\flat}} K_0^{(0)}(\kappa) , \tag{2.36}$$

then there are maps $H^1(RC_{(2)}(\mathcal{O})) \cong K_3^{(2)}(\mathcal{O})$ and $H^2(RC_{(2)}(\mathcal{O})) \to K_2^{(2)}(\mathcal{O})$. The last map is surjective by Remark refmilnork and the exact sequence

$$\cdots \to K_2^{(1)}(\kappa) \to K_2^{(2)}(\mathcal{O}) \to K22F \to K_1^{(1)}(\kappa) \to \dots$$

as $K_2^{(1)}(\kappa) = 0$. Note that the map $K_1^{(1)}(F) \to K_0^{(0)}(\kappa)$ is surjective, so that $H^3(RC_{(2)}(\mathcal{O}))$ is zero, as is $K_1^{(2)}(\mathcal{O})$.

Now let $A \subseteq K_2^{(2)}(X_{\mathcal{O}}^{\mathrm{loc}}; \square)$ be the inverse image of $\coprod_{t\in\mathcal{O}^\flat} \mathcal{O}_{\mathbb{Q}}^*$ in $\coprod_{t\in\mathcal{O}^\flat} K_1^{(1)}(F)$. Because $K_1^{(1)}(\mathcal{O}) = \mathcal{O}_{\mathbb{Q}}^*$ is equal to ker $\left(K_1^{(1)}(F) \to K_0^{(0)}(\kappa)\right)$, this means that the subcomplex

$$RC_{(2)}(\mathcal{O}) : A \to \coprod_{t\in\mathcal{O}^\flat} \mathcal{O}_{\mathbb{Q}}^* \tag{2.37}$$

of (2.36) has maps $H^1(RC_{(2)}(\mathcal{O})) \to K_3^{(2)}(\mathcal{O})$ and $H^2(RC_{(2)}(\mathcal{O})) \to K_2^{(2)}(\mathcal{O})$.

We again use the element $[u]_2$ in $K_2^{(2)}(X_{\mathcal{O}}^{\mathrm{loc}}; \square)$ for every u in $\mathcal{O}^\flat$, and put

$$\mathrm{Symb}_1(\mathcal{O}) = K_1^{(1)}(\mathcal{O}) = \mathcal{O}_{\mathbb{Q}}^* \,,$$

and

$$\mathrm{Symb}_2(\mathcal{O}) = <[u]_2 \text{ with } u \text{ in } \mathcal{O}^\flat>_{\mathbb{Q}} + (1+I)_{\mathcal{O}}^* \cup \mathcal{O}_{\mathbb{Q}}^* \,.$$

(See (2.10) for the definition of $(1+I)_{\mathcal{O}}^*$.) Observe that, if u is in $\mathcal{O}^\flat$ and v is in $\mathcal{O}_{\mathbb{Q}}^*$, then $[u]_2$ and $(1+I)_{\mathcal{O}}^* \cup v$ are in A, so we get a subcomplex of (2.37)

$$\mathit{Symb}_2(\mathcal{O}) : \mathrm{Symb}_2(\mathcal{O}) \to \coprod_{t\in\mathcal{O}^\flat} \mathcal{O}_{\mathbb{Q}}^* \,, \tag{2.38}$$

containing the acyclic subcomplex

$$(1+I)_{\mathcal{O}}^* \cup \mathcal{O}_{\mathbb{Q}}^* \to \mathrm{d}(\dots) \,. \tag{2.39}$$

We take the quotient complex of (2.38) by (2.39), to obtain the complex

$$\mathcal{M}_{(2)}(\mathcal{O}) : M_2(\mathcal{O}) \to \mathcal{O}_{\mathbb{Q}}^* \otimes \mathcal{O}_{\mathbb{Q}}^* \,, \tag{2.40}$$

with $M_2(\mathcal{O}) = \mathrm{Sym}_2(\mathcal{O})/(1+I)^* \cup \mathcal{O}_{\mathbb{Q}}^*$. Then $M_2(\mathcal{O})$ is a $\mathbb{Q}$-vector space generated by the $[u]_2$, u in $\mathcal{O}^\flat$, and $\mathrm{d}[u]_2 = (1-u) \otimes u$. (Again, we used that $\mathrm{d}(1+I)_{\mathcal{O}}^* \cup \mathcal{O}_{\mathbb{Q}}^*$ gives us exactly the right relations to change $\coprod_{t\in\mathcal{O}^\flat} \mathcal{O}_{\mathbb{Q}}^*$ into $\mathcal{O}_{\mathbb{Q}}^* \otimes \mathcal{O}_{\mathbb{Q}}^*$ because $\mathcal{O}^\flat$ generates $\mathcal{O}^*$.) Note that we now have maps

$$\mathcal{M}_{(2)}(\mathcal{O}) \leftarrow \mathit{Symb}_2(\mathcal{O}) \to RC_{(2)}(\mathcal{O}) \,,$$

with the left one a quasi isomorphism, so we obtain maps

$$H^p(\mathcal{M}_{(2)}(\mathcal{O})) \to K_{4-p}^{(2)}(\mathcal{O})$$

for $p = 1$ and 2. Again the map for $p = 1$ is an injection (cf. (2.16)). For $p = 2$ the map is a surjection by Remark 2.6 because $K_2^{(2)}(\mathcal{O}) = \ker\left(K_2^{(2)}(F) \to K_1^{(1)}(\kappa)\right)$.

By localising the base from $\mathcal{O}$ to F in (2.35) gives us (2.18), so that we get a map of complexes

$$M_2(\mathcal{O}) \to M_2(F) \,,$$

since the various steps in the constructions of the two complexes are compatible ($\mathcal{O}_{\mathbb{Q}}^* \subset F_{\mathbb{Q}}^*$, $(1+I)_{\mathcal{O}}^* \subset (1+I)^*$, etc.).

Remark 2.41. The map

$$M_2(\mathcal{O}) \to M_2(F)$$

is injective. Namely, because the construction of the complexes for $\mathcal{M}_{(2)}(\mathcal{O})$ and $\mathcal{M}_{(2)}(F)$ is compatible with the localisation from $\mathcal{O}$ to F in (2.35), we have a commutative diagram

$$\begin{tikzcd}
0 \arrow[r] & H^1(\mathcal{M}_{(2)}(\mathcal{O})) \arrow[r] \arrow[d] & M_2(\mathcal{O}) \arrow[r] \arrow[d] & \mathcal{O}_{\mathbb{Q}}^* \otimes \mathcal{O}_{\mathbb{Q}}^* \arrow[d] \\
0 \arrow[r] & H^1(\mathcal{M}_{(2)}(F)) \arrow[r] & M_2(F) \arrow[r] & F_{\mathbb{Q}}^* \otimes F_{\mathbb{Q}}^* ,
\end{tikzcd}$$

with $H^1(\mathcal{M}_{(2)}(\mathcal{O})) \subseteq K_3^{(2)}(\mathcal{O})$ and $H^1(\mathcal{M}_{(2)}(F)) \subseteq K_3^{(2)}(F)$. From the exact localisation sequence

$$\cdots \to K_3^{(1)}(\kappa) \to K_3^{(2)}(\mathcal{O}) \to K_3^{(2)}(F) \to K_2^{(1)}(\kappa) \to \dots$$

we see that $K_3^{(2)}(\mathcal{O}) \cong K_3^{(2)}(F)$, so that the map on H^1's must be injective. As $\mathcal{O}_{\mathbb{Q}}^* \otimes \mathcal{O}_{\mathbb{Q}}^* \to F_{\mathbb{Q}}^* \otimes F_{\mathbb{Q}}^*$ is clearly injective, $M_2(\mathcal{O}) \to M_2(F)$ must be injective as well. So we may think of $M_2(\mathcal{O})$ as the subspace of $M_2(F)$ generated by the $[u]_2$ with u in $\mathcal{O}^\flat \subset F^\flat$. A similar proof holds for the map $\widetilde{M}_2(\mathcal{O}) \to \widetilde{M}_2(F)$ in Remark 2.42 below.

Remark 2.42. Although not needed in this paper, we can quotient out the complex $\mathcal{M}_{(2)}(\mathcal{O})$ even further, in order to end up with a second term $\bigwedge^2 \mathcal{O}_{\mathbb{Q}}^*$ rather than $\mathcal{O}_{\mathbb{Q}}^* \otimes \mathcal{O}_{\mathbb{Q}}^*$. Namely, consider the subcomplex of $\mathcal{M}_{(2)}(\mathcal{O})$

$$<[u]_2 + [u^{-1}]_2>_{\mathbb{Q}} \to \mathrm{d}(\dots) .$$

Again the second term is in fact $\mathrm{Sym}^2(\mathcal{O}_{\mathbb{Q}}^*)$. Using the technique of the proof of [dJ95, Proposition 3.20] one checks easily that this is an acyclic subcomplex. The quotient complex is

$$\widetilde{\mathcal{M}}_{(2)}(\mathcal{O}) : \widetilde{M}_2(\mathcal{O}) \to \bigwedge^2 \mathcal{O}_{\mathbb{Q}}^* ,$$

with $\widetilde{M}_2(\mathcal{O}) = M_2(\mathcal{O})/<[u]_2 + [u_2^{-1}]>_{\mathbb{Q}}$, and $\mathrm{d}[u]_2 = (1-u) \wedge u$.

Because $\widetilde{\mathcal{M}}_{(2)}(\mathcal{O})$ is quasi isomorphic to $\mathcal{M}_{(2)}(\mathcal{O})$ we have maps

$$H^p(\widetilde{\mathcal{M}}_{(2)}(\mathcal{O})) \to K_{4-p}^{(2)}(\mathcal{O}) .$$

For $p = 1$ this is again an injection. and a surjection for $p = 2$. There is a map $\widetilde{\mathcal{M}}_{(2)}(\mathcal{O}) \to \widetilde{\mathcal{M}}_{(2)}(F)$ obtained by localizing the construction from $\mathcal{O}$ to F, which is fits into the commutative diagram

$$\begin{tikzcd}
H^p(\widetilde{\mathcal{M}}_{(2)}(\mathcal{O})) \arrow[r] \arrow[d] & H^p(\widetilde{\mathcal{M}}_{(2)}(F)) \arrow[d] \\
K_{4-p}^{(2)}(\mathcal{O}) \arrow[r] & K_{4-p}^{(2)}(F) .
\end{tikzcd}$$

2.5.2. *Construction of the complex $\mathcal{M}_{(3)}(\mathcal{O})$.* In this subsection, we will be making Assumption 2.7.

If we now try to imitate the construction of $\mathcal{M}_{(3)}(F)$ using $\mathcal{O}$ instead of F, see some differences. For example, in the construction of the spectral sequence, in codimension one, we will end up with copies of $\{t_i = u\}$ for u in $\mathcal{O}^\flat$, which look like $X_{\mathcal{O}}$, out of which we have to remove the intersections with all other such pieces of codimension one of the form $\{t_i = v\}$ for $i = 1$ and 2, and v in $\mathcal{O}^\flat$. Note that, in particular, we also cut out $t_i = v$ with u and v different elements in $\mathcal{O}^\flat$, but reducing to the same in the residue field. Then $t_i = v$ cuts out the bit in the special fibre in $t_i = u$. We therefore end up with copies of $X_F^{'\mathrm{loc}} = X_F \setminus \{t = u \text{ with } u \text{ in } \mathcal{O}^\flat\}$.

So if we do this for $\mathcal{O}$, we end up with the following spectral sequence, converging to $K_*^{(3)}(X_{\mathcal{O}}^2;\square^2) \cong K_{*+2}^{(3)}(\mathcal{O})$, see [BdJ03, (3.7)].

(2.43)

$$\begin{array}{cccc}
\vdots & \vdots & \vdots & \\
K_2^{(3)}(X_{\mathcal{O}}^{2,\mathrm{loc}};\square^2) & \left(\coprod\limits_{t\in\mathcal{O}^\flat} K_1^{(2)}(X_F^{'\mathrm{loc}};\square)\right)^2 & \coprod\limits_{t_1,t_2\in\mathcal{O}^\flat} K_0^{(1)}(F) \amalg \left(\coprod\limits_{t\in\kappa^\flat} K_0^{(1)}(X_\kappa^{\mathrm{loc}};\square)\right)^2 & \\
K_3^{(3)}(X_{\mathcal{O}}^{2,\mathrm{loc}};\square^2) & \left(\coprod\limits_{t\in\mathcal{O}^\flat} K_2^{(2)}(X_F^{'\mathrm{loc}};\square)\right)^2 & \coprod\limits_{t_1,t_2\in\mathcal{O}^\flat} K_1^{(1)}(F) \amalg \left(\coprod\limits_{t\in\kappa^\flat} K_1^{(1)}(X_\kappa^{\mathrm{loc}};\square)\right)^2 & \coprod\limits_{t_1,t_2\in\kappa^\flat} K_0^{(0)}(\kappa) \\
K_4^{(3)}(X_{\mathcal{O}}^{2,\mathrm{loc}};\square^2) & \left(\coprod\limits_{t\in\mathcal{O}^\flat} K_3^{(2)}(X_F^{'\mathrm{loc}};\square)\right)^2 & \coprod\limits_{t_1,t_2\in\mathcal{O}^\flat} K_2^{(1)}(F) \amalg \left(\coprod\limits_{t\in\kappa^\flat} K_2^{(1)}(X_\kappa^{\mathrm{loc}};\square)\right)^2 & \coprod\limits_{t_1,t_2\in\kappa^\flat} K_1^{(0)}(\kappa) \\
\vdots & \vdots & \vdots & \vdots
\end{array}$$

Here the $(\dots)^2$ corresponds to two copies, corresponding to a coproduct over t_1 in $\mathcal{O}^\flat$ or $\kappa^\flat$, and t_2 in $\mathcal{O}^\flat$ or $\kappa^\flat$. As explained before, in order to obtain $X_F^{'\mathrm{loc}}$ out of X_F, we only remove $t_i = u_j$ with u_j in $\mathcal{O}^\flat$.

Now notice that all $K_j^{(0)}(\kappa)$ are zero for $j \ge 1$, that $K_j^{(1)}(F)$ is zero for $j \ge 2$, and finally that $K_j^{(1)}(X_\kappa^{\mathrm{loc}};\square)$ is zero as well for $j \ge 2$: we consider the exact localisation sequence

$$\cdots \to K_j^{(1)}(X_\kappa^1;\square) \to K_j^{(1)}(X_\kappa^{\mathrm{loc}};\square) \to \coprod K_{j-1}^{(0)}(\kappa) \to \dots\ ,$$

and use that $K_j^{(1)}(X_\kappa^1;\square) \cong K_{j+1}^{(1)}(\kappa)$, which is zero as $K_m^{(1)}(L) = 0$ for $m \ge 2$ for any field L, as well as that $K_{j-1}^{(0)}(\kappa) = 0$ because $j-1 \ge 1$. Therefore, with $RC_{(3)}(\mathcal{O})$ the following cohomological complex in degrees 1 through 4 (corresponding to the row in (2.43) starting with $K_3^{(3)}(X^2_{\mathcal{O},\mathrm{loc}};\square^2)$):

$$\begin{aligned}RC_{(3)}(\mathcal{O}) : K_3^{(3)}(X_{\mathcal{O}}^{2,\mathrm{loc}};\square^2) \to \left(\coprod_{t\in\mathcal{O}^\flat} K_2^{(2)}(X_F^{\mathrm{loc}};\square)\right)^2 \to \\ \coprod_{t_1,t_2\in\mathcal{O}^\flat} K_1^{(1)}(F) \coprod \left(\coprod_{t\in\kappa^\flat} K_1^{(1)}(X_\kappa^{\mathrm{loc}};\square)\right)^2 \to \coprod_{t_1,t_2\in\kappa^\flat} K_0^{(0)}(\kappa)\end{aligned} \tag{2.44}$$

has maps

$$H^p(RC_{(3)}(\mathcal{O})) \to K_{6-p}^{(3)}(\mathcal{O}) \tag{2.45}$$

for $p = 2$, 3 and 4.

Remark 2.46. Note that for $p = 4$ this statement is vacuous since from the localisation sequence

$$\cdots \to K_3^{(3)}(F) \to K_2^{(2)}(\kappa) \to K_2^{(3)}(\mathcal{O}) \to K_2^{(3)}(F) \to \dots$$

and the facts that $K_2^{(3)}(F)$ is trivial, and $K_3^{(3)}(F) \to K_2^{(2)}(\kappa)$ is surjective (see Remark 2.6), it follows that $K_2^{(3)}(\mathcal{O})$ is zero.

Remark 2.47. The map

$$K_2^{(2)}(X_{\mathcal{O}}^{\mathrm{loc}};\square) \to K_2^{(2)}(X_F^{\prime\mathrm{loc}};\square) \to K_2^{(2)}(X_F^{\mathrm{loc}};\square)$$

is injective. If $K_5^{(2)}(\kappa) = 0$ then the map

$$K_3^{(3)}(X_{\mathcal{O}}^{2,\mathrm{loc}};\square^2) \to K_3^{(3)}(X_F^{\prime 2,\mathrm{loc}};\square^2)$$

is injective. If $K_4^{(2)}(F) = 0$ as well, then the map $K_3^{(3)}(X_F^{\prime 2,\mathrm{loc}};\square^2) \to K_3^{(3)}(X_F^{2,\mathrm{loc}};\square^2)$ is also injective.

Namely, we have an exact localisation sequence

$$\cdots \to K_2^{(1)}(X_\kappa^{\mathrm{loc}};\square) \to K_2^{(2)}(X_{\mathcal{O}}^{\mathrm{loc}};\square) \to K_2^{(2)}(X_F^{\prime\mathrm{loc}};\square) \to \cdots ,$$

and $K_2^{(1)}(X_\kappa^{\mathrm{loc}};\square)$ equals zero, as seen above. Also, we have an exact localisation sequence

$$\cdots \to \coprod_{t\in F^*\setminus F^\flat \bigcup\{1\}} K_2^{(1)}(F) \to K_2^{(2)}(X_F^{\prime\mathrm{loc}};\square) \to K_2^{(2)}(X_F^{\mathrm{loc}};\square) \to \cdots ,$$

and again $K_2^{(1)}(F)$ is zero.

Similarly, we have an exact sequence

$$\cdots \to K_3^{(2)}(X_\kappa^{2,\mathrm{loc}};\square^2) \to K_3^{(3)}(X^2_{\mathcal{O},\mathrm{loc}};\square^2) \to K_3^{(3)}(X_F^{\prime 2,\mathrm{loc}};\square^2) \to \cdots .$$

Now consider the spectral sequence for κ (cf. (2.18)),

$$\begin{array}{ccc}
\vdots & \vdots & \vdots \\
K_2^{(2)}(X_\kappa^{2,\mathrm{loc}};\square^2) & \coprod_{t_1\in\kappa^\flat} K_1^{(1)}(X_\kappa^{\mathrm{loc}};\square) \amalg \coprod_{t_2\in\kappa^\flat} K_1^{(1)}(X_\kappa^{\mathrm{loc}};\square) & \coprod_{t_1,t_2\in\kappa^\flat} K_0^{(0)}(\kappa) \\
K_3^{(2)}(X_\kappa^{2,\mathrm{loc}};\square^2) & \coprod_{t_1\in\kappa^\flat} K_2^{(1)}(X_\kappa^{\mathrm{loc}};\square) \amalg \coprod_{t_2\in\kappa^\flat} K_2^{(1)}(X_\kappa^{\mathrm{loc}};\square) & \coprod_{t_1,t_2\in\kappa^\flat} K_1^{(0)}(\kappa) \\
K_4^{(2)}(X_\kappa^{2,\mathrm{loc}};\square^2) & \coprod_{t_1\in\kappa^\flat} K_3^{(1)}(X_\kappa^{\mathrm{loc}};\square) \amalg \coprod_{t_2\in\kappa^\flat} K_3^{(1)}(X_\kappa^{\mathrm{loc}};\square) & \coprod_{t_1,t_2\in\kappa^\flat} K_2^{(0)}(\kappa) \\
\vdots & \vdots & \vdots
\end{array}$$

converging to $K_*^{(2)}(X_\kappa^2;\square^2) \cong K_{*+2}^{(2)}(\kappa)$. We have seen already that $K_j^{(1)}(X_\kappa^{\mathrm{loc}};\square)$ for $j \geq 2$, and we know that $K_j^{(0)}(\kappa) = 0$ for $j \geq 1$, so that we get that

$$K_3^{(2)}(X_\kappa^{2,\mathrm{loc}};\square^2) \cong K_3^{(2)}(X_\kappa^2;\square^2) \cong K_5^{(2)}(\kappa)\ ,$$

which we assume is zero. From the localisation sequence above we therefore get the second statement.

Finally, for the map $K_3^{(3)}(X_F^{'2,\mathrm{loc}};\square^2) \to K_3^{(3)}(X_F^{2,\mathrm{loc}};\square^2)$, this occurs in the spectral sequence (with $S = F^\flat \setminus \mathcal{O}^\flat$)

$$\begin{array}{ccc}
\vdots & \vdots & \vdots \\
K_2^{(3)}(X_F^{2,\mathrm{loc}};\square^2) & \coprod_{t_1\in S} K_1^{(2)}(X_F^{\mathrm{loc}};\square) \amalg \coprod_{t_2\in S} K_1^{(2)}(X_F^{\mathrm{loc}};\square) & \coprod_{t_1,t_2\in S} K_0^{(1)}(F) \\
K_3^{(3)}(X_F^{2,\mathrm{loc}};\square^2) & \coprod_{t_1\in S} K_2^{(2)}(X_F^{\mathrm{loc}};\square) \amalg \coprod_{t_2\in S} K_2^{(2)}(X_F^{\mathrm{loc}};\square) & \coprod_{t_1,t_2\in S} K_1^{(1)}(F) \\
K_4^{(3)}(X_F^{2,\mathrm{loc}};\square^2) & \coprod_{t_1\in S} K_3^{(2)}(X_F^{\mathrm{loc}};\square) \amalg \coprod_{t_2\in S} K_3^{(2)}(X_F^{\mathrm{loc}};\square) & \coprod_{t_1,t_2\in S} K_2^{(1)}(F) \\
\vdots & \vdots & \vdots
\end{array}$$

converging to $K_*^{(3)}(X_F^{'2,\mathrm{loc}};\square^2)$. The terms contributing to $K_3^{(3)}(X_F^{'2,\mathrm{loc}};\square^2)$ are $K_3^{(3)}(X_F^{2,\mathrm{loc}};\square^2)$, $\coprod_{t_1\in S} K_3^{(2)}(X_F^{\mathrm{loc}};\square) \amalg \coprod_{t_2\in S} K_3^{(2)}(X_F^{\mathrm{loc}};\square)$ and $\coprod_{t_1,t_2\in S} K_3^{(1)}(F)$. The last is evidently zero, and for the middle we consider the localisation sequence

$$\cdots \to \coprod_{t\in F^\flat} K_3^{(1)}(F) \to K_3^{(2)}(X_F;\square) \to K_3^{(2)}(X_F^{\mathrm{loc}};\square) \to \coprod_{t\in F^\flat} K_2^{(1)}(F) \to \cdots\ .$$

$K_3^{(1)}(F)$ and $K_2^{(1)}(F)$ are zero, so we get that $K_3^{(2)}(X_F^{\mathrm{loc}};\square) \cong K_3^{(2)}(X_F;\square) \cong K_4^{(2)}(F)$ which we are assuming is zero. This shows that we have an injection $K_3^{(3)}(X_F^{'2,\mathrm{loc}};\square^2) \to K_3^{(3)}(X_F^{2,\mathrm{loc}};\square^2)$.

Remark 2.48. Note that, because we can localise $\mathcal{O}$ to F, we have a natural map of the spectral sequence in (2.43) to the one in (2.18), which, at the level of the complexes (2.19) and (2.44), simply forgets the terms over κ, includes a coproduct over $\mathcal{O}^\flat$ into the corresponding coproduct over $F^\flat$, and uses the maps

$K_2^{(2)}(X_{\mathcal{O}}^{\mathrm{loc}};\square) \to K_2^{(2)}(X_F^{\mathrm{loc}};\square)$ and $K_3^{(3)}(X_{\mathcal{O}}^{2,\mathrm{loc}};\square^2) \to K_3^{(3)}(X_F^{\prime 2,\mathrm{loc}};\square^2)$. By Remark 2.47, the first one is always injective, and the second is injective if $K_5^{(2)}(\kappa)$ and $K_4^{(2)}(F)$ are zero.

Let us try to create a jewel in the crown of the scary notation in (2.44). Define $\mathrm{Symb}_n(\mathcal{O}) \subseteq K_n^{(n)}(X_{\mathcal{O}}^{n-1,\mathrm{loc}};\square^{n-1})$ for $n = 1$, 2 and 3 by setting

$$\mathrm{Symb}_1(\mathcal{O}) = \mathcal{O}_{\mathbb{Q}}^* \,,$$

$$\mathrm{Symb}_2(\mathcal{O}) = <[u]_2 \text{ with } u \text{ in } \mathcal{O}^\flat>_{\mathbb{Q}} + (1+I)_{\mathcal{O}}^* \cup \mathrm{Symb}_1(\mathcal{O}) \,,$$

as before, and

$$\mathrm{Symb}_3(\mathcal{O}) = <[u]_3 \text{ with } u \text{ in } \mathcal{O}^\flat>_{\mathbb{Q}} + (1+I)_{\mathcal{O}}^* \,\tilde{\cup}\, \mathrm{Symb}_2(\mathcal{O}) \,.$$

Again, by $\tilde{\cup}$ we denote that we use both products, coming from the two ways of projecting $X_{\mathcal{O}}^2$ to $X_{\mathcal{O}}$.

Note that for $n = 1$, $\mathrm{Symb}_1(\mathcal{O}) = \mathcal{O}_{\mathbb{Q}}^* \subseteq \mathrm{Symb}_1(F) = F_{\mathbb{Q}}^*$, and that for $n = 2$, we can view $\mathrm{Symb}_2(\mathcal{O}) \subseteq \mathrm{Symb}_2(F)$ inside $K_2^{(2)}(X_F^{\mathrm{loc}};\square)$ by Remark 2.47, as $K_2^{(2)}(X_{\mathcal{O}}^{\mathrm{loc}};\square) \subseteq K_2^{(2)}(X_F^{\mathrm{loc}};\square)$.

Because $\mathrm{d}[u]_2 = (1-u)^{-1}_{|t=u}$, and $\mathrm{d}[u]_3 = -[u]_{2|t_1=u} + [u]_{2|t_2=u}$ (where both terms lie in a copy of $K_2^{(2)}(X_{\mathcal{O}}^{\mathrm{loc}};\square)$ inside $K_2^{(2)}(X_F^{\mathrm{loc}})$, again by Remark 2.47), it follows that

(2.49)
$$\mathit{Symb}_{(3)}(\mathcal{O}) : \mathrm{Symb}_3(\mathcal{O}) \to \coprod_{t_1 \in \mathcal{O}^\flat} \mathrm{Symb}_2(\mathcal{O}) \coprod \coprod_{t_2 \in \mathcal{O}^\flat} \mathrm{Symb}_2(\mathcal{O}) \to \coprod_{t_1, t_2 \in \mathcal{O}^\flat} \mathcal{O}_{\mathbb{Q}}^*$$

is a subcomplex (in degrees 1, 2 and 3) of (2.44). Note that we used here that elements in $\mathcal{O}^\flat$ never give rise to a pole or zero over κ, so the map to $\coprod K_0^{(0)}(\kappa)$ is zero. Also, we used that an element $[u]_2$ with u in $\mathcal{O}^\flat$ under the localisation (of its construction),

$$K_2^{(2)}(X_{\mathcal{O}} \setminus \{t = u\};\square) \to K_1^{(1)}(\mathcal{O}) \to \dots$$

maps to $(1-u)^{-1}$, so under the boundary in (2.43) it never hits the $K_1^{(1)}(X_\kappa^{\mathrm{loc}};\square)$ component. Similarly, the elements in $(1+I)_{\mathcal{O}}^* \cup \mathcal{O}_{\mathbb{Q}}^*$ never hit the $K_1^{(1)}(X_\kappa^{\mathrm{loc}};\square)$.

Again, one shows that the subcomplex of (2.49) given by

$$(1+I)_{\mathcal{O}}^* \,\tilde{\cup}\, \mathrm{Symb}_2(\mathcal{O}) \to \coprod_{t_1} (1+I)_{\mathcal{O}}^* \cup \mathcal{O}_{\mathbb{Q}}^* \coprod \coprod_{t_2} (1+I)_{\mathcal{O}}^* \cup \mathcal{O}_{\mathbb{Q}}^* + \mathrm{d}(\dots) \to \mathrm{d}(\dots)$$

of (2.49) is acyclic, see [dJ95, Lemma 3.7 and Remark 3.10].

Taking the quotient complex, and the alternating part for the action of S_2 under swapping the coordinates, we finally get a complex

$$M_3(\mathcal{O}) \to M_2(\mathcal{O}) \to \mathcal{O}_{\mathbb{Q}}^* \otimes \bigwedge^2 \mathcal{O}_{\mathbb{Q}}^* \,. \tag{2.50}$$

Here

$$M_3(\mathcal{O}) = \mathrm{Symb}_3(\mathcal{O}) / \left((1+I)_{\mathcal{O}}^* \,\tilde{\cup}\, \mathrm{Symb}_2(\mathcal{O})\right)^{\mathrm{alt}}$$

and, as before,

$$M_2(\mathcal{O}) = \mathrm{Symb}_2(\mathcal{O}) / (1+I)_{\mathcal{O}}^* \cup \mathcal{O}_{\mathbb{Q}}^* \,.$$

Note that $M_n(\mathcal{O})$ is a $\mathbb{Q}$-vector space on symbols $[u]_n$ for u in $\mathcal{O}^\flat$, modulo nonexplicit relations depending on n. The maps in the complex are given by

$$\mathrm{d}[u]_3 = [u]_2 \otimes u$$

and

$$\mathrm{d}[u]_2 \otimes v = (1-u) \otimes (u \wedge v).$$

In particular, the condition for an element $\sum_i [u_i] \otimes v_i$ in $M_2(\mathcal{O}) \otimes \mathcal{O}^*_{\mathbb{Q}}$ to lie in $H^2(\mathcal{M}_{(3)}(\mathcal{O}))$ is that

$$\sum_i (1-u_i) \otimes (u_i \wedge v_i) = 0 \tag{2.51}$$

in $\mathcal{O}^*_{\mathbb{Q}} \otimes \bigwedge^2 \mathcal{O}^*_{\mathbb{Q}}$.

Again S_2 acts on the various complexes by swapping the coordinates, and we get maps

$$\mathcal{M}_{(3)}(\mathcal{O}) \leftarrow \mathit{Symb}_{(3)}(\mathcal{O})^{\mathrm{alt}} \to \mathit{RC}_{(3)}(\mathcal{O})^{\mathrm{alt}} \to \mathit{RC}_{(3)}(\mathcal{O})$$

with the left map a quasi isomorphism. Combining this with (2.45) gives us a map

$$H^p(\mathcal{M}_{(3)}(\mathcal{O})) \to K^{(3)}_{6-p}(\mathcal{O})$$

for $p = 2$ and 3, where the map for $p = 3$ is a surjection if $K_3^{(2)}(\kappa) = 0$ by Remark 2.6 and the localisation sequence

$$\cdots \to K_3^{(2)}(\kappa) \to K_3^{(3)}(\mathcal{O}) \to K_3^{(3)}(F) \to K_2^{(2)}(\kappa) \to \dots .$$

Remark 2.52. Notice that by construction (i.e., by compatibility of everything we did with the localisation of $\mathcal{O}$ to F), these maps for $p = 2$ or 3 fit into a commutative diagram

$$\begin{tikzcd}
H^p(\mathcal{M}_{(3)}(\mathcal{O})) \arrow[r] \arrow[d] & K^{(3)}_{6-p}(\mathcal{O}) \arrow[d] \\
H^p(\mathcal{M}_{(3)}(F)) \arrow[r] & K^{(3)}_{6-p}(F) .
\end{tikzcd} \tag{2.53}$$

We also note that it was proved in Remark 2.41 that the map $M_2(\mathcal{O}) \to M_2(F)$ is injective. Because we clearly have that $\mathcal{O}^*_{\mathbb{Q}} \to F^*_{\mathbb{Q}}$ is an injection, this means that, in degrees 2 and 3, $\mathcal{M}_{(3)}(\mathcal{O})$ injects into $\mathcal{M}_{(3)}(F)$.

2.5.3. *Construction of the complex $\mathcal{M}_{(3)}(\mathcal{C})$.* In this subsection we imitate the definition of the complex $\mathcal{M}_{(3)}(C)$ in Section 2.4.3, but using the complex $\mathcal{M}_{(3)}(\mathcal{O})$ rather than $\mathcal{M}_{(3)}(F)$ in the top row.

We therefore assume that we have a number field $k \subset K$, and a smooth, projective curve $\mathcal{C}$ over $k \bigcap \mathcal{O}$, and that the generic fiber $\mathcal{C}_k$ is geometrically irreducible. We put $F = k(C)$, and $\mathcal{O}$ the discrete valuation ring in F corresponding to the generic point of the special fibre of $\mathcal{C}$. We have a commutative diagram as follows.

$$\begin{tikzcd}
M_3(\mathcal{O}) \arrow[r, "\mathrm{d}"] \arrow[d] & M_2(\mathcal{O}) \otimes_{\mathbb{Q}} \mathcal{O}^*_{\mathbb{Q}} \arrow[r, "\mathrm{d}"] \arrow[d, "\partial_1"'] & \mathcal{O}^*_{\mathbb{Q}} \otimes \bigwedge^2 \mathcal{O}^*_{\mathbb{Q}} \arrow[d, "\partial_2"'] \\
0 \arrow[r] & \coprod_x \widetilde{M}_2(k(x)) \arrow[r, "\mathrm{d}"] & \coprod_x \bigwedge^2 k(x)^*_{\mathbb{Q}} .
\end{tikzcd}$$

The d's in the top row are as in $\mathcal{M}_{(3)}(\mathcal{O})$. The vertical maps, and the map in the bottom row, are given by the same formulae as before (see (2.4.3)), via the natural map $\mathcal{M}_{(3)}(\mathcal{O}) \to \mathcal{M}_{(3)}(F)$ corresponding to the localisation from $\mathcal{O}$ to F.

We let $\mathcal{M}_{(3)}(\mathcal{C})$ be the cohomological complex in degrees 1 through 4, given by the total complex associated to the double complex in the commutative diagram above. Note that therefore in particular, the condition for an element $\sum_i [u_i]_2 \otimes v_i$ in $M_2(\mathcal{O}) \otimes \mathcal{O}^*_{\mathbb{Q}}$ to define an element in $H^2(\mathcal{M}_{(3)}(\mathcal{C}))$ is

$$\sum_i (1-u_i) \otimes (u_i \wedge v_i) = 0$$

in $\mathcal{O}^*_{\mathbb{Q}} \otimes \bigwedge^2 \mathcal{O}^*_{\mathbb{Q}}$, and, for every closed point x in C,

$$\sum_i \mathrm{ord}_x(v_i)[u_i(x)]_2 = 0 \tag{2.54}$$

in $\widetilde{M}_2(k(x))$, with the convention that $[0]_2 = [1]_2 = [\infty]_2 = 0$.

The map to K-theory is similar to the map for $\mathcal{M}_{(3)}(F)$, but now we get

$$H^2(\mathcal{M}_{(3)}(\mathcal{C})) \to H^2(\mathcal{M}_{(3)}(\mathcal{O})) \to K_4^{(3)}(\mathcal{O}) \ ,$$

where the first arrow corresponds to forgetting the bottom row in $\mathcal{M}_{(3)}(\mathcal{C})$. Because this is compatible with the localisation to F (i.e, with the map $\mathcal{M}_{(3)}(\mathcal{O}) \to \mathcal{M}_{(3)}(F)$), we do know that we must end up in $K_4^{(3)}(C) \oplus K_3^{(2)}(k) \cup \mathcal{O}^*_{\mathbb{Q}}$, which equals $K_4^{(3)}(\mathcal{C}) \oplus K_3^{(2)}(k) \cup F^*_{\mathbb{Q}}$ by Remarks 2.3 and 2.5. In particular, we do get maps

$$\begin{tikzcd} H^2(\mathcal{M}_{(3)}(\mathcal{C})) \arrow[r] \arrow[d] & K_4^{(3)}(\mathcal{C}) \oplus K_3^{(2)}(k) \cup \mathcal{O}^*_{\mathbb{Q}} \arrow[d, equal] \\ H^2(\mathcal{M}_{(3)}(F)) \arrow[r] & K_4^{(3)}(C) \oplus K_3^{(2)}(k) \cup F^*_{\mathbb{Q}} \ . \end{tikzcd} \tag{2.55}$$

Remark 2.56. Just as in Remark 2.25, we can consider the projection

$$K_4^{(3)}(\mathcal{C}) \oplus K_3^{(2)}(k) \cup \mathcal{O}^*_{\mathbb{Q}} \to K_4^{(3)}(\mathcal{C})$$

to get a map

$$H^2(\mathcal{M}_{(3)}(\mathcal{C})) \to K_4^{(3)}(\mathcal{C})$$

as the composition

$$H^2(\mathcal{M}_{(3)}(\mathcal{C})) \to K_4^{(3)}(\mathcal{C}) \oplus K_3^{(2)}(k) \cup \mathcal{O}^*_{\mathbb{Q}} \to K_4^{(3)}(\mathcal{C}) \ .$$

The advantage of using the complex $\mathcal{M}_{(3)}(\mathcal{C})$ (just like the advantage of using any $\mathcal{O}$-complex over the corresponding F-complex) is that the syntomic regulator gets the input it needs on the special fibre of $\mathcal{C}$.

2.5.4. *Construction of the complex $\mathcal{C}^\bullet(\mathcal{O})$.* Now we try to imitate the construction of $\mathcal{C}^\bullet(F)$ for $\mathcal{O}$. Because we are now dealing with the two dimensional scheme $X_{\mathcal{O}}$, the localisation sequence (2.28) becomes a spectral sequence, cf. (2.35):

$$\begin{array}{ccc}
\vdots & \vdots & \vdots \\
K_2^{(3)}(X_{\mathcal{O}}^{\mathrm{loc}};\square) & \coprod_{t\in\mathcal{O}^\flat} K_1^{(2)}(F) & \coprod_{t\in\kappa^\flat} K_0^{(1)}(\kappa) \\
K_3^{(3)}(X_{\mathcal{O}}^{\mathrm{loc}};\square) & \coprod_{t\in\mathcal{O}^\flat} K_2^{(2)}(F) & \coprod_{t\in\kappa^\flat} K_1^{(1)}(\kappa) \\
K_4^{(3)}(X_{\mathcal{O}}^{\mathrm{loc}};\square) & \coprod_{t\in\mathcal{O}^\flat} K_3^{(2)}(F) & \coprod_{t\in\kappa^\flat} K_2^{(1)}(\kappa) \\
\vdots & \vdots & \vdots
\end{array} \tag{2.57}$$

converging to $K_*^{(3)}(X_{\mathcal{O}};\square) \cong K_{*+1}^{(3)}(\mathcal{O})$. Let us notice that $K_2^{(1)}(\kappa)$ and $K_3^{(1)}(\kappa)$ are zero, and that the exact localisation sequence

$$\cdots \to K_3^{(1)}(\kappa) \to K_3^{(2)}(\mathcal{O}) \to K_3^{(2)}(F) \to K_2^{(1)}(\kappa) \to K_2^{(2)}(\mathcal{O}) \to K_2^{(2)}(F) \to \ldots$$

tells us that $K_2^{(2)}(\mathcal{O}) \subseteq K_2^{(2)}(F)$ and $K_3^{(2)}(\mathcal{O}) \cong K_3^{(2)}(F)$. Therefore we get an exact sequence

$$0 \to \frac{K_4^{(3)}(\mathcal{O})}{K_3^{(2)}(\mathcal{O}) \cup \mathcal{O}_{\mathbb{Q}}^*} \to K_3^{(3)}(X_{\mathcal{O}}^{\mathrm{loc}};\square) \to \ker\left(\coprod_{t\in\mathcal{O}^\flat} K_2^{(2)}(F) \to \coprod_{t\in\kappa^\flat} K_1^{(1)}(\kappa)\right) .$$

In the middle row of the spectral sequence (2.57) above, let $B \subseteq K_3^{(3)}(X_{\mathcal{O}}^{\mathrm{loc}};\square)$ be the inverse image of $\coprod K_2^{(2)}(\mathcal{O})$ (with the coproduct over all of $\mathcal{O}^\flat$). Then we have a cohomological complex in degrees 1 and 2,

$$AC_{(3)}(\mathcal{O}) : B \to \coprod_{t\in\mathcal{O}^\flat} K_2^{(2)}(\mathcal{O}) \ , \tag{2.58}$$

and an isomorphism

$$H^1(AC_{(3)}(\mathcal{O})) \cong \frac{K_4^{(3)}(\mathcal{O})}{K_3^{(2)}(\mathcal{O}) \cup \mathcal{O}_{\mathbb{Q}}^*}$$

and a map

$$H^2(AC_{(3)}(\mathcal{O})) \to K_3^{(3)}(\mathcal{O}) \ .$$

Remark 2.59. If $K_3^{(2)}(\kappa) = 0$, or more generally, the map $K_4^{(3)}(F) \to K_3^{(2)}(\kappa)$ is surjective, then from the localization sequence

$$\cdots \to K_4^{(3)}(F) \to K_3^{(2)}(\kappa) \to K_3^{(3)}(\mathcal{O}) \to K_3^{(3)}(F) \to K_2^{(2)}(\kappa) \to \ldots$$

and Remark 2.6 we see that the map $\coprod_{t\in\mathcal{O}^\flat} K_2^{(2)}(\mathcal{O}) \to K_3^{(3)}(\mathcal{O})$, and hence the map $H^2(AC_{(3)}(\mathcal{O})) \to K_3^{(3)}(\mathcal{O})$, are surjective.

Remark 2.60. Because $K_1^{(2)}(F)$ and $K_2^{(1)}(\kappa)$ are zero, and $K_2^{(2)}(F) \to K_1^{(1)}(\kappa)$ is surjective, from (2.57) we get that there is an exact sequence

$$\mathrm{Ker}\left(\coprod_{t\in\mathcal{O}^\flat} K_2^{(2)}(F) \to \coprod_{t\in\kappa^\flat} K_1^{(1)}(\kappa)\right) \to K_2^{(3)}(X_{\mathcal{O}};\square) \to K_2^{(3)}(X_{\mathcal{O}}^{\mathrm{loc}};\square) \to 0 \ .$$

If $K_3^{(2)}(\kappa)$ is zero, or, more generally, the map $K_4^{(3)}(F) \to K_3^{(2)}(\kappa)$ surjective, then Remark 2.6 tells us that $\coprod_{t \in \mathcal{O}^\flat} K_2^{(2)}(\mathcal{O})$ surjects onto $K_2^{(3)}(X_{\mathcal{O}}; \square) \cong K_3^{(3)}(\mathcal{O})$, and we can conclude that $K_2^{(3)}(X_{\mathcal{O},\mathrm{loc}}; \square)$ is zero.

Now we consider the acyclic subcomplex

$$(1+I)^*_{\mathcal{O}} \cup K_2^{(2)}(\mathcal{O}) \to \mathrm{d}(\dots)$$

of (2.58), and quotient out to find a complex

$$\mathcal{C}^\bullet(\mathcal{O}) : \mathcal{C}^1(\mathcal{O}) \to \mathcal{C}^2(\mathcal{O}) \ ,$$

where

$$\mathcal{C}^1(\mathcal{O}) = \frac{B}{(1+I)^*_{\mathcal{O}} \cup K_2^{(2)}(\mathcal{O})} \tag{2.61}$$

and

$$\mathcal{C}^2(\mathcal{O}) = K_2^{(2)}(\mathcal{O}) \otimes \mathcal{O}^*_{\mathbb{Q}} \ .$$

We still have an isomorphism

$$H^1(\mathcal{C}^\bullet(\mathcal{O})) \cong K_4^{(3)}(\mathcal{O}) / K_3^{(2)}(\mathcal{O}) \cup \mathcal{O}^*_{\mathbb{Q}} \tag{2.62}$$

and a map

$$H^2(\mathcal{C}^\bullet(\mathcal{O})) \to K_3^{(3)}(\mathcal{O}) \ , \tag{2.63}$$

which by Remark 2.6 is a surjection if $K_4^{(3)}(F) \to K_3^{(2)}(\kappa)$ is surjective, e.g., if $K_3^{(2)}(\kappa) = 0$.

Observe that if g is in $\mathcal{O}^\flat$, and f is in $\mathcal{O}^*_{\mathbb{Q}}$, then $[g]_2 \cup (f)$ is in $\mathcal{C}^1(\mathcal{O})$, and has boundary $\{(1-g)^{-1}, f\} \otimes g = -\{(1-g), f\} \otimes g$ in $\mathcal{C}^2(\mathcal{O})$. The condition for $\sum_i [g_i]_2 \cup (f_i)$ to be in $H^1(\mathcal{C}^\bullet(\mathcal{O}))$ is therefore that

$$\sum_i \{1 - g_i, f_i\} \otimes g_i = 0$$

in $\mathcal{C}^2(\mathcal{O}) = K_2^{(2)}(\mathcal{O}) \otimes \mathcal{O}^*_{\mathbb{Q}}$.

Note that because the construction of the spectral sequence in (2.57) is compatible with localising the base from $\mathcal{O}$ to F and enlarging the coproduct from being over $\mathcal{O}^\flat$ to $F^\flat$ (in which case it becomes the localization sequence in (2.28)), and that $(1+I)^*_{\mathcal{O}}$ is contained in $(1+I)^*$, and $K_2^{(2)}(\mathcal{O}) \subseteq K_2^{(2)}(F)$, we have an obvious map of complexes,

$$\mathcal{C}^\bullet(\mathcal{O}) \to \mathcal{C}^\bullet(F) \ ,$$

which fits into the commutative diagram

$$\begin{tikzcd} H^1(\mathcal{C}^\bullet(\mathcal{O})) \arrow[r] \arrow[d] & K_4^{(3)}(\mathcal{O}) / K_3^{(2)}(\mathcal{O}) \cup \mathcal{O}^*_{\mathbb{Q}} \arrow[d] \\ H^1(\mathcal{C}^\bullet(F)) \arrow[r] & K_4^{(3)}(F) / K_3^{(2)}(F) \cup F^*_{\mathbb{Q}} \ , \end{tikzcd} \tag{2.64}$$

and similarly for H^2.

Finally, we also have a commutative diagram

$$\begin{tikzcd}
M_3(\mathcal{O}) \arrow[r] \arrow[d] & M_2(\mathcal{O}) \otimes \mathcal{O}_{\mathbb{Q}}^* \arrow[r] \arrow[d] & \mathcal{O}_{\mathbb{Q}}^* \otimes \bigwedge^2 \mathcal{O}_{\mathbb{Q}}^* \arrow[d] \\
0 \arrow[r] & \mathcal{C}^1(\mathcal{O}) \arrow[r] & \mathcal{C}^2(\mathcal{O})
\end{tikzcd} \tag{2.65}$$

as follows. We map $[u]_2 \otimes v$ to $[u]_2 \cup v$, and $u \otimes v \wedge w$ to $\{u, v\} \otimes w - \{u, w\} \otimes v$. This gives rise to a commutative diagram

$$\begin{tikzcd}
H^2(\mathcal{M}_{(3)}(\mathcal{O})) \arrow[r] \arrow[d] & K_4^{(3)}(\mathcal{O}) \arrow[d] \\
H^1(\mathcal{C}^\bullet(\mathcal{O})) \arrow[r] & K_4^{(3)}(\mathcal{O})/K_3^{(2)}(\mathcal{O}) \cup \mathcal{O}_{\mathbb{Q}}^* \, ,
\end{tikzcd} \tag{2.66}$$

cf. (2.33). Obviously, those maps are compatible with the corresponding ones in (2.32) under the localisation from $\mathcal{O}$ to F.

2.6. **A diagram.** For the convenience of the reader, we give a commutative diagram summarizing the cohomology groups of the complexes introduced, and the maps. We have kept the lay-out of the diagram in the same spirit as the relativity in the plane. Note that the outer square is only relevant if we start with a smooth proper geometrically irreducible curve over a number field $k \subset K$, with good reduction at $R \bigcap k$, and $F = k(C)$, etc.

The top half of this diagram is the top of the one in (1.15).

The vertical maps in this diagram correspond to the maps from constructions over $\mathcal{O}$ to the corresponding constructions over F. The horizontal maps are the maps on cohomology of complexes contructed in the previous subsections, and the diagonal maps correspond to the maps in (2.32), (2.53), (2.55) and (2.64).

$$\begin{tikzcd}
H^2(\mathcal{M}_{(3)}(\mathfrak{C})) \arrow[rrrrr] \arrow[dddddd] \arrow[dr] & & & & & K_4^{(3)}(\mathfrak{C}) \oplus K_3^{(2)}(k) \cup \mathcal{O}^*_{\mathbb{Q}} \arrow[dl] \arrow[dddddd] \\
& H^2(\mathcal{M}_{(3)}(\mathcal{O})) \arrow[rrr] \arrow[dddd] \arrow[dr] & & & K_4^{(3)}(\mathcal{O}) \arrow[dl] \arrow[dddd] & \\
& & H^1(\mathcal{C}^\bullet(\mathcal{O})) \arrow[r] \arrow[dd] & \frac{K_4^{(3)}(\mathcal{O})}{K_3^{(2)}(\mathcal{O})\cup\mathcal{O}^*_{\mathbb{Q}}} \arrow[dd] & & \\
& & {} \arrow[r, phantom, "(2.67)"] & {} & & \\
& & H^1(\mathcal{C}^\bullet(F)) \arrow[r] & \frac{K_4^{(3)}(F)}{K_3^{(2)}(F)\cup F^*_{\mathbb{Q}}} & & \\
& H^2(\mathcal{M}_{(3)}(F)) \arrow[rrr] \arrow[ur] & & & K_4^{(3)}(F) \arrow[ul] & \\
H^2(\mathcal{M}_{(3)}(C)) \arrow[rrrrr] \arrow[ur] & & & & & K_4^{(3)}(C) \oplus K_3^{(2)}(k) \cup F^*_{\mathbb{Q}} \arrow[ul]
\end{tikzcd}$$

Note that by Remarks 2.3 and 2.5 the right vertical map is an isomorphism.

3. Coleman integration

In this short section we briefly discuss Coleman's integration theory in the one dimensional case only. The interested reader may refer to [Bes00a] for more details.

Coleman theory is done on wide open spaces in the sense of Coleman [CdS88]. In general these are the overconvergent spaces described in section 4. In the one-dimensional case these can be described concretely in the following way. Let X be a curve over $\mathbb{C}_p$ with good reduction (there is a minor assumption that it is an extension from a discretely valued field, which will always be satisfied in our case). The rigid analytic space $X(\mathbb{C}_p)$ is set theoretically decomposed as the union $X = \cup_x U_x$ where x vary over the points in the reduction of X and U_x is the residue disc (tube in the language of Berthelot) of points reducing to x. By the assumption of good reduction each residue disc is isomorphic to a disc $|z| < 1$. A wide open space U is obtained from X by fixing a finite set of points S in the reduction and throwing away the discs inside the residue discs U_x, $x \in S$, isomorphic to $|z| < r$ for arbitrarily large $r < 1$. U should be thought of as the inverse limit of the correspondeing spaces U_r.

Coleman theory associates to U the $\mathbb{C}_p$-algebra $A_{\mathrm{col}}(U)$ and the $A_{\mathrm{col}}(U)$-modules $\Omega^i_{\mathrm{col}}(U)$ with differentials forming a complex. The key property is that this complex is exact at the one and zero forms, i.e, there is an exact sequence

$$0 \to \mathbb{C}_p \to A_{\mathrm{col}}(U) \to \Omega^1_{\mathrm{col}}(U) \to \Omega^2_{\mathrm{col}}(U) \ .$$

The space $\Omega^1_{\mathrm{col}}(U)$ contains the space $\Omega^1(U)$ of overconvergent forms on U, i.e., those forms which are rigid analytic on some U_r. Similarly, the space $A_{\mathrm{col}}(U)$ contains the space $A(U)$ of overconvergent functions. The differential extends the usual differential on the subspaces.

The whole picture extends to higher dimensions. We will only need the case where U is one-dimensional. In this case the space $\Omega^2_{\mathrm{col}}(U)$ is already 0.

Coleman functions may be interpreted as locally analytic functions on U. More precisely, again in the one-dimensional case, for $x \notin S$, the intersection of the residue disc U_x with U is U_x while for $x \in S$ it is an annulus e_x isomorphic to an annulus of the form $r < |z| < 1$. A Coleman function is analytic on each U_x and is a polynomail algebra $A(e_x)[\log(z)]$ where z is a local parameter on U_x.

We define the space $A_{\mathrm{col},1}(U)$ to be the inverse image of $\Omega^1(U) \in \Omega^1_{\mathrm{col}}(U)$ under the differential d. The space of differentials $\Omega^1_{\mathrm{col},1}(U)$ is the product $A_{\mathrm{col},1}(U) \cdot \Omega^1(U)$.

If $\omega \in \Omega^1(U_r)$ and $y, z \in U_r$ the integral $\int_z^y \omega$ is clearly well defined as $f(y) - f(z)$ where $f \in A_{\mathrm{col}}(U_r)$ and $df = \omega$. It is a basic property of Coleman integration that if X, U, ω, z, y are all defined over the subfield K, then so is the integral $\int_z^y \omega$.

4. Regulators

In this Section we compute the regulator of elements in $H^2(\mathcal{M}_{(3)}(\mathcal{O}))$ and $H^1(\mathcal{C}^\bullet(\mathcal{O}))$ as elements of $K_4^{(3)}(\mathcal{O})$ and $K_4^{(3)}(\mathcal{O})/K_3^{(2)}(\mathcal{O}) \cup \mathcal{O}^*_{\mathbb{Q}}$ respectively under the maps as in (2.67). In case the boundary of such an element is trivial, we also explain how we wish to interpret the cup product of this regulator with the cohomology class of a form ω of the second kind on C, and what are the obstacles for doing so.

We first write down the relevant spaces and the (modified) syntomic complexes computing their cohomology. For the full story the reader should consult [Bes00a].

We begin with a smooth proper relative curve $\mathcal{C}/R$. Related to that is the space $X_{\mathcal{C}} := \mathbb{P}^1_{\mathcal{C}} \setminus \{t = 1\}$. The superscript loc will denote various localizations, either $\mathcal{C}^{\mathrm{loc}}$ of $\mathcal{C}$ or $X^{\mathrm{loc}}_{\mathcal{C}}$ of $X_{\mathcal{C}}$. If the localization is not trivial, and we may and do assume this, then all localized schemes are affine.

Our goal is to compute the syntomic regulator $K_4^{(3)}(\mathcal{C}) \to H^2_{\mathrm{syn}}(\mathcal{C}, 3)$. According to [Bes00a, Proposition 8.6.3] there is an isomorphisms, commuting with the regulator, $H^2_{\mathrm{syn}}(\mathcal{C}, 3) \xrightarrow{\sim} \tilde{H}^2_{\mathrm{ms}}(\mathcal{C}, 3)$, where $\tilde{H}_{\mathrm{ms}}$ is the Gros style modified rigid syntomic cohomology, in the sense of loc. cit. From now on we will therefore concentrate on modified syntomic cohomology. We will refer to it simply as syntomic cohomology.

Let us recall one of the possible models for modified syntomic cohomology for affine schemes. Let A be an affine R-scheme. We assume we have an open embedding $A \hookrightarrow \overline{A}$, where $\overline{A}$ is proper. From the embedding $A \hookrightarrow \overline{A}$ one obtains the overconvergent space $A^\dagger$. This space can be made sense out in Grosse-Klönne's theory of overconvergent spaces [GK00] as the space whose affine ring, $\mathcal{O}(A^\dagger)$, is the weak completion, in the sense of Monsky-Washnitzer, of $\mathcal{O}(A)$. However, here we will simply think of $A^\dagger$ formally as the inverse system of strict neighborhoods of the special fiber of A in that of $\overline{A}$

We further assume that we have an R-linear endomorphism $\phi : A^\dagger \to A^\dagger$ whose reduction is a power of Frobenius, say of degree $q = p^r$. We call ϕ a Frobenius endomorphism. Standard results ([Col85, Thm A-1] or [vdP86, Thm 2.4.4.ii]) imply one always has such ϕ.

With the above data, we have

$$\tilde{H}^n_{\mathrm{ms}}(A,j) = H^n(\mathrm{MF}(F^j\Omega^\bullet(A^\dagger) \xrightarrow{1-\phi^*/q^j} \Omega^\bullet(A^\dagger))) \ .$$

Here, the filtration is the stupid filtration on the space of differentials and MF denotes the mapping fiber (Cone shifted by -1). To be more precise, one really needs to take the limit of these cohomology groups with respect to powers of ϕ, in a way explained in [Bes00a], but it is also explained there that one can ignore this point.

The cohomomology groups $\tilde{H}_{\mathrm{ms}}$ are in fact functorial with respect to arbitrary maps of schemes. This functoriality is not at all obvious from the definition except in the case where the maps extend to the dagger spaces and commute with ϕ. Fortunately, this will always be the case for us. In this situation, one may also construct relative cohomology in the obvious way (the reader is advised to look at [BdJ03, Section 5] for constructions of complexes computing relative syntomic cohomology).

To end this general review we recall that the corresponding syntomic regulator is defined by the formula

$$f \in \mathcal{O}(A)^* \subset K_1(A) \mapsto (\mathrm{dlog}(f), \log(f_0)/q) \in \tilde{H}^1_{\mathrm{ms}}(A,1) \ , \tag{4.1}$$

where $f_0 = f^q/\phi^*(f)$ and has the property that $\log(f_0)$ is in $\mathcal{O}(A^\dagger)$. We also recall from[Bes00a, Definition 6.5] that the cup product $\tilde{H}^\bullet_{\mathrm{ms}}(A,i) \times \tilde{H}^\bullet_{\mathrm{ms}}(A,j) \to \tilde{H}^\bullet_{\mathrm{ms}}(A,i+j)$ is given by

$$\begin{aligned}(\omega_1,\epsilon_1) \cup (\omega_2,\epsilon_2) = \Bigg(&\omega_1 \wedge \omega_2, \\ &\epsilon_1 \wedge \left(\gamma + (1-\gamma)\frac{\phi^*}{q^j}\right)\omega_2 \\ &+(-1)^{\deg(\omega_1)}\left(\left((1-\gamma) + \gamma\frac{\phi^*}{q^i}\right)\omega_1\right) \wedge \epsilon_2\Bigg) \ .\end{aligned} \tag{4.2}$$

for some constant γ, which can be taken arbitrarily (producing homotopic products).

We now write these constructions for the affine schemes we are considering. To simplify notation we write U for $(\mathcal{C}^{\mathrm{loc}})^\dagger$, U' for $(X^{\mathrm{loc}}_{\mathcal{C}})^\dagger$, and X_U for $(X_{\mathcal{C}^{\mathrm{loc}}})^\dagger$. We may localize such that $U' \subset X_U$. We fix a Frobenius endomorphism $\phi : U \to U$. We can then take the Frobenius endomorphism for X_U to be the product of ϕ with the map $t \mapsto t^q$ and for U' the restriction of this endomorphism to U'. Since $t \mapsto t^q$ fixes 0 and ∞ we can use the embedding of U in U' at $t = 0$ and $t = \infty$. With this we have the following models for syntomic cohomology.

$$\tilde{H}^i_{\mathrm{ms}}(X^{\mathrm{loc}}_{\mathcal{C}}, i) = \frac{\{(\omega,\epsilon),\ \omega \in \Omega^1(U'),\ \epsilon \in \Omega^{i-1}(U'),\ d\omega = 0,\ d\epsilon = \left(1 - \frac{\phi^*}{q^i}\right)(\omega)\}}{\{(0,d\epsilon),\ \epsilon \in \Omega^{i-2}(U')\}} \tag{4.3}$$

For $i = 1, 2$. Now, for the relative one we can write, by throwing away terms which are forced to be 0,

(4.4)

$$\tilde{H}^2_{\mathrm{ms}}(X^{\mathrm{loc}}_{\mathcal{C}}, \square, 2) = \frac{\left\{ (\omega, \epsilon, \epsilon_\infty, \epsilon_0), \begin{array}{c} \omega \in \Omega^2(U'),\ \epsilon \in \Omega^1(U'),\ \epsilon_s \in \mathcal{O}(U), s = 0, \infty, \\ d\omega = 0,\ d\epsilon = \left(1 - \dfrac{\phi^*}{q^i}\right)(\omega),\ d\epsilon_s = \epsilon|_{\{t=s\}},\ s = 0, \infty \end{array} \right\}}{\left\{ \left(0, d\epsilon, \epsilon|_{\{t=\infty\}}, \epsilon|_{\{t=0\}}\right),\ \epsilon \in \mathcal{O}(U') \right\}}$$

The map between $\tilde{H}^2_{\mathrm{ms}}(X_{\mathcal{C}}, \square, 2)$ and $\tilde{H}^i_{\mathrm{ms}}(X_{\mathcal{C}}, i)$ remembers only ω and ϵ. Since U' is two dimensional and therefore does not support forms of degree 3, we also have

(4.5)

$$\tilde{H}^3_{\mathrm{ms}}(X^{\mathrm{loc}}_{\mathcal{C}}, \square, 3) = \frac{\{(\epsilon, \epsilon_\infty, \epsilon_0),\ \epsilon \in \Omega^2(U'),\ \epsilon_s \in \Omega^1(U), s = 0, \infty,\ d\epsilon = 0,\ d\epsilon_s = \epsilon|_{\{t=s\}},\ s = 0, \infty\}}{\left\{ \left(d\epsilon, \epsilon|_{\{t=\infty\}}, \epsilon|_{\{t=0\}}\right),\ \epsilon \in \Omega^1(U') \right\}}$$

If we replace U' by X_U we obtain a model for $\tilde{H}^3_{\mathrm{ms}}(X_{\mathcal{C}^{\mathrm{loc}}}, \square, 3)$

The last model is

$$\tilde{H}^2_{\mathrm{ms}}(\mathcal{C}^{\mathrm{loc}}, 3) = \frac{\{\epsilon \in \Omega^1(U),\ d\epsilon = 0\}}{\{d\epsilon\ , \epsilon \in \mathcal{O}(U)\}} \tag{4.6}$$

This is of course just the first de Rham cohomology of U. However, the "correct" isomorphism with this cohomology is not the obvious one but rather the one twisted by $1 - \phi^*/q^3$, i.e.,

$$[\eta] \mapsto [(1 - \phi^*/q^3)\eta] \tag{4.7}$$

(for an explanation of this see [Bes00a, Proposition 10.1.3]). Here, and in what follows, we denote the cohomology class of an element in square brackets.

We need a formula for the cup product $\tilde{H}^2_{\mathrm{ms}}(X^{\mathrm{loc}}_{\mathcal{C}}, \square, 2) \times \tilde{H}^1_{\mathrm{ms}}(X^{\mathrm{loc}}_{\mathcal{C}}, 1) \to \tilde{H}^3_{\mathrm{ms}}(X^{\mathrm{loc}}_{\mathcal{C}}, \square, 3)$ in terms of the models (4.4), (4.3) and (4.5) respectively. Using the formula for a cup product between a cone and a complex and (4.2) with $\gamma = 0$ we find the following formula:

$$(\omega, \epsilon, \epsilon_\infty, \epsilon_0) \cup (\eta, h) = (h\omega + \epsilon \wedge \frac{\phi^*}{q}\eta, \epsilon_\infty \eta, \epsilon_0 \eta)\ . \tag{4.8}$$

Suppose now that f and g are in $\mathcal{O}^*(\mathcal{C}^{\mathrm{loc}})$ (see Subsection 2.5.4). To compute the regulator of $[g]_2 \cup (f)$ we start with $[g]_2$ in $K_2^{(2)}(X^{\mathrm{loc}}_{\mathcal{C}}, \square)$. It maps in $K_2^{(2)}(X^{\mathrm{loc}}_{\mathcal{C}})$ to $-\frac{t-g}{t-1} \cup (1-g)$, by pulling back along g the corresponding result for the universal elements [BdJ03, Proposition 6.7].

Lemma 4.9. *We have in $\tilde{H}^2_{\mathrm{ms}}(X^{\mathrm{loc}}_{\mathcal{C}}, 2)$ that $-ch(\frac{t-g}{t-1} \cup (1-g)) = (\omega_g, \epsilon_g)$, in the model* (4.3) *with*

$$\omega_g = -\operatorname{dlog}(\frac{t-g}{t-1}) \wedge \operatorname{dlog}(1-g)$$

and

$$\epsilon_g = \frac{1}{q}\log(1-g)_0 \operatorname{dlog}\left(\frac{t-g}{t-1}\right) - \frac{1}{q^2}\log\left(\frac{t-g}{t-1}\right)_0 \operatorname{dlog}\phi^*(1-g)$$

Proof. This follows from the formula (4.1) for the regulators of functions, the compatibility of ch with cup products and the cup product formula (4.2). □

Proposition 4.10. *We have in* $\tilde{H}^2_{\mathrm{ms}}(X^{\mathrm{loc}}_{\mathcal{C}}, \square, 2)$*, using the model* (4.4),

$$ch([g]_2) = [\omega_g, \epsilon_g, 0, \Theta(g)] \ ,$$

where

$$d\Theta(g) = \epsilon_g|_{t=0} = \frac{1}{q}\log(1-g)_0 \,\mathrm{dlog}\, g - \frac{1}{q^2}\log g_0 \,\mathrm{dlog}\,\phi^*(1-g) \ . \tag{4.11}$$

Proof. We are looking for a closed four-tuple, whose first two coordinates represent the cohomology class of (ω_g, ϵ_g). It is easy to see that we may assume that the first two coordinates are indeed (ω_g, ϵ_g). Then the closeness condition implies that the differentials of the next two coordinates give the restriction to $t = \infty$ and $t = 0$ respectively of ϵ_g. These are respectively 0 and $\epsilon_g|_{t=0}$, so the result is clear. □

Remark 4.12. 1. One can show that there exist a function Θ on $\mathbb{P}^1$ such that $\Theta(g)$ is indeed the composition of Θ and g, but we will not need to use this.
2. The determination of the regulator at this stage is incomplete, since we have only determined $\Theta(g)$ up to a constant. It will turn out that for the regulator computation this is irrelevant. For the computation of the boundary this becomes much trickier. We in fact failed to determine the boundary of the regulator directly. When we need this towards the end of Section 9 for the proof of Theorem 1.5, we will use a trick to overcome this difficulty, which in particular forces us to assume working over a number field at that stage.

Proposition 4.13. *The regulator of* $[g]_2 \cup (f)$ *in* $\tilde{H}^3_{\mathrm{ms}}(X^{\mathrm{loc}}_{\mathcal{C}}, \square, 3)$ *is represented by the following element in the model* (4.5)

$$\epsilon(g,f) := \left(\frac{1}{q}\log f_0 \omega_g + \frac{1}{q}\epsilon_g \wedge \phi^* \,\mathrm{dlog}\, f, 0, \frac{1}{q}\Theta(g)\phi^* \,\mathrm{dlog}\, f\right)$$

Proof. This follows again from the compatibility of the regulator with cup products and from the formulas for the cup product in relative syntomic cohomology (4.8). □

Suppose now that $\alpha = \sum_i [g_i]_2 \cup (f_i)$ belongs to

$$H^1(\mathcal{C}^\bullet(\mathcal{O})) \cong K_4^{(3)}(\mathcal{O})/K_3^{(2)}(\mathcal{O}) \cup \mathcal{O}^*_{\mathbb{Q}} \ ,$$

see (2.62). Note that α is only determined up to an element in $(1+I)^*_{\mathcal{O}} \cup \mathcal{O}^*_{\mathbb{Q}}$, see (2.58) and (2.61). A term in the latter space consists explicitly of elements of the form

$$\delta = \sum_j \delta_{1,j} \cup \delta_{2,j} \tag{4.14}$$

with $\delta_{1,j} \in K_1^{(1)}(X^{\mathrm{loc}}_{\mathcal{C}}, \square)$ and $\delta_{2,j} \in K_2^{(2)}(\mathcal{C}^{\mathrm{loc}})$, for all possible localizations. Therefore, for an appropriately chosen $\mathcal{C}^{\mathrm{loc}}$, there exists $\beta \in K_3^{(3)}(X_{\mathcal{C}^{\mathrm{loc}}}, \square)$ whose restriction to $(X^{\mathrm{loc}}_{\mathcal{C}}, \square)$ is $\alpha + \delta$, where δ is as in (4.14). If we write $ch(\beta) = [\epsilon, \epsilon_\infty, \epsilon_0]$, with the ϵ's living on X_U, then we have $[\epsilon, \epsilon_\infty, \epsilon_0]|_{(X^{\mathrm{loc}}_{\mathcal{C}}, \square)} = \sum [\epsilon(g_i, f_i)] + ch(\delta)$. Writing this explicitly this means that

$$(\epsilon, \epsilon_\infty, \epsilon_0)|_{(U', \square)} = \sum \epsilon(g_i, f_i) + ch(\delta) + (d\lambda, \lambda|_{\{t=\infty\}}, \lambda|_{\{t=0\}})$$

for some $\lambda \in \Omega^1(U')$ and where now $ch(\delta)$ means any form representing this class.

The isomorphism $\tilde{H}^3_{\mathrm{ms}}(X_{\mathcal{C}^{\mathrm{loc}}}, \square, 3) \cong \tilde{H}^2_{\mathrm{ms}}(\mathcal{C}^{\mathrm{loc}}, 3)$ is obtained by integration from 0 to ∞. More precisely it is given by $[\epsilon, \epsilon_\infty, \epsilon_0] \mapsto [(\int_0^\infty \epsilon) - (\epsilon_\infty - \epsilon_0)]$, where the integration is only with respect to the variable t. Note that we are integrating forms on X_U.

For forms on U' we may do Coleman integration instead (Section 3). This technique was introduced in [BdJ03, Section 5]. Note that we only discussed Coleman integration over $\mathbb{C}_p$. The extension of scalars of U and the fibers of $U' \to U$, to $\mathbb{C}_p$ are wide open space in the sense of Coleman so one can do Coleman integration on them. By abuse of notation we will continue to denote this extension of scalars by the same letters. Coleman integration will be the same as ordinary integration if the forms extend to X_U. The theory of Coleman integration is not sufficiently developed yet to tell us that what we do makes sense in general, so we must be careful to check that it makes sense for the particular forms we are working with.

Now we check what happens to the term $\epsilon(g, f)$ under this integration. The integral of the first term is

$$\begin{aligned}
&\int_0^\infty \frac{1}{q} \log f_0 \omega_g + \frac{1}{q} \epsilon_g \wedge \phi^* \operatorname{dlog} f = \frac{1}{q} \log f_0 \int_0^\infty \omega_g + \frac{1}{q} \left(\int_0^\infty \epsilon_g \right) \wedge \phi^* \operatorname{dlog} f \\
=&\frac{1}{q} \log f_0 \log g \operatorname{dlog}(1-g) - \frac{1}{q^2} \log(1-g)_0 \log g \phi^* \operatorname{dlog} f \ .
\end{aligned}$$

The last equality follows because $\int_0^\infty \operatorname{dlog} \frac{t-g}{t-1} = -\log g$ and the term involving $\log(\frac{t-g}{t-1})_0$ vanishes because it does not involve a dt. Adding the term a_0 we obtain

$$\begin{aligned}
\int_0^\infty \epsilon(g, f) =& \frac{1}{q} \log f_0 \log g \operatorname{dlog}(1-g) \\
&- \frac{1}{q^2} \log(1-g)_0 \log g \phi^* \operatorname{dlog} f \\
&+ \frac{1}{q} \Theta(g) \phi^* \operatorname{dlog} f \ .
\end{aligned} \tag{4.15}$$

Note that this integral belongs to $\Omega^1_{\mathrm{col},1}(U)$, in the notation of Section 3.

Lemma 4.16. *We have $\int_0^\infty ch(\delta) = 0$ for any δ in $(1+I)^*_{\mathcal{O}} \cup K_2^{(2)}(\mathcal{O})$.*

Proof. As in (4.14) δ is a sum of terms of the form $\delta_1 \cup \delta_2$ with $\delta_1 \in K_1^{(1)}(X^{\mathrm{loc}}_{\mathcal{C}}, \square)$ and $\delta_2 \in K_2^{(2)}(\mathcal{C}^{\mathrm{loc}})$. That $\int_0^\infty$ vanishes on these elements follows from the proof of [BdJ03, Proposition 7.2]. $\square$

Now we deal with the term $(d\lambda, \lambda|_{\{t=\infty\}}, \lambda|_{\{t=0\}})$.

Proposition 4.17. *Suppose that $X^{\mathrm{loc}}_{\mathcal{C}}$ is obtained from $X_{\mathcal{C}^{\mathrm{loc}}}$ by removing the graphs of $t = h_j(x)$ for $j = 1, \dots, n$. Assume further that the reductions of $t = h_i(x)$ are disjoint (which we can achieve by shrinking $\mathcal{C}^{\mathrm{loc}}$). Then there are $a_j(x), a(x) \in \mathcal{O}(U)$ such that we have*

$$\int_0^\infty (d\lambda, \lambda|_{\{t=\infty\}}, \lambda|_{\{t=0\}}) = d(a + \sum_j a_j \log(h_j)) \ .$$

In particular, it belongs to $\Omega^1_{\mathrm{col},1}(U)$.

Proof. We have global coordinates x and t on U' so we can write $\lambda = f(x,t)dx + g(x,t)dt$. Then

$$d\lambda = \left(\frac{\partial g}{\partial x} - \frac{\partial f}{\partial t}\right) dx \wedge dt\ .$$

Therefore

$$\int_{t=0}^{t=\infty} d\lambda = \left(\int_{t=0}^{t=\infty} \frac{\partial g}{\partial x} dt\right) dx - (f(x,\infty) - f(x,0))dx\ .$$

But the second term is exactly $\lambda|_{t=\infty} - \lambda|_{t=0}$ so we find

$$\int_0^\infty [d\lambda, \lambda|_{\{t=\infty\}}, \lambda|_{\{t=0\}}] = d\left(\int_{t=0}^{t=\infty} g(x,t)dt\right)\ .$$

Consider now the two-form $\gamma = g(x,t)dx \wedge dt \in \Omega^2(U')$. This is closed so represents a cohomology class in $H^2_{\mathrm{rig}}((X^{\mathrm{loc}}_{\mathcal{C}})_\kappa/K)$. We have a short exact sequence

$$H^2_{\mathrm{rig}}((X_{\mathcal{C}^{\mathrm{loc}}})_\kappa/K) \to H^2_{\mathrm{rig}}((X^{\mathrm{loc}}_{\mathcal{C}})_\kappa/K \xrightarrow{\mathrm{Res}} \oplus_i H^1_{\mathrm{rig}}((\mathcal{C}^{\mathrm{loc}})_\kappa/K)\ ,$$

where the map $\mathrm{Res} = \oplus_j \mathrm{Res}_j$ is the sum of the boundary maps on the reductions of $t = h_j(x)$, composed with the pullback under the isomorphisms of these graphs with $(\mathcal{C}^{\mathrm{loc}})_\kappa$. Suppose that $\mathrm{Res}_j(\gamma)$ is the cohomology class of $a_j(x)dx \in \Omega^1(U)$. Let $\gamma_j := a_j(x)dx \wedge \mathrm{dlog}(t - h_j(x))$. Clearly $\mathrm{Res}_l(\gamma_j) = 0$ if $l \neq j$. We claim that $\mathrm{Res}_j(\gamma_j) = \mathrm{Res}_j(\gamma)$. This can be seen easily by applying the map $(x,t) \to (x, t - h_j(x))$, transforming γ_j to $a_j(x)dx \wedge \mathrm{dlog}(t)$. Thus, $\gamma - \sum_j \gamma_j$ extends to $H^2_{\mathrm{rig}}((X_{\mathcal{C}^{\mathrm{loc}}})_\kappa/K)$ and its integral is a holomorphic one form on U. Let this form be $a(x)dx$. Since $\int_{t=0}^{t=\infty} \gamma_j = \pm a_j(x)\log(h_j(x))dx$ we find $\pm\int_{t=0}^{t=\infty} \gamma = (a(x) + \sum a_j(x)\log(h_j(x)))dx$ and dividing by dx we find $\int_{t=0}^{t=\infty} g(x,t)dt = \pm(a(x) + \sum a_j(x)\log(h_j(x)))$. This completes the proof. □

These results give us a strategy for breaking the regulator into a sum of terms, each depending on the pairs (g_i, f_i), as follows: Suppose that ω is a form of the second kind on C and let $[\omega]$ be its cohomology class in $H^1_{\mathrm{dR}}(C/K)$.

Definition 4.18. A functional $L_\omega : \Omega^1_{\mathrm{col},1}(U) \to \mathbb{C}_p$ will be called good if it has the following properties:

- It kills terms of the forms da and $d(a\log f)$ for $a, f \in \mathcal{O}(U)$.
  It kills all terms of the form $\int_0^\infty [d\lambda, \lambda|_{\{t=\infty\}}, \lambda|_{\{t=0\}}]$.
- If η is in $\Omega^1(U)$ and its cohomology class is the restiction to U of a class $[\eta]$ in $H^1(C)$, then we have $L_\omega(\eta) = [\eta] \cup [\omega]$.

Proposition 4.19. *Suppose that an element β in $K_4^{(3)}(\mathcal{C}^{\mathrm{loc}})$ maps to $\sum_i [g_i]_2 \cup (f_i)$ in $H^1(\mathcal{C}^\bullet(\mathcal{O}))$ under the natural map $K_4^{(3)}(\mathcal{C}^{\mathrm{loc}}) \to K_4^{(3)}(\mathcal{O}) \to K_4^{(3)}(\mathcal{O})/K_3^{(2)}(\mathcal{O}) \cup \mathcal{O}^*_{\mathbb{Q}}$, see* (2.62)*, and that $ch(\beta) = [\eta_0]$ in the model* (4.6)*. Then we have, for a good functional L_ω,*

$$L_\omega(\eta_0) = \sum_i L_\omega\left(\int_0^\infty \epsilon(g_i, f_i)\right)\ .$$

Proof. We must first show that the map

$$K_4^{(3)}(\mathcal{C}^{\mathrm{loc}}) \xrightarrow{ch} \tilde{H}^2_{\mathrm{ms}}(\mathcal{C}^{\mathrm{loc}}, 3) \xrightarrow{\eta_0 \mapsto L_\omega(\eta_0)} \mathbb{C}_p$$

factors via $K_4^{(3)}(\mathcal{O})/K_3^{(2)}(\mathcal{O}) \cup \mathcal{O}_{\mathbb{Q}}^*$. By further localizing, it suffices to show that the map above vanishes on elements of the form $\gamma \cup f$ with $\gamma \in K_3^{(2)}(\mathcal{C}^{\mathrm{loc}})$ and $f \in \mathcal{O}^*(\mathcal{C}^{\mathrm{loc}})$. We have

$$\tilde{H}^1_{\mathrm{ms}}(\mathcal{C}^{\mathrm{loc}}, 2) = \{(0,\epsilon), \epsilon \in \mathcal{O}(U),\ d\epsilon = 0\} = \{(0,\epsilon), \epsilon \in K\} \tag{4.20}$$

Thus $ch(\gamma) = (0,\alpha)$ for some $\alpha \in K$. On the other hand, by (4.1) we have $ch(f) = (\operatorname{dlog} f, \log(f_0)/q)$ (here f_0) does not matter). Using (4.2) we obtain, in the model (4.6)

$$ch(\gamma \cup f) = (0,\alpha) \cup (\operatorname{dlog} f, \log(f_0)/q) = \alpha \operatorname{dlog} f\ .$$

The factorization thus follows from first property of the good functional (with $a = 1$). Next, by Proposition 4.17 the first property also implies that L_ω kills all terms of the form $\int_0^\infty [d\lambda, \lambda|_{\{t=\infty\}}, \lambda|_{\{t=0\}}]$. The result now follows immediately from the discussion above. ☐

Remark 4.21. There is a final wrinkle here because of the normalization (4.7) for the syntomic regulator. For β as in the Corollary, the regulator of β is in fact $[\eta]$ with $\left(1 - \frac{\phi^*}{q^3}\right)\eta = \eta_0$ Thus, once we have the functional L_ω we will be able to compute $L_\omega(\eta_0)$ but will in fact want $L_\omega(\eta)$. Fortunately, it is easy to see (and will be explained) that if we know $L_\omega(\eta_0)$ for *all* ω, then we also know $L_\omega(\eta)$ for all ω. Now, if β extends to $K_4^{(3)}(\mathcal{C})$, then $[\eta]$ extends to C to yield the regulator of β on $\mathcal{C}$. By the second condition on L_ω (which was not used yet!!) we obtain the required cup product $[\eta] \cup [\omega]$, In fact, as in previous computations, the result with η is much simpler than with η_0, confirming the "correctness" of our normalization.

5. WISHES

This Section is highly speculative. It contains no formal proofs. Nevertheless, we feel it is vital for the understanding of a significant portion of the computations to come. It also suggests interesting research directions into a more canonical representation of syntomic cohomology, one that would make the computations in the syntomic case equivalent to the complex case.

We want to follow a strategy that proved very successful in computing syntomic regulators on K_2 of curves (see the discussion after Proposition 5.2 in [Bes00b]). We argue heuristically, in some make believe world where syntomic cohomology looks much more like Deligne cohomology from the computational standpoint, and get a formula for the regulator. Then we try to relate this formula with the formula we obtained in the previous Section and see what needs to be proved to show that the two formulas are equivalent. That the make believe formula turns out to be correct is a strong indication that one should be able to turn the make believe computation into a rigorous one.

The make believe computation is based on the following assumptions:

- The "cohomology" is given by the pairs (ω, h) where ω is an i-form and h is an $i-1$ form with $dh = \omega$. Of course h is not an actual form but something like a Coleman form, for example a Coleman function.
- The "regulator" of a function f is the pair $(\operatorname{dlog} f, \log f)$.
- The cup product is given by $(\omega_1, h_1) \cup (\omega_2, h_2) = (\omega_1 \wedge \omega_2, \omega_1 \wedge h_2 \text{ or } h_1 \wedge \omega_2)$.

With these rules, we can redo the computation from the previous Section in this make believe language: We have in $\tilde{H}^2_{\mathrm{ms}}(X^{\mathrm{loc}}_{\mathcal{C}}, 2)$ that $-ch(\frac{t-g}{t-1} \cup (1-g)) = (\omega_g, \varepsilon_g)$

with ω_g as in Lemma 4.9 and $\varepsilon_g = -\log(1-g)\,\mathrm{dlog}(\frac{t-g}{t-1})$. Since the restriction of ε_g to $t=0$ is $-\log(1-g)\,\mathrm{dlog}(g) = d\,\mathrm{Li}_2(g)$ we have, following the proof of Proposition 4.10, that $ch([g]_2) \in \tilde{H}^2_{\mathrm{ms}}(X^{\mathrm{loc}}_{\mathbb{C}}, \square, 2)$ equals $[\omega_g, \varepsilon_g, 0, \mathrm{Li}_2(g)]$. Cupping with $(\mathrm{dlog}\, f, \log f)$ we get

$$ch([g]_2 \cup (f)) = \text{"}\epsilon\text{"}(g,f) = [-\log f\,\mathrm{dlog}(\frac{t-g}{t-1}) \wedge \mathrm{dlog}(1-g)), 0, 0]\ .$$

If we now integration from 0 to ∞ we get

$$\int_0^\infty \text{"}\epsilon\text{"}(g,f) = \log f \log g\,\mathrm{dlog}(1-g)\ .$$

We now compare this with $\int_0^\infty \epsilon(g,f)$ of (4.15). Continueing to mimic the discussion of the K_2 in [Bes00b], the former version should be an untwisted version of the latter, i.e., without the "twist" by $(1-\frac{\phi^*}{q^3})$. To see this, we use the formalism described in [Bes00b, Remark 3.1] to get

$$\begin{aligned}\left(1-\frac{\phi^*}{q^3}\right)[\log(f)\log(g)\,\mathrm{dlog}(1-g)] = \ &\frac{1}{q}\log(f_0)\log(g)\,\mathrm{dlog}(1-g)\\ &+\frac{1}{q^2}\log\phi^*(f)\log(g)\,\mathrm{dlog}(1-g)_0\\ &+\frac{1}{q^3}\log(g_0)\log\phi^*(f)\phi^*\,\mathrm{dlog}(1-g)\end{aligned} \tag{5.1}$$

This already begins to look similar to $\int_0^\infty \epsilon(g,f)$ but there are differences. We want to argue that the difference is "exact". This can not be taken to simply mean being the differential of something, since in Coleman's theory every form is integrable. Experience has shown that things are exact if they are the differential of a product of functions. We will use two such assertions. To each one will correspond a precise statement in the following Sections, which will be justified by the techniques we will introduce. To remind ourselves where these occured, we will call them "Wishes", and mark them explicitly. The first one is

Wish 5.2. We have in cohomology

$$\Theta(g)d\log\phi^*(f) = -d\Theta(g)\log\phi^*(f)$$

Using this Wish we can write the term $\frac{1}{q}\Theta(g)d\log\phi^*(f)$ in (4.15) as

$$\begin{aligned}&-\frac{1}{q}d\Theta(g)\log\phi^*(f)\\ &= -\frac{1}{q}\left(\frac{1}{q}\log(1-g)_0\,\mathrm{dlog}\, g - \frac{1}{q^2}\log g_0\,\mathrm{dlog}\,\phi^*(1-g)\right)\log\phi^*(f)\\ &= -\frac{1}{q^2}\log(1-g)_0\,\mathrm{dlog}\, g\log\phi^*(f) + \frac{1}{q^3}\log g_0\,\mathrm{dlog}\,\phi^*(1-g)\log\phi^*(f)\ ,\end{aligned}$$

so we obtain

$$\begin{aligned}\int_0^\infty \epsilon(g,f) = \ &\frac{1}{q}\log f_0\log g\,\mathrm{dlog}(1-g) - \frac{1}{q^2}\log(1-g)_0\log g\phi^*\,\mathrm{dlog}\, f\\ &-\frac{1}{q^2}\log(1-g)_0\,\mathrm{dlog}\, g\log\phi^*(f) + \frac{1}{q^3}\log g_0\,\mathrm{dlog}\,\phi^*(1-g)\log\phi^*(f)\ .\end{aligned}$$

Comparing this with $\left(1-\frac{\phi^*}{q^3}\right)(\log(f)\log(g)\operatorname{dlog}(1-g))$ given in (5.1) we see that the first and last terms are the same, and that therefore we get our desired equality, "twisted" by $1-\frac{\phi^*}{q^3}$ if we get our second Wish to come true.

Wish 5.3. We have in cohomology

$$\log(1-g)_0\log g\phi^*\operatorname{dlog} f+\log(1-g)_0\operatorname{dlog} g\log\phi^*(f)+\operatorname{dlog}((1-g)_0)\log g\log\phi^*(f)=0\ .$$

In the next two Sections we will introduce triple indices. The Wishes described above correspond to precise results states with the use of triple indices, which we can indeed prove.

6. The triple index, local theory

We briefly recall first the theory of the "local index" from [Bes00b, Section 4]. In our new context this should be called the double index. To make things slightly simpler, we work in an algebraic context. The transition to working with annuli is straightforward.

Let K be a field of characteristic 0. We consider the algebra $A_{\log}:=K((z))[\log(z)]$ of polynomials over the formal variable $\log(z)$, over the field of finite to the left Laurent power series in z. We further consider the module of differentials $A_{\log}\cdot dz$. It is an easy excercise in integration by parts to see that every form in $A_{\log}\cdot dz$ has an integral in $A_{\log}$ in a unique way up to a constant. We distinguish in $A_{\log}$ the subfield $\mathsf{Mer}:=K((z))$ of meromorphic functions and the subspace $A_{\log,1}=\mathsf{Mer}+K\cdot\log(z)$ consisting exactly of all functions whose differential is in $\mathsf{Mer}\cdot dz$. To $F\in A_{\log,1}$ we can associated the residue of its differential $\operatorname{Res} dF\in K$. If $F\in A_{\log,1}$, then $F\in\mathsf{Mer}$ if and only if $\operatorname{Res} dF=0$.

Definition 6.1 ([Bes00b, Proposition 4.5])**.** The double index is the unique anti symmetric bilinear form $\langle\ ,\ \rangle : A_{\log,1}\times A_{\log,1}\to K$ such that $\langle F,G\rangle=\operatorname{Res} FdG$ whenever this last expression makes sense.

We recall that the construction of this index is essentially trivial: one notices that the anti-symmetry forces $\langle\log(z),\log(z)\rangle=0$ and that $\langle F,G\rangle=-\operatorname{Res} GdF$ whenever this expression makes sense. Then one writes $F=\alpha\log(z)+f$, $G=\beta\log(z)+g$ with $f,g\in\mathsf{Mer}$ and then one uses the bilinearity to write $\langle F,G\rangle$ as a sum of terms that can be computed.

The triple index turns out to be a bit more complicated. First of all we need to explain on which data it is evaluated:

- Three functions F,G,H in $A_{\log,1}$.
- For each two functions R and S out of F,G,H a choice of $\int RdS$ (i.e., a function in $A_{\log}$ whose differential is RdS) and of $\int SdR$ in such a way that

$$\int RdS+\int SdR=RS\ . \tag{6.2}$$

As it will turn out this information is a bit redundant: clearly $\int RdS$ determines $\int SdR$. Also it will turn out that the index will be independent of $\int FdG$. Still, this symmetric data is very convenient. To not carry around too much notation, we will simply denoted this data by $(F,G;H)$, where the additional choices should be understood from the context. In particular, any permutation of F,G,H induces an obvious permutation of the additional data. Also, if $(F_i,G;H)$, $i=1,2$ are given with all their additional data then there is a natural choice of data for $(F_1+$

$F_2, G; H)$, and similary in the second and third positions. If we do need to indicate a change in the auxiliary data we will write this as $(F, G; H|I_{FdG}, \cdots)$, where the subscript FdG indicates that I is an integral of FdG.

Proposition 6.3. *There exist a unique function from data as above to K, denoted $(F, G; H) \mapsto \langle F, G; H\rangle$, called the triple index, such that the following conditions are satisfied.*

1. *Trilinearity - the triple index is linear in each of the three variables, which means that $\langle \alpha_1 F_1 + \alpha_2 F_2, G; H\rangle = \alpha_1\langle F_1, G; H\rangle + \alpha_2\langle F_2, G; H\rangle$ provided that all auxiliary data are chosen in the way indicated above, and similary for linearity in G and H.*
2. *Symmetry - we have $\langle F, G; H\rangle = \langle G, F; H\rangle$, again with the choice of auxiliary data indicated above.*
3. *Triple identity - We have, again with the obvious additional choices,*
$$\langle F, G; H\rangle + \langle F, H; G\rangle + \langle G, H; F\rangle = 0.$$
4. *Reduction to the double index - If $G \in \mathsf{Mer}$ then $\langle F, G; H\rangle = \left\langle F, \int GdH\right\rangle$, where $\int GdH$ is taken from the auxiliary data and is in $A_{\log,1}$ because by assumption $GdH \in \mathsf{Mer}\cdot dz$.*

Proof. We first show that the dependency on the choices of integrals is forced by the properties of the triple index.

Lemma 6.4. *Suppose that the triple index exists. We then have the following change of constant formulae:*

1. *If C is a constant, then*
$$\langle F, G; H|(I + C)_{GdH}, (J - C)_{HdG}\rangle = \langle F, G; H|I_{GdH}, J_{HdG}\rangle - C \cdot \operatorname{Res} dF$$
$$\langle F, G; H|(I + C)_{FdH}, (J - C)_{HdF}\rangle = \langle F, G; H|I_{FdH}, J_{HdF}\rangle - C \cdot \operatorname{Res} dG$$
2. *The triple index is independent of the integral $\int FdG$.*

Proof. We use the trilinearity. Consider the data $(F, 0; H)$, where the additional data is the same for F and H but we take the integral of $0dH$ to be C, hence we are forced to take that of $Hd0$ to be $-C$. We take $\int 0dF = 0$. The trilinarity implied that $\langle F, G; H\rangle$ and $\langle F, 0; H\rangle$ gives the left hand side of the formula. But reduction to the double index means that $\langle F, 0; H\rangle = \langle F, C\rangle = -\operatorname{Res} CdF$. An identical argument proves the second case. Finally, if in the above argument we take instead $\int 0dF = D$ and $\int 0dH = 0$, we see from exactly the same argument that the integral is independent of the auxiliary choice $\int FdG$. □

We now check that the triple index is uniquely defined on all data where at least one of F, G, H is in Mer. Clearly in this case we can use Reduction to the double index together with symmetry and the triple formula to compute the index, so it is clearly unique. The following Lemma gives existence.

Lemma 6.5. *Consider the following recepy:*

1. *If $G \in \mathsf{Mer}$ define $\langle F, G; H\rangle = \left\langle F, \int GdH\right\rangle$,*
2. *If $F \in \mathsf{Mer}$ define $\langle F, G; H\rangle = \langle G, F; H\rangle$ where the last expression is defined as in 1,*
3. *If $H \in \mathsf{Mer}$ define $\langle F, G; H\rangle = -(\langle F, H; G\rangle + \langle G, H; F\rangle)$ where each of these terms is defined as in 1.*

Then this recepy gives a well defined $\langle F, G; H\rangle$ in all cases where at least one of F, G and H is in Mer *and restricted to this subset it satisfies all properties of the triple index.*

Proof. To show that this expression is well defined we need to consider what happens when two of F, G, H are in Mer: If $F, G \in \mathsf{Mer}$ we check that $\left\langle F, \int GdH\right\rangle = \left\langle G, \int FdH\right\rangle$. This follows because by the definition of the double index both expressions equal $\operatorname{Res} FGdH$. Next we check that if $G, H \in \mathsf{Mer}$ then

$$\begin{aligned}
&\left\langle F, \int GdH\right\rangle + \left\langle F, \int HdG\right\rangle + \left\langle G, \int HdF\right\rangle \\
&= \langle F, GH\rangle + \left\langle G, \int HdF\right\rangle \text{ by bilinearity of the double index and (6.2)} \\
&= -\operatorname{Res} GHdF + \operatorname{Res} GHdF = 0\ .
\end{aligned}$$

Thus we find that we have a well defined expression. We need to check that all properties of the expected triple index hold in this case. Trilinearity is essentially clear from the bilinearity of the double index. Symmetry is also easy: if F or G are in Mer then symmetry follows from the first two rules. If H is in Mer then the expression in (3) is clearly symmetric in F and G. The triple identity is forced by (3) and the reduction to the double index is an immediate consequence of our check that the triple index is well defined. ☐

Note that the proof of Lemma 6.4 applies verbatim for this partial triple index, so we know the dependency on the choices of integrals.

To extend the triple index to all F, G and H we first check the case where $F = G = H = \log(z)$. Then we can arrange that all auxiliary data equal $(1/2)\log^2(z)$. The triple formula implies immediately that (with this data)

$$\langle \log(z), \log(z); \log(z)\rangle = 0. \tag{6.6}$$

We can now demonstrate uniqueness for the triple index. Suppose $F_i = \alpha_i \log(z) + f_i$, $i = 1, 2, 3$ where $\alpha_i \in K$ and $f_i \in \mathsf{Mer}$. Choose some auxiliary data $\int RdS$ for any two R and S out of f_i and $\alpha_i \log(z)$, where we continue to take $\int \log(z) \operatorname{dlog}(z) = (1/2)\log^2(z)$. Using trilinearity and (6.6) we can write $\langle F_1, F_2; F_3\rangle$, with some choice of auxiliary data, as the sum with some coefficients of triple indices where at least one of the entries is in Mer which are therefore computable by previous considerations. Now we can use change of constant to write $\langle F_1, F_2; F_3\rangle$ with arbitrary auxiliary data. This shows uniqueness and gives a formula for the general index. We need to check that this formula is well defined, which given the fact that all the summands are well defined thanks to Lemma 6.5 amounts to checking independence of the choices of the auxiliary data. This is Just a tedious formal check: suppose for example that we add C to $\int \alpha_1 \log(z) df_3$, and correspondingly subtract C from $\int f_3\alpha_1 \operatorname{dlog} z$. This will have the effect that $\int F_1 dF_3$ will be added a C and $\int F_3 dF_1$ will be subtracted a C. This procedure will subtract $\alpha_2 C = C \operatorname{Res} dF_2$ from $\langle \alpha_1, \alpha_2 \log(z); f_3\rangle$ and will not change any of the other indices. This shows that the change does not alter the index.

It remains to check that our formula satisfies all the properties for the triple index. First the change of constant formula of Lemma 6.4 is clear because we used it in the definition and we showed that the formula we get is well defined. Now given change of constant it easy to see that it is enough to check trilinearity, symmetry and triple identity for one choice of auxiliary data. The derivation of these three formulas is then completely formal. Finally, reduction to the double index can only occur if at least one α_i is 0. But in this case we clearly get the triple index for the case where $F_i \in \mathsf{Mer}$ so we know this formula already. $\square$

To compute the triple index in some concrete situations, which will be needed later, we introduce the notion of the constant term.

Definition 6.7. The constant term, with respect to the variable z is the linear functional $c_z : A_{\log} \to K$, first defined on Mer by

$$c_z(\sum a_n z^n) = a_0$$

and then in general by

$$c_z(\sum_{i=0}^{\infty} f_i(z)\log^i(z)) = c_z(f_0)\ .$$

Note that the unlike the triple index, the constant term definitely depends on the choice of the local parameter z. For example, for $\alpha \in K$ and the function $f(z) = \log(z) = \log(\alpha z) - \log(\alpha)$ we have $c_z(f) = 0$ but $c_{\alpha z}(f) = -\log(\alpha)$.

Proposition 6.8. *Let F, G and H be three functions in $A_{\log}$ whose differentials lie in $\mathsf{Mer}dz$ and have at most simple poles at 0. The choice of integrals $\int FdH$ and $\int GdH$ gives an auxiliary data for the computation of $\langle F, G; H\rangle$ and with respect to this choice we have*

$$\langle F, G; H\rangle = c_z(F)\cdot c_z(G)\cdot \operatorname{Res} dH - \operatorname{Res} dF \cdot c_z(\int GdH) - \operatorname{Res} dG \cdot c_z(\int FdH)$$

Proof. We have a bilinear map

$$(F, H) \to \int' FdH := \text{ unique } \int FdH \text{ with } c_z(\int FdH) = 0\ .$$

Therefore, we see that the map

$$(F, G, H) \to \langle F, G; H\rangle' := \left\langle F, G; H | \int' FdH_{FdH}, \int' GdH_{GdH} \right\rangle$$

is trilinear and symmetric in F and G. By Lemma 6.4 it suffices to prove that

$$\langle F, G, H\rangle' = c_z(F)\cdot c_z(G)\cdot \operatorname{Res} dH \tag{6.9}$$

and as both sides are trilinear and symmetric in F and G, and as $F = a\log(z) + f(z)$ with $f(z)$ holomorphic and similary for G and H, it suffices to treat the following cases:

(1) When f, g and h are holomorphic we have

$$\langle f, g, h\rangle' = \operatorname{Res} fgdh = 0 = c_z(f)c_z(g)\operatorname{Res} dh$$

since $\operatorname{Res} dh = 0$.

(2) Suppose $F = G = H = \log(z)$. Since $c_z(\log^2(z)/2) = 0$ we see that $\langle \log(z), \log(z); \log(z)\rangle'$ is the local index computed with all auxiliary data set equal to $\log^2(z)/2$, and this we know is 0 by (6.6). On the other hand the right hand side of (6.9) is also zero

since $c_z(\log(z)) = 0$.
(3) If g and h are holomorphic we have

$$\langle \log(z), g; h\rangle' = \left\langle \log(z), \int' gdh \right\rangle = -\operatorname{Res}(\int' gdh) \operatorname{dlog} z = (\int' gdh)(0) = 0 ,$$

which equals $c_z(\log(z))c_z(g) \operatorname{Res} dh$ as required.
(4) if f and g are holomorphic we find

$$\langle f, g; \log(z)\rangle' = \operatorname{Res} fg \operatorname{dlog} z = fg(0) = c_z(f)c_z(g) \operatorname{Res} \operatorname{dlog} z .$$

(5) If g is holomorphic and $a = c_z(g)$ we see that $\int'(g-a) \operatorname{dlog} z = \int' g \operatorname{dlog} z - a\log(z)$. Using this we find

$$\begin{aligned}\langle \log(z), g; \log(z)\rangle &= \left\langle \log(z), \int' g \operatorname{dlog} z \right\rangle = \left\langle \log(z), \int' (g-a) \operatorname{dlog} z \right\rangle \\ &= -\operatorname{Res}\left(\int' (g-a) \operatorname{dlog} z\right) \operatorname{dlog} z = 0\end{aligned}$$

since $\int'(g-a) \operatorname{dlog} z$ is holomorphic and has constant term 0. This again equals the right hand side.
(6) The final case is for $\langle \log(z), \log(z); h\rangle$ with h holomorphic. Here, since $c_z(h\log(z)) = 0$, we have the equation $\int' h \operatorname{dlog} z + \int' \log(z) dh = h\log(z)$. We therefore immediately deduce this last case from the previous one and the tripple identity. ☐

7. The triple index, global theory

At this point we will switch for convenience to assuming that our ground field is $\mathbb{C}_p$. Suppose now that we consider an open annulus $V \cong \{r < |z| < s\}$ with a parameter z. Then exactly the same analysis gives us a triple index on V.

The uniqueness of the triple index immediately implies (compare [Bes00b, Lemma 4.6]) the following result.

Lemma 7.1. *If $\phi : V \to V$ is an endomorphism of degree n, let $\phi^*(F, G; H)$ be defined in the obvious way, pulling back by ϕ all the auxiliary data. Denote this data simply by $(\phi^*F, \phi^*G; \phi^*H)$. Then we have the formula*

$$\langle \phi^*F, \phi^*G; \phi^*H\rangle = n\langle F, G; H\rangle.$$

Consider now a wide open space U over $\mathbb{C}_p$ with annuli ends set $\mathsf{End}(U)$. We will denote the triple index with respect to the end e by the subscript e. When we are given 3 Coleman functions F, G and H on U, such that their differentials are in $A(U)$, we may choose Coleman integrals for all forms RdS when R and S are among F, G and H, and we may do so in such a way that $\int RdS + \int SdR = RS$ globally. This allows us to compute $\langle F, G; H\rangle_e$ at each end e and we may consider the global triple index

$$\langle F, G; H\rangle_{\mathrm{gl}} = \sum_{e \in \mathsf{End}(U)} \langle F, G; H\rangle_e$$

Lemma 7.2. *The expression $\langle F, G; H\rangle_{\mathrm{gl}}$ is independent of the auxiliary choices, so depends only on F, G and H.*

Proof. Since the possible integrals differ from one another by a global constant, if we change for example $\int GdH$ by a constant C, the change of constant formula implies that the global triple index changes by $\sum_e C \operatorname{Res}_e dF = C \sum_e \operatorname{Res}_e dF = C \cdot 0 = 0$. ☐

Unlike the global double index, the global triple index does not depend solely on the cohomology classes of $dF, \cdots$, and not even just on the differentials of the functions. For example, if C is a constant we have the formula $\langle F, C; H\rangle_{\mathrm{gl}} = \sum_e \left\langle F, \int CdH \right\rangle_e = C \sum_e \langle F, H\rangle_e$. However, we do have the following.

Lemma 7.3. *If C is a constant then $\langle F, G; C\rangle_{\mathrm{gl}} = 0$.*

Proof. Indeed,

$$\begin{aligned}\langle F, G; 1\rangle_{\mathrm{gl}} &= -\langle F, 1, G\rangle_{\mathrm{gl}} - \langle G, 1, F\rangle_{\mathrm{gl}} \text{ by the triple identity}\\ &= -\left\langle F, \int dG \right\rangle_{\mathrm{gl}} - \left\langle G, \int dF \right\rangle_{\mathrm{gl}} \text{ by reduction to the double index}\\ &= -\langle F, G\rangle_{\mathrm{gl}} - \langle G, F\rangle_{\mathrm{gl}} = 0 \ ,\end{aligned}$$

where the last two equalities follow because the global double index is independent of the choice of the integral and by the antisymmetry of the double index. ☐

The lemma suggests that the global triple index is quite an interesting creature. It deserves further study. For our purposes we only need the following results:

Proposition 7.4. *Let $F, G, H \in A_{\mathrm{col}}(U)$ with $dF, dG, dH \in \Omega^1(U)$ and suppose that $[dF]$ and $[dG]$ are eigenvectors for Frobenius with eigenvalue q. Then $\langle F, G; H\rangle_{\mathrm{gl}} = 0$.*

Proof. We begin by establishing the following formulae. If $r \in A(U)$ then

$$\langle F, r, H\rangle_{\mathrm{gl}} = \sum_e \left\langle F, \int rdH \right\rangle_e = 0 \tag{7.5}$$

where the last equality follows from [Bes00b, Corollary 4.11]. Similarly we find that if also $s \in A(U)$ then $\langle s, G, H\rangle_{\mathrm{gl}} = 0$. Now if $h \in A(U)$, then

$$\langle F, G; h\rangle_{\mathrm{gl}} = -\langle F, h; G\rangle_{\mathrm{gl}} - \langle G, h; F\rangle_{\mathrm{gl}} = 0 \ ,$$

by application of (7.5). This last formula shows that for fixed F and G the function $H \mapsto \langle F, G; H\rangle_{\mathrm{gl}}$ depends only on the cohomology class of dH, $[dH] \in H^1(U)$. Let ϕ be a Frobenius lift on U. The assumption on F and G implies the existence of $r, s \in A(U)$ such that $\phi^* F = qF + r$ and $\phi^* G = qG + s$. Using this we can compute

$$\begin{aligned}q\langle F, G; H\rangle_{\mathrm{gl}} &= \langle \phi^* F, \phi^* G; \phi^* H\rangle_{\mathrm{gl}}\\ &= \langle qF + r, qG + s; \phi^* H\rangle_{\mathrm{gl}} = q^2 \langle F, G; \phi^* H\rangle_{\mathrm{gl}} \ ,\end{aligned}$$

using bilinearity and (7.5). This shows that the functional $[dH] \mapsto \langle F, G; H\rangle_{\mathrm{gl}}$ is an eigenvector for the action of ϕ^* with eigenvalue $1/q$. Such a functional must be 0 because the eigenvalues of ϕ^* on $H^1(U)$ are either q or Weil numbers of weight 1. ☐

Note that this proposition applies in particular when F and G are of the form $r + \log(f)$ where $r, f \in A(U)$. This follows since by [CdS88, Lemma 2.5.1], $\log(f^q/\phi^*(f))$ is in $A(U)$.

Proposition 7.6. *Suppose* $\omega \in \Omega^1(U)$ *has trivial residues on all residue ends, so that its Coleman integral* F_ω *is in fact analytic on the ends. Let* F, G, H *be Coleman functions on* U *whose differentials are holomorphic and represent eigenvectors for Frobenius with eigenvalue* q*. Then*

$$\sum_e \left\langle F, G; \int F_\omega dH \right\rangle_e + \sum_e \left\langle F, H; \int F_\omega dG \right\rangle_e + \sum_e \left\langle G, H; \int F_\omega dF \right\rangle_e \tag{7.7}$$

equals zero.

Proof. Note that the expression above makes sense since on each residue end e the form $F_\omega dH$ is analytic, so the corresponding triple index is defined, and similarly with H replaced by F and F. Note also that this is of course not a global index in the sens of this section, since $F_\omega dH$ is not holomorphic. The strategy for the proof is the same as for Proposition 7.4. First we notice that if F_ω is in fact holomorphic, then the identity holds by Proposition 7.4. It follows that the expression factors via the cohomology class $[\omega]$. Suppose now that we replace F by a holomorphic function u. We then have

$$\sum_e \left\langle u, G; \int F_\omega dH \right\rangle_e = \sum_e \left\langle G, \int F_\omega u dH \right\rangle_e$$
$$\sum_e \left\langle u, H; \int F_\omega dG \right\rangle_e = \sum_e \left\langle H, \int F_\omega u dG \right\rangle_e$$

by reduction to the double index, and

$$\begin{aligned}
&\sum_e \left\langle G, H; \int F_\omega du \right\rangle_e = \sum_e \left\langle G, H; F_\omega u - \int u\omega \right\rangle_e \\
&= \sum_e \langle G, H; F_\omega u \rangle_e \quad \text{by Proposition 7.4} \\
&= -\sum_e \langle G, F_\omega u; H \rangle - \sum_e \langle H, F_\omega u; G \rangle \quad \text{by the triple identity} \\
&= -\sum_e \left\langle G, \int F_\omega u dH \right\rangle_e - \sum_e \left\langle H, \int F_\omega u dG \right\rangle_e
\end{aligned}$$

by reducing to the double index again as F_ω is analytic. This shows that if we replace F by u in the formula to be proved we indeed get 0. Similarly we get the same result if we replace G by a holomorphic v, H by a holomorphic w, or if we do 2 or 3 of these replacements at the same time. Now, exactly as in the proof of Proposition 7.4, writing the right hand side of (7.7) as $T(F, G, H, \omega)$, we easily get from the previous computation that

$$\begin{aligned}
qT(F, G, H, \omega) &= T(\phi^* F, \phi^* G, \phi^* H, \phi^* \omega) \\
&= q^3 T(F, G, H, \phi^* \omega)
\end{aligned}$$

which shows that the functional $\gamma([\omega]) := T(F, G, H, \omega)$ satisfies $\gamma(\phi^*[\omega]) = q^{-2}\gamma([\omega])$, so that $\gamma(q^2\phi^* - \mathrm{id})[\omega]) = 0$. By the theory of Weil numbers, it follows that $\gamma = 0$. This proves what we want. ☐

8. The proof of Theorem 1.10

In this section we prove Theorem 1.10. We will use the theory of the triple index, for which it is more convenient for us to work over $\mathbb{C}_p$. For technical reasons, the syntomic regulator itself must be developed over a discretely values field. However, since we have formulas for the regulator that make sense over $\mathbb{C}_p$ as well, we work from now until the end of this paper over $\mathbb{C}_p$. Now that we have at our disposal the triple index, we can interpret our make believe computation from Section 5 in such a way that it will become true. We continue with the notation of the previous section, so U is a wide open space over $\mathbb{C}_p$.

The first thing that the triple index allows us to do is to extend the cup product to some Coleman differential forms. We first need a lemma.

Lemma 8.1. *The map* $\Omega^1_{\mathrm{col},1}(U) \to H^1(U) \otimes \Omega^1(U)$ *given by*

$$\sum F_{\omega_i} \eta_i \to \sum [\omega_i] \otimes \eta_i$$

is well defined.

Proof. This is [Bes02, Corollary 6.2]. ☐

Proposition 8.2. *There is a unique bilinear map*

$$\ll\, ,\, \gg : A_{\mathrm{col},1}(U) \otimes \Omega^1_{\mathrm{col},1}(U) \to \mathbb{C}_p$$

such that we have, for any F*,* G*,* H *in* $A_{\mathrm{col},1}(U)$*,*

$$\ll F, GdH \gg = \langle F, G; H \rangle_{\mathrm{gl}} \,. \tag{8.3}$$

Proof. By definition, $\Omega^1_{\mathrm{col},1}(U)$ is generated by forms like GdH so uniqueness is clear. To show the existence we first note that by Lemma 7.3 the right hand side depends only on dH. This shows that $\ll\, ,\, \gg$ is well defined as a map $A_{\mathrm{col},1}(U) \otimes A_{\mathrm{col},1}(U) \otimes \Omega^1(U) \to \mathbb{C}_p$, where the tensors are taken over $\mathbb{C}_p$. Lemma 8.1 shows that the kernel of the map $G \otimes dH \to GdH$ from $A_{\mathrm{col},1}(U) \otimes \Omega^1(U)$ to $\Omega^1_{\mathrm{col},1}(U)$ is contained in $A(U) \otimes \Omega^1(U)$ so it is enough to observe that that if g in $A(U)$ then $\langle F, g; H \rangle_{\mathrm{gl}} = \left\langle F, \int gdH \right\rangle_{\mathrm{gl}}$ indeed depends only on the form gdH. ☐

The interest in the pairing $\ll\, ,\, \gg$ lies in the fact that restricted to $A_{\mathrm{col},1}(U) \otimes \Omega^1(U)$ it is given by

$$\ll F, dG \gg = \langle F, G \rangle_{\mathrm{gl}} \,.$$

The pairing on the right was studied in [Bes00b]. It is known to depend only on dF and if dF, dG give cohomology classes that extend to C it is simply given by the cup product. This proves part of the following result.

Proposition 8.4. *Let* ω *in* $\Omega^1(U)$*, such that* $[\omega]$ *extends to* C*, and let* $F = F_\omega$ *in* $A_{\mathrm{col},1}(U)$ *be a Coleman integral of* ω*. The functional* $L_\omega(\eta) = \ll F, \eta \gg$ *on* $\Omega^1_{\mathrm{col},1}(U)$ *is good in the sense of Definition 4.18.*

Proof. Note that we are not claiming that this functional is independent of the choice of the constant of integration. The only property we need to prove is that L_ω vanishes on forms of type $d(a \log f)$, with a and f in $A(U)$. This is easily established:

$$\begin{aligned}\ll F, d(a\log f)\gg &= \ll F, a\,\mathrm{dlog}\, f\gg + \ll F, \log f da\gg\\ &= \langle F, a; \log f\rangle_{\mathrm{gl}} + \langle F, a; \log f\rangle_{\mathrm{gl}}\\ &= \langle a, \log f; F\rangle_{\mathrm{gl}} = 0\end{aligned}$$

by Proposition 7.4. □

The last proposition, together with Proposition 4.19, suggest that we need to compute the expression $\ll F, \int_0^\infty \epsilon(g,f)\gg$. We will manipulate this, by “making our wishes come true”, in the form of the following proposition.

Proposition 8.5. *Let F be as in Proposition 8.4 and let $g, f \in \mathcal{O}^*(\mathcal{C}^{\mathrm{loc}})$ with $g \neq 1$. Let $\int_0^\infty \epsilon(g,f)$ be as in* (4.15)*. Then we have*

$$\ll F, \int_0^\infty \epsilon(g,f)\gg = \sum_e \mathcal{T}(g,f,F)_e\ , \tag{8.6}$$

where

$$\begin{aligned}\mathcal{T}(g,f,F)_e =&\frac{1}{q}\left\langle \log f_0, \log g; \int F\,\mathrm{dlog}(1-g)\right\rangle_e\\ &+\frac{1}{q^2}\left\langle \log\phi^*(f), \log(g); \int F\,\mathrm{dlog}(1-g)_0\right\rangle_e\\ &+\frac{1}{q^3}\left\langle \log\phi^*(f), \log(g_0); \int F\phi^*\,\mathrm{dlog}(1-g)\right\rangle_e\ .\end{aligned}\tag{8.7}$$

Proof. We have by (4.15) and (8.3)

$$\begin{aligned}\ll F, \int_0^\infty \epsilon(g,f)\gg = \sum_e \Big(&\frac{1}{q}\left\langle F, \log g; \int \log f_0\,\mathrm{dlog}(1-g)\right\rangle_e\\ &-\frac{1}{q^2}\left\langle F, \log g; \int \log(1-g)_0 d\log\phi^*(f)\right\rangle_e\\ &+\frac{1}{q}\langle F, \Theta(g); d\log\phi^*(f)\rangle_e\Big)\ .\end{aligned}$$

Note that dF is in $\Omega^1(U)$ and has trivial residues along all annuli ends. It follows that F is holomorphic on each annuli end.

At every annulus e we obtain the identities

$$\begin{aligned}\left\langle F, \log g; \int \log f_0\,\mathrm{dlog}(1-g)\right\rangle_e &= \left\langle \log(g), \int F\log f_0\,\mathrm{dlog}(1-g)\right\rangle_e\\ &= \left\langle \log f_0, \log g; \int F\,\mathrm{dlog}(1-g)\right\rangle_e\\ \left\langle F, \log g; \int \log(1-g)_0 d\log\phi^*(f)\right\rangle_e &= \left\langle \log g, \int F\log(1-g)_0 d\log\phi^*(f)\right\rangle_e\\ &= \langle \log g, F\log(1-g)_0; \log\phi^*(f)\rangle_e\end{aligned}$$

and

$$\langle F, \Theta(g); d\log\phi^*(f)\rangle_e = \mathrm{Res}_e\, F\Theta(g)d\log\phi^*(f) = \langle \log\phi^*(f), \Theta(g)F\rangle_e$$

so we obtain

$$
\begin{aligned}
\ll F, \int_0^\infty \epsilon(g,f) \gg = \sum_e \Big(& \frac{1}{q}\left\langle \log f_0, \log g; \int F \operatorname{dlog}(1-g)\right\rangle_e \\
& -\frac{1}{q^2}\langle \log g, F\log(1-g)_0; \log\phi^*(f)\rangle_e \\
& -\frac{1}{q}\langle \log\phi^*(f), \Theta(g)F\rangle_e \Big) \,.
\end{aligned}
$$

To equate this with the right hand side of (8.6) we now realize our wishes one by one. First we notice that the first summands in each expression are identical. The realization of the first wish corresponds to the formula

$$
\begin{aligned}
& \sum_e \langle \log\phi^*(f), \Theta(g)F\rangle_e \\
&= \sum_e \left\langle \log\phi^*(f), \int F d\Theta(g) \right\rangle_e + \sum_e \left\langle \log\phi^*(f), \int \Theta(g) dF \right\rangle_e \\
&= \sum_e \left\langle \log\phi^*(f), \int F d\Theta(g) \right\rangle_e \,,
\end{aligned}
$$

as the second sum on the second line vanishes by [Bes00b, Corollary 4.11]. Now we may use the formula (4.11) for $d\Theta(g)$ to write this as

$$
\begin{aligned}
\sum_e \Big(& \frac{1}{q}\langle \log\phi^*(f), F\log(1-g)_0; \log(g)\rangle_e \\
& -\frac{1}{q^2}\left\langle \log\phi^*(f), \log(g_0); \int F\phi^* \operatorname{dlog}(1-g)\right\rangle_e \Big) \,,
\end{aligned}
$$

so the left hand side of (8.6) becomes

$$
\begin{aligned}
\sum_e \Big(& \frac{1}{q}\left\langle \log f_0, \log g; \int F \operatorname{dlog}(1-g)\right\rangle_e \\
& -\frac{1}{q^2}\langle \log g, F\log(1-g)_0; \log\phi^*(f)\rangle_e \\
& -\frac{1}{q^2}\langle \log\phi^*(f), F\log(1-g)_0; \log(g)\rangle_e \\
& +\frac{1}{q^3}\left\langle \log\phi^*(f), \log(g_0); \int F\phi^* \operatorname{dlog}(1-g)\right\rangle_e \Big) \,.
\end{aligned}
$$

Now the last term also agrees with the last term of the right hand side of (8.6) and we are left with verifying the realization of the second wish in the form of

$$
\begin{aligned}
\sum_e \Big(& \langle \log g, F\log(1-g)_0; \log\phi^*(f)\rangle_e \\
& +\langle \log\phi^*(f), F\log(1-g)_0; \log(g)\rangle_e \\
& +\left\langle \log\phi^*(f), \log(g); \int F \operatorname{dlog}(1-g)_0\right\rangle_e \Big) = 0 \,.
\end{aligned}
$$

If the last triple index is replaced by $\langle \log\phi^*(f), \log(g); F\log(1-g)_0\rangle_e$ the result is an immediate consequence of the triple identity, and indeed we have

$$\begin{aligned}&\sum_e \left\langle \log\phi^*(f), \log(g); \int F\,\mathrm{dlog}(1-g)_0 \right\rangle_e \\ = &\sum_e \langle \log\phi^*(f), \log(g); F\log(1-g)_0\rangle_e \\ &-\sum_e \left\langle \log\phi^*(f), \log(g); \int \log(1-g)_0 dF \right\rangle_e ,\end{aligned}$$

and the last sum is 0 by Proposition 7.4. □

Proposition 8.8. *Let G be such that $dG \in \Omega^1(U)$ and G holomorphic on annuli ends. Then, with the notation of Proposition 8.5, we have*

$$\mathcal{T}(g,f,\phi^*G)_e = \left\langle \log(f), \log(g); \int (\phi^* - \frac{1}{q^2}) G\,\mathrm{dlog}(1-g) \right\rangle_e .$$

Proof. Let $F = \phi^*G$. We replace in (8.7) each term of the form h_0 by $q\log(h) - \log\phi^*(h)$. Then we get

$$\begin{aligned}\mathcal{T}(g,f,F)_e =&\frac{1}{q}\left\langle q\log(f) - \log\phi^*(f), \log g; \int F\,\mathrm{dlog}(1-g) \right\rangle_e \\ &+\frac{1}{q^2}\left\langle \log\phi^*(f), \log(g); q\int F\,\mathrm{dlog}(1-g) - \int F d\log\phi^*(1-g) \right\rangle_e \\ &+\frac{1}{q^3}\left\langle \log\phi^*(f), q\log(g) - \log\phi^*(g); \int F\phi^*\,\mathrm{dlog}(1-g) \right\rangle_e\end{aligned}$$

which after some cancelations gives

$$\begin{aligned}= &\left\langle \log(f), \log(g); \int F\,\mathrm{dlog}(1-g) \right\rangle_e \\ &-\frac{1}{q^3}\left\langle \log\phi^*(f), \log\phi^*(g); \int F d\log\phi^*(1-g) \right\rangle_e\end{aligned}$$

and after substituting ϕ^*G for F

$$\begin{aligned}&=\left\langle \log(f), \log(g); \int \phi^*G\,\mathrm{dlog}(1-g) \right\rangle_e \\ &-\frac{1}{q^2}\left\langle \log(f), \log(g); \int G\,\mathrm{dlog}(1-g) \right\rangle_e \\ = &\left\langle \log(f), \log(g); \int (\phi^* - \frac{1}{q^2}) G\,\mathrm{dlog}(1-g) \right\rangle_e\end{aligned}$$

as required. □

Theorem 8.9. *1. Suppose that an element $\beta \in K_4^{(3)}(\mathcal{C}^{\mathrm{loc}})$ maps to $\sum_i [g_i]_2 \cup f_i$ in $H^1(\mathcal{C}^\bullet(\mathcal{O}))$ under the natural map*

$$K_4^{(3)}(\mathcal{C}^{\mathrm{loc}}) \to K_4^{(3)}(\mathcal{O}) \to K_4^{(3)}(\mathcal{O})/K_3^{(2)}(\mathcal{O}) \cup \mathcal{O}_{\mathbb{Q}}^* , \tag{8.10}$$

see (2.62)*, and that* $ch(\beta) = [\eta] \in \tilde{H}^2_{\mathrm{ms}}(\mathcal{C}^{\mathrm{loc}}, 3)$ *in the model* (4.7)*. Let* ω *in* $\Omega^1(U)$ *have trivial residues along all annuli ends of* U*. Then*

$$\langle F_\omega, F_\eta \rangle_{\mathrm{gl}} = \sum_i \sum_e \left\langle \log(f_i), \log(g_i); \int F_\omega \operatorname{dlog}(1-g_i) \right\rangle_e ,$$

where F_ω *and* F_η *are any Coleman integrals of* ω *and* η *respectively.*

2. In particular, The composed map

$$K_4^{(3)}(\mathcal{C}^{\mathrm{loc}}) \xrightarrow{ch} \tilde{H}^2_{\mathrm{ms}}(\mathcal{C}^{\mathrm{loc}}, 3) \xrightarrow{[\eta] \mapsto \langle F_\omega, F_\eta \rangle_{\mathrm{gl}}} \mathbb{C}_p \tag{8.11}$$

factors via (8.10)

Proof. First one easily checks that the validity of the formula depends only on the cohomology class of ω. Since the operator $\phi^* - 1/q^2$ is invertible on $H^1(U)$ we can assume that $\omega = (\phi^* - 1/q^2)\mu$ with μ in $\Omega^1(U)$ and that $F_\omega = (\phi^* - 1/q^2)G$ with G a Coleman integral of μ. Notice that G satisfies the condition of Proposition 8.8. Let η_0 be $ch(\beta) \in \tilde{H}^2_{\mathrm{ms}}(\mathcal{C}^{\mathrm{loc}}, 3)$ in the model (4.6) so that by (4.7) we have $\eta_0 = (1 - \phi^*/q^3)\eta$. We can take the Coleman integral of η_0 to be $F_{\eta_0} = (1 - \phi^*/q^3)F_\eta$. Let $F = \phi^* G$. By Proposition 8.4 the functional $L_\omega(\eta) = \ll F, \eta \gg$ is good in the sense of Definition 4.18. It follows that we may apply Proposition 4.19 to obtain

$$\begin{aligned}
\ll F, \eta_0 \gg &= \sum_i \ll F, \int_0^\infty \epsilon(g_i, f_i) \gg \\
&= \sum_i \sum_e \mathcal{T}(g_i, f_i, F)_e \quad \text{by Proposition 8.5} \\
&= \sum_i \sum_e \left\langle \log(f), \log(g); \int (\phi^* - \frac{1}{q^2}) G \operatorname{dlog}(1-g) \right\rangle_e
\end{aligned}$$

by Proposition 8.8. On the other hand, we have

$$\begin{aligned}
\ll F, \eta_0 \gg &= \langle F, F_{\eta_0} \rangle_{\mathrm{gl}} \\
&= \left\langle F, (1 - \frac{\phi^*}{q^3}) F_\eta \right\rangle_{\mathrm{gl}} \\
&= \left\langle \phi^* G, F_\eta - \frac{\phi^*}{q^3} F_\eta \right\rangle_{\mathrm{gl}} \\
&= \langle \phi^* G, F_\eta \rangle_{\mathrm{gl}} - \left\langle \frac{1}{q^2} G, F_\eta \right\rangle_{\mathrm{gl}} \\
&= \left\langle (\phi^* - \frac{1}{q^2}) G, F_\eta \right\rangle_{\mathrm{gl}} ,
\end{aligned}$$

so our formula was proved with $(\phi^* - \frac{1}{q^2})G$ as required. ☐

Proof of Theorem 1.10. This is now an easy consequence of Theorem 8.9. Suppose $\beta \in K_4^{(3)}(\mathcal{C}^{\mathrm{loc}})$ is the localization of $\alpha \in K_4^{(3)}(\mathcal{C})$. Then $[\eta] = ch(\beta)$ is the restriction of $ch(\alpha)$ and we may therefore assume that η extends to a form of the second kind on C. It follows from [Bes00b, Proposition 4.10] that $\langle F_\omega, F_\eta \rangle_{\mathrm{gl}} = [\omega] \cup [\eta]$. Thus, composing (8.11) with the map $K_4^{(3)}(C) \xrightarrow{\sim} K_4^{(3)}(\mathcal{C}) \to K_4^{(3)}(\mathcal{C}^{\mathrm{loc}})$ precisely gives us the map (1.11) from Theorem 1.10. Both the factorization and the explicit formula now follow from Theorem 8.9. ☐

Remark 8.12. As we have the map $H^2(\mathcal{M}_{(3)}(\mathcal{O})) \to H^1(\mathcal{C}^\bullet(\mathcal{O}))$ given by mapping $[g]_2 \otimes f$ to $[g]_2 \cup (f)$, see (2.65), Theorem 8.9 gives us a formula for the regulator on $H^2(\mathcal{M}_{(3)}(\mathcal{O}))$ as well, see (1.15).

9. End of the proofs

In this Section we prove Theorems 1.5 and 1.3. To this end, we let ω be a holomorphic form on C, and we let $\sum_i [g_i]_2 \otimes f_i$ be an element of $H^2(\mathcal{M}_{(3)}(\mathcal{O}))$.

First we rewrite the formula in Theorem 8.9 in this case. Following the argument in the proof of [Bes00b, Proposition 5.5] using Proposition 7.4 to replace the sum over annuli ends by a sum over points, we obtain

$$\sum_e \langle F_\eta, F_\omega \rangle_e = \sum_i \sum_{x \in C} \left\langle \log(f_i), \log(g_i); \int F_\omega \operatorname{dlog}(1-g_i) \right\rangle_x .$$

We again extend scalars to $\mathbb{C}_p$, so in particular points are $\mathbb{C}_p$ valued. Fix a local parameter at each point x, which we will call z_x, or, whenever there is no cause for confusion, simply z. Consider a single point x in C. We recall that with respect to the local parameter z at x we define, for a rational function f, $\bar{f}(x) = (f/z^{\operatorname{ord}_x(f)})(x)$. For such a function f we have $c_z(\log(f)) = \log(\bar{f}(x))$. We also have $\operatorname{Res}_x(F_\omega \operatorname{dlog}(f)) = \operatorname{ord}_x(f) \cdot F_\omega(x)$. Thus, using Proposition 6.8, we obtain

$$\begin{aligned} \sum_e \langle F_\eta, F_\omega \rangle_e = \sum_i \sum_{x \in C} \Big[& \operatorname{ord}_x(1-g_i) F_\omega(x) \log \bar{f}_i(x) \log \bar{g}_i(x) \\ & - \operatorname{ord}_x(f_i) c_z \left(\int \log(g_i) F_\omega \operatorname{dlog}(1-g_i) \right) \\ & - \operatorname{ord}_x(g_i) c_z \left(\int \log(f_i) F_\omega \operatorname{dlog}(1-g_i) \right) \Big] . \end{aligned} \tag{9.1}$$

Let A (respectively B) be the subgroup of $k(C)^*$ generated by the f_i and g_i (respectively by the $1-g_i$). By choosing bases for A and B and then choosing appropriate integrals we can arrange it so that for each f in A and h in B an integral $\int \log(f) F_\omega \operatorname{dlog} h$ is chosen such that the map $(f,h) \to \int \log(f) F_\omega \operatorname{dlog} h$ is bilinear. Since the overall sum in (9.1) is independent of the choice of integrals, we may and do assume from now on that the integrals there are chosen as above.

Lemma 9.2. *For every x in C we have*

$$\sum_i \operatorname{ord}_x(f_i) c_z \left(\int \log(g_i) F_\omega \operatorname{dlog}(1-g_i) \right) = \sum_i \operatorname{ord}_x(g_i) c_z \left(\int \log(f_i) F_\omega \operatorname{dlog}(1-g_i) \right) .$$

Proof. With the choices above the map

$$(f,g,h) \to \operatorname{ord}_x(f) c_z (\int \log(g) F_\omega \operatorname{dlog}(h)) - \operatorname{ord}_x(g) c_z (\int \log(f) F_\omega \operatorname{dlog}(h))$$

is trilinear and antisymmetric with respect to f and g. The Lemma follows since $\sum (1-g_i) \otimes (g_i \wedge f_i) = 0$. □

We recall that the function $L_2(x)$ is defined by $L_2(x) = \operatorname{Li}_2(x) + \log(x)\log(1-x)$ and that we have $dL_2(x) = \log(x) \operatorname{dlog}(1-x)$. Note that this last form is holomorphic in the residue disc of 1 and as a consequence so is $L_2(x)$.

Lemma 9.3. *Let g be a rational function. The constant term at x of $L_2(g)$ equals $L_2(g(x))$ if $g(x) \neq 0, \infty$, equals 0 if $g(x) = 0$ and equals $\log^2(\bar{g}(x))/2$ if $g(x) = \infty$, where $\bar{g}$ is computed with respect to the same local parameter as the constant term. In addition, the expansion of $L_2(g)$ with respect to any local parameter contains no summands of the form $Const \cdot z^n$ with $n < 0$.*

Proof. This is clear if $g(x) \neq 0, \infty$. Suppose $g(x) = 0$. Since Li_2 is holomorphic near 0 and has value 0 there, we see that the constant term and terms of the form z^n for $n < 0$ are the same as in $\log(g)\log(1-g)$. Near x we have $\log(g(z)) = \mathrm{ord}_x(g)\log(z) +$ a holomorphic function in z. Also, $\log(1-g)$ is holomorphic near x with value 0 there. Thus the result is clear. Finally, by [Col82, Proposition 6.4], we have $L_2(g) + L_2(1/g) = \log^2(g)/2$ so the result at $g(x) = \infty$ is deduced from that of $1/g$ when $g(x) = \infty$. □

Lemma 9.4. *For any point x in C and for any choice of a Coleman integral $\int L_2(g)\omega$ the quantity $c_z(\int L_2(g)\omega)$ is independent of the choice of the local parameter z at x.*

Proof. Let f_ω be the unique Coleman integral of ω that vanishes at x. We may choose a Coleman integral $\int f_\omega dL_2(g)$ in such a way that the integration by parts formula

$$\int L_2(g)\omega = L_2(g)f_\omega - \int f_\omega dL_2(g)$$

holds. It is therefore sufficient to show that the constant term of each of the summands on the right is independent of the parameter. From the last assertion in Lemma 9.3 and the fact that $f_\omega(x) = 0$ it is easy to see that the constant term of the first summand is 0. For the second summand we have

$$\begin{aligned}\int f_\omega dL_2(g) &= \int f_\omega \log(g)\,\mathrm{dlog}(1-g)\\ &= \log(g)\int f_\omega\,\mathrm{dlog}(1-g) - \int\Big(\int f_\omega\,\mathrm{dlog}(1-g)\Big)\,\mathrm{dlog}(g)\end{aligned}$$

for appropriate choices of integrals. As $f_\omega\,\mathrm{dlog}(1-g)$ is holomorphic at x, we may arrange it so that $\int f_\omega\,\mathrm{dlog}(1-g)$ vanishes at x. Then in the last formula the first term has constant term 0 while the second term is holomorphic at x hence its constant term is independent of z. □

Using the last Lemma we may set

$$\int L_2(g)\omega|_x := c_z\left(\int L_2(g)\omega\right)$$

with respect to any parameter z at x. Using this we can define $\int_D L_2(g)\omega$ for any divisor D of degree zero. If we change $\int L_2(g)\omega$ by a constant, its value at x in the above sense will change by the same constant. Thus when D has degree 0 the integral $\int_D L_2(g)\omega$ does not depend on the constant of integration even if D and the divisor of g have a common support. This explains the general definition of the integral in Theorem 1.5.

Lemma 9.5. *Choose integrals such that the integration by parts formula*

$$\int \log(g)F_\omega\,\mathrm{dlog}(1-g) = F_\omega L_2(g) - \int L_2(g)\omega$$

is satisfied. Then we have at a point x and with respect to the local parameter z,

$$c_z(\int \log(g)F_\omega \operatorname{dlog}(1-g)) = F_\omega(x)c_z(L_2(g)) - \int L_2(g)\omega|_x$$

Proof. One just applies c_z to the integration by parts formula and observes that by Lemma 9.3 we have

$$c_z(F_\omega L_2(g)) = F_\omega(x)c_z(L_2(g))\ .$$

□

Proof of Theorem 1.5. Consider (9.1). By Lemma 9.2 we can choose our integrals such that for each point x the sum over i of each of the last two terms is identical. The term $\operatorname{ord}_x(f_i)c_z(\int \log(g_i)F_\omega \operatorname{dlog}(1-g_i))$ is computed in Lemma 9.5 and Lemma 9.3. Substituting the results we see that we have the equation

$$\begin{aligned}\sum_e \langle F_\eta, F_\omega\rangle_e = \sum_i \Big[&\sum_{x\in C}(\operatorname{ord}_x(1-g_i)F_\omega(x)\log \bar{f}_i(x)\log \bar{g}_i(x)) \\ &+2\int_{(f_i)} L_2(g_i))\omega \\ &-2\sum_{x\in C}\operatorname{ord}_x(f_i)F_\omega(x)\times \begin{cases}0 & g_i(x)=0\\ L_2(g_i(x)) & g(x)\neq 0,\infty\\ \log^2(\bar{g}_i)(x))/2 & g_i(x)=\infty\end{cases}\Big]\ .\end{aligned}$$

In the first sum over x only terms with $g_i(x)=\infty$ can be non zero. Thus there is never a contribution from x where $g_i(x)=0$ and the right hand side becomes

(9.6)
$$\sum_i \left[2\int_{(f_i)} L_2(g_i)\omega - 2\sum_{g_i(x)\neq 0,\infty}\operatorname{ord}_x(f_i)F_\omega(x)L_2(g_i(x)) + \sum_{g_i(x)=\infty}F_\omega(x)\alpha_x(f_i,g_i)\right],$$

with

$$\begin{aligned}\alpha_x(f,g) &= \operatorname{ord}_x(1-g)\log \bar{f}(x)\log \bar{g}(x) - \operatorname{ord}_x(f)\log^2 \bar{g}(x)\\ &= \log \bar{g}(x)\left(\operatorname{ord}_x(1-g)\log \bar{f}(x) - \operatorname{ord}_x(f)\log \bar{g}(x)\right)\end{aligned}$$

which, since $g(x)=\infty$ hence $\operatorname{ord}_x(1-g)=\operatorname{ord}_x(g)$ and $\bar{g}(x) = -\overline{1-g}(x)$, can also be written as

$$= \log \overline{1-g}(x)\left(\operatorname{ord}_x(g)\log \bar{f}(x) - \operatorname{ord}_x(f)\log \bar{g}(x)\right)\ .$$

For each x in C, the expression $\beta_x(f,g,h) := \log \overline{h}(x)\left(\operatorname{ord}_x(g)\log \bar{f}(x) - \operatorname{ord}_x(f)\log \bar{g}(x)\right)$ is trilinear in f, g and h and antisymmetric in f and g. Thus, again because $\sum_i (1-g_i)\otimes(g_i\wedge f_i)\otimes = 0$ (see (2.26)), we see that

$$\sum_i \beta_x(f_i,g_i,1-g_i) = 0\ . \tag{9.7}$$

If $g_i(x)=0$ then $\beta_x(f_i,g_i,1-g_i)=0$ while if $g(x)\neq 0,\infty$ then $\beta_x(f_i,g_i,1-g_i) = -\operatorname{ord}_x(f_i)\log g_i(x)\log(1-g_i(x))$, where we set the value of $\log(x)\log(1-x)$ at 1 to be 0, which is consistent with taking limits and with what follows. Thus, summing (9.7) multiplied by $F_\omega(x)$ over all x in C we see that

$$\sum_i \sum_{g_i(x)=\infty} F_\omega(x)\alpha_x(f_i,g_i) = \sum_i \sum_{g_i(x)\neq 0,\infty}\operatorname{ord}_x(f_i)F_\omega(x)\log g_i(x)\log(1-g_i(x))\ .$$

Substituting this in (9.6) we obtain

$$\begin{aligned}\sum_e \langle F_\eta, F_\omega\rangle_e = 2\sum_i \Big[\int_{(f_i)} & L_2(g_i)\omega \\ & - \sum_{g_i(x)\neq 0,\infty} \mathrm{ord}_x(f_i) F_\omega(x) \left(L_2(g_i(x)) - \frac{1}{2}\log g_i(x)\log(1-g_i(x))\right)\Big] ,\end{aligned}$$

which, as $L_2(z) - \log(z)\log(1-z)/2 = L_{\mathrm{mod},2}(z)$ by definition, becomes

$$\sum_e \langle F_\eta, F_\omega\rangle_e = 2\sum_i \int_{(f_i)} L_2(g_i)\omega - 2\sum_i \sum_{g_i(x)\neq 0,\infty} \mathrm{ord}_x(f_i) F_\omega(x) L_{\mathrm{mod},2}(g_i(x)) .$$

This formula finishes the proof of Theorem 1.3. In order to finish the proof of Theorem 1.5, we now only have to show that, if $\sum_i [g_i]_2 \otimes f_i$ is in $H^2(\mathcal{M}_{(3)}(\mathcal{C}))$ rather than just in $H^2(\mathcal{M}_{(3)}(\mathcal{O}))$, and $F = k(C)$ with k a number field, then for every closed point x in C,

$$\sum_i \mathrm{ord}_x(f_i) F_\omega(x) L_{\mathrm{mod},2}(g_i(x))$$

equals zero. For this we use that we have an element in $H^2(\mathcal{M}_{(3)}(\mathcal{C}))$ rather than just $H^2(\mathcal{M}_{(3)}(\mathcal{O}))$, so that (2.54) holds for $\sum_i [g_i]_2 \otimes f_i$, i.e.,

$$\sum_i \mathrm{ord}_x(f_i)[g_i(x)]_2$$

is zero in $H^1(\widetilde{\mathcal{M}}_{(2)}(k(x))) \subset \widetilde{M}_2(k(x))$, with the convention that $[0]_2 = [1]_2 = [\infty]_2 = 0$.

$F_\omega(x)$ is just a constant, so we only have to show that for any number field $L \subset K$ (like $k(x)$ as we are assuming all of the f_i and g_i are defined over the fixed number field k), the map

$$\begin{aligned} H^1(\widetilde{\mathcal{M}}_{(2)}(L)) &\to K \\ \sum_i [a_i]_2 &\mapsto \sum_i L_{\mathrm{mod},2}(a_i) \end{aligned}$$

is well defined. It is conjectured in [BdJ03, Conjecture 1.14] that this map is the syntomic regulator map, as composition (with $\mathcal{O}_L$ the ring of integers in L)

$$H^1(\widetilde{\mathcal{M}}_{(2)}(L)) \to K_3^{(2)}(L) \cong K_3^{(2)}(\mathcal{O}_L) \to H^1_{\mathrm{syn}}(\mathcal{O}_L, 2) \cong K ,$$

which would of course imply what we need. But using other results, we can show that the map

$$\begin{aligned} \widetilde{M}_2(L) &\to K \\ [a]_2 &\mapsto L_{\mathrm{mod},2}(a) \end{aligned}$$

is well defined, which will prove what we want.

Namely for any field L of characteristic zero, let $B_2'(L)$ be the free $\mathbb{Q}$-vector space on elements $\{b\}_2$ with b in F, $b \neq 0, 1$, modulo the five term relation

$$\{b\}_2 + \{c\}_2 + \left\{\frac{1-b}{1-bc}\right\}_2 + \{1-bc\}_2 + \left\{\frac{1-c}{1-bc}\right\}_2 = 0 . \tag{9.8}$$

It is shown in [dJ00, Lemma 5.2] that there is a map $B_2'(L) \to \widetilde{M}_2(L)$, given by sending $\{b\}_2$ to $[b]_2$. In case L is a number field, this was already done on page 240 of [dJ95] (where the relations were not made explicit and the group was called $B_2(L)$), and the map was shown to be an isomorphism in that case. Finally, in [Col82, Corollaries 6.4(ii),(iii) and 6.5b] Coleman shows that $L_{\mathrm{mod},2}$ (which is called D there) satisfies

$$L_{\mathrm{mod},2}(z^{-1}) = -L_{\mathrm{mod},2}(z)$$
$$L_{\mathrm{mod},2}(1-z) = -L_{\mathrm{mod},2}(z)$$

as well as (with signs corrected)

$$L_{\mathrm{mod},2}(xy) = L_{\mathrm{mod},2}(x)+L_{\mathrm{mod},2}(y)+L_{\mathrm{mod},2}\left(\frac{x}{x-1}(1-y)\right)+L_{\mathrm{mod},2}\left(\frac{y}{y-1}(1-x)\right) .$$

Substituting $x=(bc)^{-1}$, $y=c$ in the last relation and using the first two, one sees that $L_{\mathrm{mod},2}$ satisfies the relation corresponding to (9.8). Therefore it induces a map

$$\widetilde{M}_2(L) \cong B_2'(L) \to K$$

mapping $[b]_2$ to $L_{\mathrm{mod},2}(b)$. This finishes the proof of Theorem 1.5. ☐

References

[BdJ03] A. Besser and R. de Jeu. The syntomic regulator for the K-theory of fields. *Ann. Sci. Ecole Norm. Sup. (4)*, 36(6):867–924, 2003.

[Bes00a] A. Besser. Syntomic regulators and p-adic integration I: rigid syntomic regulators. *Israel Journal of Math.*, 120:291–334, 2000.

[Bes00b] A. Besser. Syntomic regulators and p-adic integration II: K_2 of curves. *Israel Journal of Math.*, 120:335–360, 2000.

[Bes02] A. Besser. Coleman integration using the Tannakian formalism. *Math. Ann.*, 322(1):19–48, 2002.

[Blo90] S. Bloch, November 1990. Letter to C. Deninger.

[Blo00] Spencer J. Bloch. *Higher regulators, algebraic K-theory, and zeta functions of elliptic curves*, volume 11 of *CRM Monograph Series*. American Mathematical Society, Providence, RI, 2000.

[CdS88] R. Coleman and E. de Shalit. p-adic regulators on curves and special values of p-adic L-functions. *Invent. Math.*, 93(2):239–266, 1988.

[Col82] R. Coleman. Dilogarithms, regulators, and p-adic L-functions. *Invent. math.*, 69:171–208, 1982.

[Col85] R. Coleman. Torsion points on curves and p-adic abelian integrals. *Annals of Math.*, 121:111–168, 1985.

[dJ95] R. de Jeu. Zagier's conjecture and wedge complexes in algebraic K-theory. *Compositio Math.*, 96(2):197–247, 1995.

[dJ96] R. de Jeu. On $K_4^{(3)}$ of curves over number fields. *Invent. Math.*, 125(3):523–556, 1996.

[dJ00] R. de Jeu. Towards regulator formulae for the K-theory of curves over number fields. *Compositio Math.*, 124(2):137–194, 2000.

[GK00] E. Grosse-Klönne. Rigid analytic spaces with overconvergent structure sheaf. *J. Reine Angew. Math.*, 519:73–95, 2000.

[GS99] H. Gillet and C. Soulé. Filtrations on higher algebraic K-theory. In *Algebraic K-theory (Seattle, WA, 1997)*, volume 67 of *Proc. Sympos. Pure Math.*, pages 89–148. Amer. Math. Soc., Providence, RI, 1999.

[Har77] G. Harder. Die Kohomologie S-arithmetischer Gruppen über Funktionenkörpern. *Invent. Math.*, 42:135–175, 1977.

[Qui73] D. Quillen. Higher algebraic K-theory. I. In *Algebraic K-theory, I: Higher K-theories*, volume 341 of *Lecture Notes in Mathematics*, pages 85–147. Springer, Berlin, 1973.

[Sou85] C. Soulé. Opérations en K-théorie algébrique. *Canad. J. Math.*, 37(3):488–550, 1985.

[vdP86] M. van der Put. The cohomology of Monsky and Washnitzer. *Mém. Soc. Math. France (N.S.)*, (23):4, 33–59, 1986. Introductions aux cohomologies p-adiques (Luminy, 1984).

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