ON THE SYNTOMIC REGULATOR FOR K_1 OF A SURFACE

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ABSTRACT. We consider elements of $K_1(S)$, where S is a proper surface over a p-adic field with good reduction, which are given by a formal sum $\sum(Z_i, f_i)$ with Z_i curves in S and f_i rational functions on the Z_i in such a way that the sum of the divisors of the f_i is 0 on S. Assuming compatibility of pushforwards in syntomic and motivic cohomologies our result computes the syntomic regulator of such an element, interpreted as a functional on $H^2_{\mathrm{dR}}(S)$, when evaluated on the cup product $\omega \cup [\eta]$ of a holomorphic form ω by the first cohomology class of a form of the second kind η. The result is $\sum_i \langle F_\eta, \log(f_i); F_\omega \rangle_{\mathrm{gl}, Z_i}$, where F_ω and F_η are Coleman integrals of ω and η respectively and the symbol in brackets is the global triple index, as defined in our previous work.

1. INTRODUCTION

In this paper we write down a formula for the syntomic regulator on the first algebraic K-group of a smooth complete surface over the ring of integers of a p-adic field. Formulas for the syntomic regulator are to be thought of as p-adic analogues of formulas for the Beilinson regulator [Bei85]. Several such formulas exists [Bes00c, BdJ03, BdJ04] for zero and one-dimensional varieties.

In this paper, motivic cohomology is considered in the sense of Beilinson, i.e., it is an eigenspace of the Adams operations on Algebraic K-theory. More precisely, if S is a proper variety over a field K we have

$$H^{2i-j}_{\mathcal{M}}(S, \mathbb{Q}(i)) \subset K_j(S) \otimes \mathbb{Q} .$$

Soulé defined étale regulator maps,

$$\mathrm{reg}_{\text{ét}} : H^{2i-j}_{\mathcal{M}}(S, \mathbb{Q}(i)) \to H^1(K, H^{2i-j-1}_{\text{ét}}(\bar{S}, \mathbb{Q}_p(i))) .$$

From now on we assume that K is a p-adic field, i.e., a finite extension of the field $\mathbb{Q}_p$ of p-adic numbers, with ring of integers $\mathcal{O}_K$ and residue field κ. Bloch and Kato [BK90] constructed an exponential map

$$\exp : H^{2i-j-1}_{\mathrm{dR}}(S/K)/F^i \to H^1(K, H^{2i-j-1}_{\text{ét}}(\bar{S}, \mathbb{Q}_p(i))) .$$

Suppose now that S is a surface. We want to compute the regulator for elements in the higher Chow group $CH^2(S, 1)$. We recall that these are given as formal sums

$$\theta = \sum (Z_i, f_i) , \tag{1.1}$$

where Z_i are curves in S and the f_i are rational functions on the Z_i with the property that in S we have $\sum \mathrm{div}(f_i) = 0$.

We have an isomorphism $CH^2(S, 1) \otimes \mathbb{Q} \cong H^3_{\mathcal{M}}(S, \mathbb{Q}(2))$. We find

$$\mathrm{reg}_{\text{ét}}(\theta) \in H^1(K, H^2_{\text{ét}}(\bar{S}, \mathbb{Q}_p(2))).$$

Our goal will be to show that under some conditions we can find an element $\operatorname{reg}_{\mathrm{syn}}(\theta) \in H^2_{\mathrm{dR}}(S/K)/F^2$ such that $\exp(\operatorname{reg}_{\mathrm{syn}}(\theta)) = \operatorname{reg}_{\text{ét}}(\theta)$ and to describe explicitly this $\operatorname{reg}_{\mathrm{syn}}(\theta)$.

Our first assumption, which we make from now onward, is that S has a smooth integral model $\mathcal{S}$ over $\mathcal{O}_K$. Our results rely on the work of Niziol [Niz97] (in which S can be of arbitrary dimension), which shows that the integral classes in $K_j(S)$, i.e., those which come from the K-theory of $\mathcal{S}$, map under the étale Chern character $\mathrm{ch}_{\text{ét}}$ to elements in the image of exp. More precisely, one has a commutative diagram

$$\begin{array}{ccc} K_j(\mathcal{S}) & \longrightarrow & K_j(S) \\ \downarrow \mathrm{ch}_{\mathrm{syn}} & & \downarrow \mathrm{ch}_{\text{ét}} \\ H^{2i-j-1}_{\mathrm{dR}}(S/K)/F^i & \xrightarrow{\exp} & H^1(K, H^{2i-j-1}_{\text{ét}}(\bar{S}, \mathbb{Q}_p(i))) \end{array}$$

where $\mathrm{ch}_{\mathrm{syn}}$ is the syntomic Chern character [Gro94, Niz97, Bes00b]. We recall that the regulator is easily derived from the Chern character. Thus, computing the syntomic regulator is essentially the same as computing the étale regulator for integral elements, except that one gets the answer in terms of the usually more tractable de Rham cohomology.

To use this result we therefore need to impose a certain integrality relation on θ.

Assumption 1. *The normalizations X_i of the Z_i admit smooth $\mathcal{O}_K$-models $\mathcal{X}_i$ and their maps to S extend to give maps $g_i : \mathcal{X}_i \to \mathcal{S}$. The functions f_i, viewed as rational functions on the $\mathcal{X}_i$, have divisors which do not contain the special fiber (note that this last condition can easily be achieved by multiplying the f_i by appropriate constants).*

Under this assumption we will indeed find the required

$$\operatorname{reg}_{\mathrm{syn}}(\theta) \in H^2_{\mathrm{dR}}(S/K)/F^2 \cong \operatorname{Hom}(F^1 H^2_{\mathrm{dR}}(S/K), K)$$

where the last isomorphism is given by Poincaré duality. Our formula describes the element $\operatorname{reg}_{\mathrm{syn}}(\theta)$ as a functional on (certain elements of) $F^1 H^2_{\mathrm{dR}}(S/K)$.

In the complex case one has a rather similar situation: Suppose for a moment that $K = \mathbb{C}$. The real Beilinson regulator maps θ to an element of the Deligne cohomology group $H^3_{\mathcal{D}}(S, \mathbb{R}(2))$. Beilinson obtains the following formula for the regulator as a functional on $F^1 H^2_{\mathrm{dR}}(S/\mathbb{C})$.

$$\operatorname{reg}(\theta)(\omega) = \frac{1}{2\pi\sqrt{-1}} \sum \int_{Z_i - Z_i^{\mathrm{sing}}} \omega \log|f_i| \ .$$

In the p-adic case we do not have a good notion of an integral over a Riemann surface, so the formula looks somewhat different. To begin with, we only describe the values of the functional $\operatorname{reg}_{\mathrm{syn}}(\theta)$ on decomposable elements $\mu = \omega \cup [\eta]$, where $\omega \in F^1 H^1_{\mathrm{dR}}(S/K)$ is a holomorphic form on S and $[\eta] \in H^1_{\mathrm{dR}}(S/K)$ is the class of a form of the second kind η on S.

To explain the formula we arrive at, we need to recall briefly the theory of Coleman integration and the theory of local indices (see Section 2 for more details). Coleman integration is a way of assigning in a canonical up to constant and functorial way a locally analytic primitive to differential forms on varieties over p-adic fields. In particular, we may choose for ω and η Coleman integrals F_ω and F_η respectively. We note that the theory is functorial so that we may pullback Coleman functions with respect to morphisms of $\mathcal{O}_K$-schemes.

Local indices [Bes00c, BdJ04] are a generalization of the notion of residue. Consider the field of Laurent series $K((z))$ and the polynomial ring $K((z))[\log(z)]$ in the formal variable $\log(z)$, which admits a differential by setting $d(\log z) = dz/z$. The triple index associates to 3 polynomials of degree at most 1 in $K((z))[\log(z)]$ with a constant linear coefficient, F, G, H, together with some auxiliary data, a formal quantity $\langle F, G; H\rangle$ which among other things equals $\operatorname{Res} FGdH$ when this expression makes sense (i.e., when no logs are involved). For further details see Section 2.

Suppose now that F, G and H are Coleman integrals of meromorphic differential forms on a curve X above K. Near each closed point of X these functions are of the type discussed above so that a triple index $\langle F, G; H\rangle$ may be defined. While this index depends on certain auxiliary data it turns out that by using Coleman integration to consistently choose the auxiliary data, the sum of these indices over all points of X, which we denote $\langle F, G; H\rangle_{\mathrm{gl}}$, does not.

We can finally state our main Theorem. We need to assume a conjecture, Conjecture 4.2, about compatibility of pushforward maps in syntomic and motivic cohomology. This conjecture will likely be settled in future work of Chiarellotto and coauthors, or by future work of the author.

Theorem 1.1. *Assume Conjecture 4.2 holds true. Let $\mathcal{S}$ be a proper smooth surface over the ring of integers $\mathcal{O}_K$ of the p-adic field K with generic fiber S and let θ be the element $\theta = \sum(Z_i, f_i) \in CH^2(S, 1)$ satisfying the integrality Assumption 1. Then, there exists an element $\operatorname{reg}_{\mathrm{syn}}(\theta) \in H^2_{\mathrm{dR}}(S/K)/F^2$, mapping to $\operatorname{reg}_{\text{ét}}(\theta)$ under $\exp$, such that, viewed as a functional on $F^1H^2_{\mathrm{dR}}(S/K)$ and evaluated on the element $\omega \cup [\eta]$ with $\omega \in F^1H^1_{\mathrm{dR}}(S/K)$ and $[\eta] \in H^1_{\mathrm{dR}}(S/K)$ represented by the form of the second kind η, it gives*

$$\operatorname{reg}_{\mathrm{syn}}(\theta)(\omega \cup [\eta]) = \sum_i \langle g_i^* F_\eta, \log(f_i); g_i^* F_\omega\rangle_{\mathrm{gl}, X_i}$$

where $g_i : X_i \to S$ is the normalization of Z_i, the global triple index is computed on the rigid analytic curve associated with X_i and the two functions F_η and F_ω are Coleman integrals of η and ω respectively on S.

It may be asked what has been gained by replacing the mysterious syntomic regulator by the mysterious Coleman integration and triple index. A possible answer is that the latter are somewhat less mysterious than the former. The triple indexes are rather easy to compute. Coleman integrals are in general rather difficult to compute. However, we now have a way of numerically computing them on hyperelliptic curves [Gut06, BBK10] and they furthermore enjoy functorial properties that may help compute them, e.g., by using Hecke correspondences on modular curves. On the other hand, we know of no direct method for computing the syntomic regulator for the type of elements we are considering here.

The results of this paper are used by Langer [Lan10] for constructing elements in $CH^2(E\times E, 1)$ which are indecomposable but their syntomic cohomology classes are decomposable.

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2. Preliminaries

In this section we collect various preliminary materials required for the rest of the text. The references are [Bes00a, Bes00b, Bes00c, BdJ04].

We begin by describing syntomic cohomology, finite polynomial cohomology and the relation between the two. To be precise, we will describe the version of syntomic cohomology introduced in [Bes00b] under the title "modified syntomic cohomology". This has the advantage of being easier to describe and work with. Although ultimately one is interested in the true syntomic cohomology, the two are often the same (see [Bes00b, Proposition 8.6.3]).

Let $\mathcal{X}$ be a smooth scheme over $\mathcal{O}_K$ with generic fiber X and special fiber $\mathcal{X}_s$. In [Bes00b, Sections 4-5] we associated with $\mathcal{X}$ complexes $\mathbb{R}\Gamma_{\mathrm{rig}}(\mathcal{X}_s/K)$ and $\mathbb{R}\Gamma_{\mathrm{dR}}(X/K)$ together with complexes $F^n\mathbb{R}\Gamma_{\mathrm{dR}}(X/K)$ for every non-negative integer n. These complexes are functorial with respect to morphisms of κ (respectively K) schemes. Their cohomologies give respectively the rigid cohomology (in the sense of Berthelot [Ber97b]), $H_{\mathrm{rig}}(\mathcal{X}_s/K)$, the de Rham cohomology, $H_{\mathrm{dR}}(X/K)$, and the pieces of its Hodge filtration. There are canonical maps $F^n\mathbb{R}\Gamma_{\mathrm{dR}} \to \mathbb{R}\Gamma_{\mathrm{dR}}$ and also a *specialization map* [BB04], functorial in $\mathcal{X}$,

$$\mathrm{sp}: \mathbb{R}\Gamma_{\mathrm{dR}}(X_K/K) \to \mathbb{R}\Gamma_{\mathrm{rig}}(\mathcal{X}_s/K)\ . \tag{2.1}$$

We now fix a *Frobenius endomorphism* φ of $\mathcal{X}_s$ [Bes00b, Definition 8.1], i.e., a k-linear endomorphism obtained by extension of scalars from an r-th power of the absolute Frobenius over a model of $\mathcal{X}_s$ defined over a finite field for some r. The number $q = p^r$ is then called the degree of φ.

Definition 2.1. For every polynomial P with coefficients in $\mathbb{Q}$ we define the *syntomic P-complex* of $\mathcal{X}$ to be

$$\mathbb{R}\Gamma_{\mathrm{f},P}(\mathcal{X},n) := \mathrm{MF}\left(F^n\mathbb{R}\Gamma_{\mathrm{dR}}(X/K) \xrightarrow{P(\varphi^*)} \mathbb{R}\Gamma_{\mathrm{rig}}(\mathcal{X}_s/K)\right)\ , \tag{2.2}$$

where the notation $P(\varphi^*)$ is a shorthand for the composition

$$F^n\mathbb{R}\Gamma_{\mathrm{dR}}(X/K) \to \mathbb{R}\Gamma_{\mathrm{dR}}(X/K) \xrightarrow{\mathrm{sp}} \mathbb{R}\Gamma_{\mathrm{rig}}(\mathcal{X}_s/K) \xrightarrow{P(\varphi^*)} \mathbb{R}\Gamma_{\mathrm{rig}}(\mathcal{X}_s/K)\ .$$

The i-th cohomology of $\mathbb{R}\Gamma_{\mathrm{f},P}(\mathcal{X},n)$ will be denoted by $H^i_{\mathrm{f},P}(\mathcal{X},n)$.

To explain how these cohomologies are going to be used, let us begin by observing that from the long exact sequence associated with the Cone(see [Bes00a, (12)] we have the following short exact sequence

$$0 \to H^{i-1}_{\mathrm{rig}}(\mathcal{X}_s/K)/P(\varphi^*)(F^nH^{i-1}_{\mathrm{dR}}(X/K)) \to H^i_{\mathrm{f},P}(\mathcal{X},n) \to F^nH^i_{\mathrm{dR}}(X/K)^{P(\varphi^*)=0} \to 0\ . \tag{2.3}$$

In [Bes00a, Definition 2.3] it is explained how, when $P|Q$, there exists a map $\mathbb{R}\Gamma_{\mathrm{f},P}(\mathcal{X},n) \to \mathbb{R}\Gamma_{\mathrm{f},Q}(\mathcal{X},n)$ such that the induced map on cohomology fits into the following commutative diagram

(2.4)
$$\begin{array}{ccccc}
H^{i-1}_{\mathrm{rig}}(\mathcal{X}_s/K)/P(\varphi^*)(F^n H^{i-1}_{\mathrm{dR}}(X/K)) & \longrightarrow & H^i_{\mathrm{f},P}(\mathcal{X},n) & \longrightarrow & F^n H^i_{\mathrm{dR}}(X/K)^{P(\varphi^*)=0} \\
\downarrow{\scriptstyle Q/P(\varphi^*)} & & \downarrow & & \downarrow \\
H^{i-1}_{\mathrm{rig}}(\mathcal{X}_s/K)/Q(\varphi^*)(F^n H^{i-1}_{\mathrm{dR}}(X/K)) & \longrightarrow & H^i_{\mathrm{f},Q}(\mathcal{X},n) & \longrightarrow & F^n H^i_{\mathrm{dR}}(X/K)^{Q(\varphi^*)=0}
\end{array}$$

Let us now consider the special case of the polynomial $P_i(t) = 1 - t^i/q^{ni}$. For $P = P_1$ the two extreme terms in the short exact sequence 2.3 can be identified with certain extension groups as follows: Let $\mathrm{MF}^{\mathrm{f}}_K$ be the category whose objects are triples (V, V', ι) where V and V' are vector spaces over K, V is filtered, V' has an endomorphism, and $\iota : V \to V'$ is a map which need not be an isomorphism in general. The definition of morphisms is clear. For each smooth $\mathcal{O}_K$-scheme $\mathcal{X}$ and index i the triple consisting of $H^i_{\mathrm{dR}}(X/K)$ with its filtration, $H^i_{\mathrm{rig}}(\mathcal{X}_s/K)$ with its Frobenius and the specialization map form an object of this category. Let $\mathbf{1}$ be the obvious trivial one dimensional object in $\mathrm{MF}^{\mathrm{f}}_K$. Twisting in $\mathrm{MF}^{\mathrm{f}}_K$ consists of shifting the filtration and dividing the Frobenius by a power of q. Then,

$$F^n H^i_{\mathrm{dR}}(X/K)^{\varphi^*=q^n} = \mathrm{Ext}^0_{\mathrm{MF}^{\mathrm{f}}_K}(\mathbf{1}, H^i(\mathcal{X})(n))$$
$$H^{i-1}_{\mathrm{rig}}(\mathcal{X}_s/K)/1-\varphi^*/q^n)(F^n H^{i-1}_{\mathrm{dR}}(X/K)) = \mathrm{Ext}^1_{\mathrm{MF}^{\mathrm{f}}_K}(\mathbf{1}, H^{i-1}(\mathcal{X})(n)) \ .$$

Deligne cohomology, or rather the more refined absolute Hodge cohomology of Beilinson, behaves in exactly the same way with respect to the category of mixed Hodge structures. The cohomology theory H_{f,P_1} is therefore a p-adic version of Deligne cohomology.

Remark 2.2. To make a better version of p-adic Deligne cohomology one should refine the above construction. One thing is certainly to replace the linear Frobenius by a semi-linear one. This leads to the definition of syntomic cohomology [Bes00b, Definition 6.1]. There should be even more refined versions, but they get progressively harder to work with.

Definition 2.3. Order $\mathbb{Q}_p[t]$ by division and let $\mathcal{P}$ be a directed subset of $\mathbb{Q}_p[t]$. Then we define

$$\mathbb{R}\Gamma_{\mathcal{P}}(\mathcal{X},n) := \varinjlim_{P\in\mathcal{P}} \mathbb{R}\Gamma_{\mathrm{f},P}(\mathcal{X},n) \ ,$$

with associated cohomology $H^i_{\mathcal{P}}(\mathcal{X},n)$.

From the commutative diagram (2.4) we get the short exact sequence

(2.5) $$H^{i-1}_{\mathrm{rig}}(\mathcal{X}_s/K)/F^n H^{i-1}_{\mathrm{dR}}(X/K)) \xrightarrow{\iota} H^i_{\mathcal{P}}(\mathcal{X},n) \xrightarrow{\pi} F^n H^i_{\mathrm{dR}}(X/K)^{\mathcal{P}(\varphi^*)=0} \to 0$$

where the term on the right stands for the union of the kernels of all $P(\varphi^*)$ for all $P \in \mathcal{P}$ and the map on the left is induced by the composition of the leftmost map in (2.3) with the map $P(\varphi^*)$. We note that a similar sequence exists for cohomology with supports and relative cohomology. We have immediately

Lemma 2.4. *If $P(\varphi^*)$ is invertible on $H^{i-1}_{\mathrm{rig}}(\mathcal{X}_s/K)$ for all $P \in \mathcal{P}$ then the sequence* (2.5) *is exact on the left as well.*

We have the relation $P_i|P_j$ for $i < j$ and consequently we have maps $\mathbb{R}\Gamma_{\mathrm{f},P_i}(\mathcal{X},n) \to \mathbb{R}\Gamma_{\mathrm{f},P_j}(\mathcal{X},n)$ as above.

Definition 2.5. The modified syntomic complex of $\mathcal{X}$ is

$$\mathbb{R}\Gamma_{\mathrm{ms}}(\mathcal{X},n) := \varinjlim_{i} \mathbb{R}\Gamma_{\mathrm{f},P_i}(\mathcal{X},n)\ .$$

In other words, it is $\mathbb{R}\Gamma_{\mathcal{P}}(\mathcal{X},n)$ where $\mathcal{P}=\{P_i\}$.

Clearly, changing from P_1 to P_i corresponds to changing φ to φ^i. Since all Frobenius endomorphisms identify after taking a sufficiently large power we see that the limiting process makes H_{ms} independent of the choice of the Frobenius endomorphism and consequently one can show [Bes00b, Lemma 8.5] that it is functorial in $\mathcal{X}$. On the other hand, in most interesting cases one does not lose too much with this limit. Indeed, we have the following result.

Proposition 2.6. *suppose that $\mathcal{X}$ is proper over $\mathcal{O}_K$ and $2n \neq i, i-1$ or $i-2$. Then both $H^i_{\mathrm{ms}}(\mathcal{X},n)$ and $H^i_{\mathrm{syn}}(\mathcal{X},n)$ are isomorphic to $H^{i-1}_{\mathrm{dR}}(X/K)/F^nH^{i-1}_{\mathrm{dR}}(X/K)$.*

Proof. This is just part 3 of Proposition 8.6 in [Bes00b], together with the isomorphism between de Rham and rigid cohomologies for proper smooth $\mathcal{X}$'s. We just note that in the case of H_{ms} the isomorphism follows easily by weight considerations from Lemma 2.4 □

In [Bes00a] we embedded modified syntomic cohomology in a cohomology theory which we termed "finite polynomial",

$$H^i_{\mathrm{ms}}(\mathcal{X},n) \to H^i_{\mathrm{fp}}(\mathcal{X},n,2n)\ . \tag{2.6}$$

To define this new cohomology theory we simply replace the limit over the polynomials P_i by a limit over a larger monoid, consisting of all polynomials "of a given weight".

Definition 2.7. The finite polynomial complex $\mathbb{R}\Gamma_{\mathrm{fp}}(\mathcal{X},n,m)$ is $\varinjlim_P \mathbb{R}\Gamma_{\mathrm{f},P}(\mathcal{X},n)$, where the limit is over the multiplicative monoid (ordered by division) $\mathcal{P}_m$ of all monic rational polynomials "of weight m", i.e, of those polynomials such that all of their roots have complex absolute value $q^{m/2}$, with q the degree of φ. The associated cohomologies are denoted $H^i_{\mathrm{fp}}(\mathcal{X},n,m)$.

Specializing the above discussion we get from (2.5) a short exact sequence

$$H^{i-1}_{\mathrm{rig}}(\mathcal{X}_s/K)/F^nH^{i-1}_{\mathrm{dR}}(X/K)) \to H^i_{\mathrm{fp}}(\mathcal{X},n,m) \to F^nH^i_{\mathrm{dR}}(X/K)^{w=m} \to 0 \tag{2.7}$$

where the $w=m$ stands for the part of weight m, i.e., which when mapped to rigid cohomology lands in the weight m part. We further get from Lemma 2.4 the following.

Lemma 2.8. *If $H^{i-1}_{\mathrm{rig}}(\mathcal{X}_s/K)$ has no part of weight m then the sequence (2.7) is exact on the left as well. In particular, if $\mathcal{X}$ is proper we get a short exact sequence*

$$0 \to H^{i-1}_{\mathrm{dR}}(X/K)/F^n \to H^i_{\mathrm{fp}}(\mathcal{X},n,i) \to F^nH^i_{\mathrm{dR}}(X/K) \to 0\ . \tag{2.8}$$

An important property of finite polynomial cohomology theory is the existence of cup products

$$H^{i_1}_{\mathrm{fp}}(\mathcal{X},n_1,m_1) \times H^{i_1}_{\mathrm{fp}}(\mathcal{X},n_2,m_2) \to H^{i_1+i_2}_{\mathrm{fp}}(\mathcal{X},n_1+n_2,m_1+m_2) \tag{2.9}$$

which is compatible with the cup product on modified syntomic cohomology via (2.6). It can be obtained by taking direct limits of products

$$\mathbb{R}\Gamma_{\mathrm{f},P}(\mathcal{X},n_1) \times \mathbb{R}\Gamma_{\mathrm{f},P}(\mathcal{X},n_2) \to \mathbb{R}\Gamma_{\mathrm{f},P*Q}(\mathcal{X},n_1+n_2)$$

defined as follows [Bes00a, Definition 4.1 and Remark 4.3]: The polynomial $P * Q$ is defined by the formula

$$(\prod(1-\alpha_i t)) * (\prod(1-\beta_j t)) = \prod(1-\alpha_i\beta_j t)$$

and has the property that there exist polynomials $a(t,s)$ and $b(t,s)$ such that

$$P * Q(ts) = a(t,s)P(t) + b(t,s)Q(s). \tag{2.10}$$

Let two variable polynomials act on $\mathbb{R}\Gamma_{\mathrm{rig}}(\mathcal{X}_s/K) \otimes \mathbb{R}\Gamma_{\mathrm{rig}}(\mathcal{X}_s/K)$ by letting t act as $\varphi^* \otimes \mathrm{id}$ and s as $\mathrm{id} \otimes \varphi^*$. Represent an element of $\mathbb{R}\Gamma_{\mathrm{f},P}(\mathcal{X},n)$ by a pair (x,y) with $x \in F^n\mathbb{R}\Gamma_{\mathrm{dR}}(X/K)$ and $y \in \mathbb{R}\Gamma_{\mathrm{rig}}(\mathcal{X}_s/K)$ of degrees i and $i-1$ respectively, the cup product can be given by the formula

(2.11)

$$(x_1,y_1) \cup (x_2,y_2) = \left(x_1 \cup x_2, (-1)^{\deg x_1} \bigcup (b(t,s)(x_1 \otimes y_2) + a(t,s)(y_1 \otimes x_2))\right) ,$$

where we are implicitly using the specialization map sp from (2.1) to identify elements in the de Rham complex with those in the rigid complex. The following formal result which follows easily from the structure of the product (see [Bes00a, Proposition 2.5.3]) is very useful.

Lemma 2.9. *The induced product on finite polynomial cohomology has the following projection formula with respect to the short exact sequence* (2.5)*:*

$$\iota(x) \cup y = \iota(x \cup \mathrm{sp}\,\pi(y)) .$$

In [Bes00a] we showed that for proper $\mathcal{X}$ the cup product yields Poincaré duality in the appropriate dimensions and twists. This was used to define a pushforward in finite polynomial cohomology and the main result of the paper was that this was compatible with the pushforward of cycles via the cycle class map to syntomic cohomology composed with the map (2.6). In the non-proper case the situation is more complicated but finite polynomial cohomology is still useful.

The following proposition shows the way finite polynomial cohomology can be used to compute regulators

Proposition 2.10. *Suppose $\mathcal{X}$ is proper of relative dimension d over $\mathcal{O}_K$ and suppose we are given n and i such that $2n \neq i, i-1$, and*

$$\alpha \in H^i_{\mathrm{ms}}(\mathcal{X},n) \cong H^{i-1}_{\mathrm{dR}}(X/K)/F^n$$

using Proposition 2.6. Suppose that $\mu \in F^m H^j_{\mathrm{dR}}(X/K)$. Let β be a lift of μ to $H^j_{\mathrm{fp}}(\mathcal{X},m,j)$ in (2.8)*. Then, viewing α in $H^i_{\mathrm{fp}}(\mathcal{X},n,2n)$ via* (2.6)

(1) *the cup product*

$$\alpha \cup \beta \in H^{i+j}_{\mathrm{fp}}(\mathcal{X},n+m,2n+j)$$

is the image of $\alpha \cup \mu \in H^{i+j-1}_{\mathrm{dR}}(X/K)/F^{m+n}$ under the leftmost map of (2.7)*.*

(2) *In particular, if $i+j = 2d+1$ and $n+m = d+1$, define the product*

$$\tilde{\cup} : H^i_{\mathrm{fp}}(\mathcal{X},n,2n) \times H^j_{\mathrm{fp}}(\mathcal{X},m,j) \xrightarrow{\cup} H^{2d+1}_{\mathrm{fp}}(\mathcal{X},d+1,2n+j) \cong H^{2d}_{\mathrm{dR}}(X/K)/F^{d+1} \cong K$$

Then we have the equality $\alpha \tilde{\cup} \beta = \alpha \cup \mu$ in K.

Proof. This follows easily from Lemma 2.9. Note that the condition $2n \neq i-1$ guarantees that $2n+j \neq 2d$, hence the isomorphism in the description of $\tilde{\cup}$ holds by Lemma 2.8. □

Remark 2.11. For the Proposition above to work the use of finite polynomial cohomology is essential, since it may not be possible to lift μ to an element of H_{ms}.

We now turn to Coleman integration theory. For this, we first introduce another modification of the cohomology theories above, by replacing the algebraic de Rham complex and its filtrations by the rigid complex and its filtrations (when we apply this to modified syntomic cohomology as in Definition 2.5 this is what we called the Gros style modified syntomic cohomology in [Bes00b], see discussion on the filtrations there). We obtain cohomology theories, for a directed set of polynomials $\mathcal{P}$,

$$\tilde{H}_{\mathrm{f},\mathcal{P}}(\mathcal{X},n)\;,$$

and obvious maps, induced by the specialization map

$$H_{\mathrm{f},\mathcal{P}}(\mathcal{X},n) \to \tilde{H}_{\mathrm{f},\mathcal{P}}(\mathcal{X},n)\;. \tag{2.12}$$

The cohomology theory $\tilde{H}_{\mathrm{f},P}$ can be given an explicit representations in the affine case. Suppose that $\mathcal{Y}$ is affine and smooth over $\mathcal{O}_K$. Following [Bes00b] we associate with $\mathcal{Y}$ a "basic wide open space" U in the terminology of Coleman [Col89, p. 219], or an overconvergent space (with good reduction) in the terminology of Grosse-Klönne [GK00]. It has an underlying affinoid subvariety U'. We may assume that our Frobenius endomorphism φ lifts to an endomorphism ϕ of U. The de Rham complex of U is quasi-isomorphic to $\mathbb{R}\Gamma_{\mathrm{rig}}(\mathcal{Y}_s/K)$ and the filtration discussed above corresponds to the stupid filtration on that complex. Thus, we have the following representation of the cohomologies above.

(2.13)

$$\begin{aligned}\tilde{H}^i_{\mathrm{f},P}(\mathcal{Y},n) &= H^i(MF(F^n\Omega^\bullet(U) \xrightarrow{P(\phi^*)} \Omega^\bullet(U)))\\ &= \{(\omega,f), \omega\in F^n\Omega^i(U), f\in\Omega^{i-1}(U), d\omega=0, df=P(\phi^*)\omega\}/\sim\end{aligned}$$

where MF stands for the mapping fiber, $F^n\Omega^i=\Omega^i$ if $n\le i$ and 0 otherwise, and $\sim$ is an appropriate equivalence relation. It is important to note that the map (2.5) will send f to $(0,P(\phi^*)f)$ in this representation.

Remark 2.12.
(1) The map (2.12) is an isomorphism in some interesting cases. An obvious example is when $\mathcal{X}$ is smooth and proper. Another case is when $\mathcal{X}$ is affine, $i=2\dim X$ and $n>\dim X$. In this case $H^i_{\mathrm{dR}}(X/K)$ vanishes as well as the n'th filtered part of $H^{i-1}_{\mathrm{dR}}(X/K)$ and so the short exact sequence (2.3) implies that for each polynomial P we have $H^i_{\mathrm{f},P}(\mathcal{X},n)\cong H^{i-1}_{\mathrm{rig}}(\mathcal{X}_s/K)$ and the same holds with the tilded version.
(2) The formulas for the cup product apply also for the tilded version.
(3) The tilded version is cruder, but it is sometimes more comfortable in computations, and in describing the relation with Coleman integration.

Keeping the data above consider the cohomology group $\tilde{H}^1_{\mathcal{P}_1\cdot\mathcal{P}_2}(\mathcal{Y},1)$, notation as in Definition 2.3. Thus, in the explicit description (2.13), an element $\tilde{\omega}$ in this cohomology group is given by a pair (ω,f), where ω is a closed one form on U while f is a function on U such that the relation $P(\phi^*)\omega=df$ holds for some polynomial P with weights 1 and 2. Since $H^1(U)$ has weights 1 and 2 any closed form ω lifts to such an element while the function f is unique up to a constant. It follows that we have a short exact sequence

$$0 \to K \to \tilde{H}^1_{\mathcal{P}_1\cdot\mathcal{P}_2}(\mathcal{Y},1) \to \Omega^1(U)^{d=0} \to 0 .$$

Coleman shows that the data (ω, f) defines a unique locally analytic function F_ω on the geometric points of U', which satisfies the two relations $df = \omega$ and $P(\phi^*)F_\omega = f$. As we vary f by a constant F_ω changes in the same way, and F_ω, defined up to a constant, is called the Coleman integral of ω. Coleman shows the surprising fact that F_ω is independent of the choice of ϕ. In [Bes00a] the values of F_ω are interpreted as pullbacks of $\tilde{\omega}$ via sections $\mathcal{O}_K \to \mathcal{Y}$, which make this fact slightly less surprising. Given a map $g : \mathcal{Y}' \to \mathcal{Y}$ we may pullback the Coleman function F_ω to g^*F_ω along the map induced by g on the generic fiber. It is part of the theory developed in [Bes00a] that this pullback corresponds to the pullback map on finite polynomial cohomology.

Definition 2.13. For a basic wide open U, the space of Coleman integrals of holomorphic forms on U is denoted $A_{\mathrm{Col},1}(U)$. The space $\Omega^1_{\mathrm{Col},1}(U)$ is the product, inside the space of locally analytic differential forms on U', of $A_{\mathrm{Col},1}(U)$ with $\Omega^1(U)$.

Remark 2.14.
(1) If $\mathcal{Y}$ is affine, then a class in $H^1_{\mathcal{P}_1\cdot\mathcal{P}_2}(\mathcal{Y},1)$ can be viewed, via the map (2.12), as a Coleman integral of a form of the third kind.
(2) If $\mathcal{X}$ is a smooth proper $\mathcal{O}_K$-scheme and ω is a holomorphic form on its generic fiber X, then we may lift ω to an element

$$\tilde{\omega} \in H^1_{\mathrm{fp}}(\mathcal{X},1,1) = H^1_{\mathcal{P}_1}(\mathcal{X},1)$$

unique up to a constant. For any affine $\mathcal{Y} \in \mathcal{X}$ the restriction of $\tilde{\omega}$ to $\mathcal{Y}$, viewed in $H^1_{\mathcal{P}_1\cdot\mathcal{P}_2}(\mathcal{Y},1)$ via the map coming from (2.4), is a Coleman function of $\omega|_{\mathcal{Y}}$. Since these are compatible on intersections they glue up to give a global Coleman integral of ω, still unique up to a constant, which is canonically associated to $\tilde{\omega}$ and whose values at points can again be deduced via pullbacks.

We note that the theory of Coleman integration generalizes to give iterated integrals [Col82, CdS88, Bes02] (see also [Vol03] for a related but somewhat different theory) but the approach via polynomials does not generalize this way. The (algebra) of all iterated Coleman integrals on U is denoted $A_{\mathrm{Col}}(U)$.

Let us now consider the 1-dimensional case. The difference $U - U'$ is then a disjoint union of annuli e_i, called the *annuli ends* of U. After choosing a branch of the p-adic logarithm it is possible to extend the Coleman integral to a function on U instead of on U'. The iterated integrals extend as well. Such an integral restricted to an annulus end e_i is not an analytic function but rather a polynomial in $\log(z)$, where z is a local parameter on the annulus, where the coefficients are analytic functions.

To end this section we recall the definition and properties of the double and triple indices, discussed in [Bes00c, BdJ04] respectively. The theory has a local side and a global side. For the local theory recall from the introduction that the ring $K((z))[\log(z)]$ admits a differential onto its associated module of differentials $K((z))[\log(z)]dz$ by setting $d\log(z) = dz/z$. We let $A_{\log,1}$ equal $d^{-1}(K((z))dz)$, which is the same as the polynomials in $\log(z)$ of degree ≤ 1 and with a constant linear coefficient.

In [Bes00c, Proposition 4.5] we defined the notion of the *double index* to be the unique K-bilinear and antisymmetric map

$$\langle \cdot, \cdot \rangle : A_{\log,1} \times A_{\log,1} \to K$$

with the property that $\langle F, G \rangle = \operatorname{Res} F dG$ whenever $F \in K((z))$ (existence and uniqueness are easy, see loc. cit.).

In [BdJ04, Section 6] we extended this notion to define a *triple index*. This is a function on triples $F, G, H \in A_{\log,1}$, but together with choices of integrals in $A_{\log}$ of all the pairs RdS with distinct R and S, and satisfying the condition that $\int RdS + \int SdR = SR$ (these choices will be called the auxiliary data). Dropping all the auxiliary data from the notation the triple index, denoted $\langle \cdot, \cdot; \cdot \rangle$, satisfies the following relations.

(1) Trilinearity and Symmetry in the first two indices
(2) Triple identity $\langle F, G; H \rangle + \langle F, H; G \rangle + \langle H, G; F \rangle = 0$
(3) Reduction to double index $\langle F, G; H \rangle = \langle F, \int GdH \rangle$ when $G \in A$ (and the auxiliary integral $\int GdH$ is used).

The formulation of the first two properties requires keeping track of the auxiliary data, which is however quite obvious. Existence and uniqueness of this index [BdJ04, Proposition 6.3] is now somewhat less obvious.

Remark 2.15. The theory of the double and triple index extend without any difficulty to the case where the field of Laurent series $K((z))$ is replaced by the ring of rigid analytic functions on an annulus $\{r < |z| < s\}$. As explained in the proof of Proposition 5.5 of [Bes00c]and in [BdJ04, Remark 8.11], if dF, dG and dH extend to the disc $\{|z| < s\}$ as meromorphic analytic functions, then the triple (and double) index on the annulus may be replaced by the sum of the indices at all points of $\{|z| \le r\}$.

The global theory of the indices is a partial generalization of the residue theorem. Suppose that we have a wide open U in a curve X as before, with annuli ends e_i, and that we have Coleman functions F, G and H in $A_{\mathrm{Col},1}(U)$. Restricted to each annuli end these functions are in the appropriate $A_{\log}$ and hence local indices on the e_i may be defined. For the triple index auxiliary data must be chosen and we may and do choose them globally in $A_{\mathrm{Col}}(U)$ and then restricting to the e_i. The sum of the local indices on the e_i is called the *global index*

$$\langle F, G \rangle_{\mathrm{gl}} = \sum_i \langle F, G \rangle_{e_i} \ , \ \langle F, G; H \rangle_{\mathrm{gl}} = \sum_i \langle F, G; H \rangle_{e_i} \ .$$

These constructions satisfy certain nice properties. The double index $\langle F, G \rangle_{\mathrm{gl}}$ depends in fact only on the cohomology classes in $H^1_{\mathrm{dR}}(U)$ of the forms dF and dG. The triple index $\langle F, G; H \rangle_{\mathrm{gl}}$ depends only on F, G and H and not on the auxiliary choices made. We furthermore have the following vanishing result, which is a special case of [BdJ04, Proposition 7.4]

Lemma 2.16. *If F, G and f are analytic on U and f non-vanishing, then* $\langle F, \log(f) + G; H \rangle_{\mathrm{gl}} = 0$.

Proof. We have

$$\langle F, \log(f) + G; H \rangle_{\mathrm{gl}} = \langle \log(f) + G, \int FdH \rangle_{\mathrm{gl}} = 0$$

where this last equality follows with $\log(f) = 0$ by the fact that the global double index factors via the cohomology classes as mentioned before and with $G = 0$ by [Bes00c, Corollary 4.11] □

In [BdJ04, Proposition 8.2] we showed that the triple index can be used to define a pairing between Coleman forms and functions on a curve. More precisely, we have a well defined pairing

$$\langle\langle \cdot, \cdot \rangle\rangle : A_{\mathrm{Col},1}(U) \otimes \Omega^1_{\mathrm{Col},1}(U) \to K \ , \ \langle\langle F, GdH \rangle\rangle = \langle F, G; H \rangle_{\mathrm{gl}} \ , \tag{2.14}$$

and the ensuing remark shows the following important fact.

Proposition 2.17. *If $\theta \in \Omega^1(U)$, then $\langle\langle F, \theta \rangle\rangle = \langle F, F_\theta \rangle_{\mathrm{gl}}$. In particular, $\langle\langle F, \theta \rangle\rangle = 0$ if θ is exact.*

We finally note that the triple index has the following equivariance property (see [BdJ04, Lemma 7.1]): If $\alpha : e \to e$ is an endomorphisms of degree n, then the formula

$$\langle \alpha^* F, \alpha^* G; \alpha^* H \rangle_e = n \langle F, G; H \rangle_e$$

holds. In particular, when we specialize to the case $\alpha = \phi$ and apply to the pairing $\langle\langle \bullet, \bullet \rangle\rangle$ we obtain the following formula.

$$\langle\langle \phi^* F, \phi^* \theta \rangle\rangle = q \langle\langle F, \theta \rangle\rangle \ . \tag{2.15}$$

Consequently, if $P(t) = \sum_{i=0}^n a_i t^i$, then

$$\left\langle\!\!\left\langle (\phi^*)^n F, \sum a_i (\phi^*)^i \theta \right\rangle\!\!\right\rangle = \sum a_i q^i \langle\langle (\phi^*)^{n-i} F, \theta \rangle\rangle \ .$$

In other words,

$$\langle\langle (\phi^*)^n F, P(\phi^*)\theta \rangle\rangle = \langle\langle Q(\phi^*) F, \theta \rangle\rangle \ , \ Q(t) = t^n P(q/t) \ . \tag{2.16}$$

3. Strategy of the proof

To see how the required formula can be proved, we first must recall how the combination $\sum (Z_i, f_i)$ gives rise to an element in $H^3_{\mathcal{M}}(S, \mathbb{Q}(2))$. Following the Brown-Gersten spectral sequence this can be seen as follows. Let $Z = \cup Z_i$ be the union of all the Z_i's and let let $Z' \subset Z$ be the part of the non-singular locus of Z where all the f_i are invertible. The f_i then define an element $\alpha \in K_1(Z') = K_1'(Z')$. The condition that $\sum (f_i) = 0$ in S exactly means that α maps to 0 under the map in the long exact sequence

$$\cdots \to K_1'(Z - Z') \to K_1'(Z) \to K_1'(Z') \to K_0'(Z - Z') \to \cdots$$

and consequently there is an element $\tilde{\alpha} \in K_1'(Z)$ mapping to α. Our required element θ is then obtained by pushing forward $\tilde{\alpha}$ to $K_1'(S) = K_1(S)$ and projecting on the corresponding eigenspace. The element $\tilde{\alpha}$, and consequently its pushforward to S, are unique up to the pushforward of an element in $K_1'(Z - Z')$, but this image is killed by the projection. We recall that the regulator is derived from the Chern character defined already on K-theory, and that this Chern character kills this image as well. Thus, for our purposes it is sufficient to consider θ as the pushforward of any $\tilde{\alpha}$ as above.

Let $Y \subset S$ be the complement of $Z - Z'$. By base change we may assume that $S - Y$ is a finite set of K-rational points. Base changing to Y we obtain an element α', which is the pushforward of α by $Z' \to Y$. The commutation of pushforward with pullback to an open subscheme shows that α' is the restriction of θ to Y.

Other elements in $K_1(S)$ which map to α' differ from θ by an element coming from $K_1(S-Y)$. Since $S-Y$ is just a finite number of points it is easy to see that all these lifts of α' have the same image under the étale Chern character map. Thus, we can take θ to be any lift of α'.

We now switch over to the integral side. Since Y was obtained from S by removing the images of K-rational sections, which extend to sections $\mathcal{O}_K \to \mathcal{S}$ since $\mathcal{S}$ is proper, we obtain an integral model $\mathcal{Y}$ for Y by removing the images of these sections from $\mathcal{S}$.

We now impose the integrality assumption, Assumption 1 from the introduction. Thus, the Z_i are images of the generic fibers of $g_i : \mathcal{X}_i \to \mathcal{S}$. We may consider the f_i as rational functions on the $\mathcal{X}_i$. Recall that we assume that their divisors do not contain the special fiber.

Let $\mathcal{X}$ be the disjoint union of the $\mathcal{X}_i$ with the obvious map $g : \mathcal{X} \to \mathcal{S}$. Let $\mathcal{X}' = \mathcal{X} \times_{\mathcal{S}} \mathcal{Y}$ and let $\boldsymbol{\alpha} \in K_1(\mathcal{X}') = K_1'(\mathcal{X}')$ be the element induced by the f_i, which is invertible on $\mathcal{X}'$ by our assumption. Since both $S-Y$ and $\mathcal{S}-\mathcal{Y}$ are just a disjoint union of a finite number of copies of K and $\mathcal{O}_K$ respectively, their K_0 are the same. This implies that since α' lifts to $K_1(S)$, $g_*(\boldsymbol{\alpha})$ extends to an element $\boldsymbol{\theta} \in K_1(\mathcal{S})$. The restriction of $\boldsymbol{\theta}$ to S further restricts to α' in $K_1(Y)$. This restriction can therefore be taken to be our θ. By Niziol's Theorem discussed in the introduction, we already see that $\mathrm{reg}_{\text{ét}}(\theta)$ is in the image under exp of $\mathrm{ch}_{\mathrm{syn}}(\boldsymbol{\theta})$, which we may term $\mathrm{reg}_{\mathrm{syn}}(\theta)$. This element therefore satisfies the first requirement in Theorem 1.1.

To compute $\mathrm{reg}_{\mathrm{syn}}(\theta)$ we assume Conjecture 4.2 that tells us that the Gysin (pushforward) maps in syntomic cohomology, commute with the same maps in K-theory. This will allow us to argue that $\mathrm{reg}(\boldsymbol{\theta})$ has the property that its restriction to $\mathcal{Y}$ is $g_*(\mathrm{reg}(\boldsymbol{\alpha}))$. It will turn out (Remark 6.1) that this property suffices to characterize $\mathrm{reg}(\boldsymbol{\theta})$. To compute the cup product with $\omega \cup \eta$ we will use a projection formula in finite polynomial cohomology to transform this cup product to a computation on $\mathcal{X}$ which is then carried out. The Gysin formalism and the projection formula will be developed in Sections 4 and 5.

4. Pushforwards in syntomic cohomology

In this and the next section we establish the technical results in syntomic cohomology which are required for the proof. The key result is the projection formula in syntomic and finite polynomial cohomology, Theorem 5.2. We will also need to discuss syntomic homology, pushforwards. The discussion overlaps to a large extent with that of [CCM10]. These two sections are completely formal. The only possible novelty is a homological algebra setup making the projection formula particularly transparent.

The idea for defining syntomic homology is as follows: Define complexes with compact support for the rigid, de Rham and filtered de Rham cohomologies, define the corresponding homological complexes as their duals and then make the same construction that manufactured syntomic cohomology. The exact same idea works for all other versions, including finite polynomial.

The reader is advised to consult [Bes00b], where many of the homological algebra constructions are explained in detail.

Let us begin with rigid cohomology. For a k-scheme X we fix a compactification $\overline{X}$ with complement Z and we may define the rigid complex with compact supports functor,

$$\mathbb{R}\Gamma_{\mathrm{rig},c}(X/K) = MF(\mathbb{R}\Gamma_{\mathrm{rig}}(\overline{X}/K) \to \mathbb{R}\Gamma_{\mathrm{rig}}(Z/K)) \ .$$

This can be given a more explicit representation [Ber97a] and be shown to be independent of the choice of compactification. Furthermore, there exists a product

$$\mathbb{R}\Gamma_{\mathrm{rig}}(X/K) \times \mathbb{R}\Gamma_{\mathrm{rig},c}(X/K) \to \mathbb{R}\Gamma_{\mathrm{rig},c}(X/K) \ ,$$

and Poincaré duality is satisfied: If $\dim X = d$, then there is a trace map isomorphism $\mathrm{tr} : H^{2d}_{\mathrm{rig},c}(X/K) \xrightarrow{\sim} K$ and the pairing

$$\langle \bullet, \bullet \rangle : H^i_{\mathrm{rig}}(X/K) \times H^{2d-i}_{\mathrm{rig},c}(X/K) \xrightarrow{\cup} H^{2d}_{\mathrm{rig},c}(X/K) \xrightarrow{\mathrm{tr}} K \tag{4.1}$$

is perfect.

The de Rham situation is completely analogous. For a K-scheme X we fix a compactification $\overline{X}$ with complement Z and we may define the de Rham complex with compact supports functor

$$\mathbb{R}\Gamma_{\mathrm{dR},c}(X/K) = MF(\mathbb{R}\Gamma_{\mathrm{dR}}(\overline{X}/K) \to \mathbb{R}\Gamma_{\mathrm{dR}}(Z/K)) \ .$$

In [BCF04] cup products and Poincaré duality are established following a method of Deligne. We review this in our case (the theory applies also when X is not smooth) to show the compatibility with the filtration. Deligne defined for a coherent sheaf $\mathcal{F}$ on X a pro-coherent sheaf (an inverse system of coherent sheaves) $j_!\mathcal{F}$ on $\overline{X}$ by

$$j_!\mathcal{F} = (\mathcal{J}^n\overline{\mathcal{F}})_n \ ,$$

where $\overline{\mathcal{F}}$ is any coherent extension of $\mathcal{F}$ to $\overline{X}$ and $\mathcal{J}$ is the ideal of definition for Z. This is independent of the coherent extension. Now, one simply puts

$$\mathbb{R}\Gamma_{\mathrm{dR},c}(X/K) = \mathbb{R}\Gamma(\overline{X}, j_!(\Omega^\bullet_X)) \ .$$

This definition can be seen to be equivalent to the previous one [BCF04, Proposition 1.3]. The cup product is now induced by the map of sheaves $j_*\mathcal{G} \otimes j_!\mathcal{F} \to j_!(\mathcal{G} \otimes \mathcal{F})$ defined because "an arbitrary pole around Z can be canceled by a sufficiently high power of zero along Z".

Now let $\mathcal{X}$ be an $\mathcal{O}_K$-scheme. There are two kinds of maps between the rigid cohomology of its special fiber and de Rham cohomology of its generic fiber. First of all, if $\mathcal{X}$ is smooth there is a specialization map [Bes00b, BCF04],

$$\mathrm{sp} : \mathbb{R}\Gamma_{\mathrm{dR}}(X/K) \to \mathbb{R}\Gamma_{\mathrm{rig}}(\mathcal{X}_s/K) \ .$$

On the other hand, in complete generality there is a cospecialization map [BB04, BCF04],

$$\mathrm{cosp} : \mathbb{R}\Gamma_{\mathrm{rig},c}(\mathcal{X}_s/K) \to \mathbb{R}\Gamma_{\mathrm{dR},c}(X/K) \ . \tag{4.2}$$

These maps are compatible with Poincaré duality in the obvious way. In particular, if we define the rigid and de Rham homological complexes as the (homological) duals of the complexes with compact supports,

$$\mathbb{R}\Gamma^{\mathrm{rig}}(Y/K) := \mathcal{H}\mathrm{om}(\mathbb{R}\Gamma_{\mathrm{rig},c}(Y/K), K) \ ,$$
$$\mathbb{R}\Gamma^{\mathrm{dR}}(X/K) := \mathcal{H}\mathrm{om}(\mathbb{R}\Gamma_{\mathrm{dR},c}(X/K), K) \ .$$

we get quasi-isomorphisms in the smooth of dimension d case

$$\mathbb{R}\Gamma^{2d-\bullet}_{\mathrm{rig}}(Y/K) \xrightarrow{\sim} \mathbb{R}\Gamma^{\mathrm{rig}}(Y/K) \ ,$$
$$\mathbb{R}\Gamma^{2d-\bullet}_{\mathrm{dR}}(X/K) \xrightarrow{\sim} \mathbb{R}\Gamma^{\mathrm{dR}}(X/K) \ .$$

One has to define the Frobenius correctly as we now explain (see also [CCM10, Remark 5.6]). Suppose $\dim Y = d$. Since the cup product is functorial, and since φ^* acts as q^d on $H^{2d}_{\mathrm{rig},c}(Y/K)$, the pairing (4.1) satisfies the relation $\langle \varphi^* x, \varphi^* y\rangle = q^d\langle x, y\rangle$. Thus, if we identify $H^{2d-i}_{\mathrm{rig}}(Y/K)$ with the dual of $H^i_{\mathrm{rig},c}(Y/K)$, the map φ^* on $H^{2d-i}_{\mathrm{rig}}(Y/K)$ is identified with q^d times the dual of the map $(\varphi^*)^{-1}$. Since we will want the quasi-isomorphism between the homological and cohomological complexes to respect Frobenius, it is easy to see that Frobenius on the homological complexes should be the dual of the inverse of φ^* on the complex with compact supports. It is further easily verified that this is also what is required to make Frobenius commute with the covariant maps induced by proper morphisms.

We do not know if the map φ^* has an inverse (this depends on the particular complex used). It is a quasi-isomorphism so it has an inverse in the derived category. As was shown in [Bes00b], there is no problem in practice in considering the cone of such a morphism. This appears to be hidden in the approach of [CCM10]. Since we are interested in taking cones for polynomials applied to φ^* though, it looks easier to just replace $\mathbb{R}\Gamma_{\mathrm{rig},c}(Y/K)$ by a quasi-isomorphic complex on which φ^* is invertible. This is easily achieved by "perfectifying" as follows (we also perfectify the complex without compact supports for future use).

Definition 4.1. The complexes $\mathbb{R}\Gamma_{\mathrm{rig},p}(Y/K)$ and $\mathbb{R}\Gamma_{\mathrm{rig},cp}(Y/K)$ are defined as

$$\mathbb{R}\Gamma_{\mathrm{rig},p}(Y/K) := \varinjlim_n \mathbb{R}\Gamma_{\mathrm{rig}}(Y/K)\ ,\ \mathbb{R}\Gamma_{\mathrm{rig},cp}(Y/K) := \varinjlim_n \mathbb{R}\Gamma_{\mathrm{rig},c}(Y/K)\ .$$

In each case, the same complex is placed in each degree and the connecting map is φ^*.

There is an induced φ^* on $\mathbb{R}\Gamma_{\mathrm{rig},p}(Y/K)$ and on $\mathbb{R}\Gamma_{\mathrm{rig},cp}(Y/K)$ which is invertible There are obvious maps, compatible with Frobenius,

$$\mathbb{R}\Gamma_{\mathrm{rig}}(Y/K) \to \mathbb{R}\Gamma_{\mathrm{rig},p}(Y/K)\ ,\ \mathbb{R}\Gamma_{\mathrm{rig},c}(Y/K) \to \mathbb{R}\Gamma_{\mathrm{rig},cp}(Y/K)\ .$$

Finally, there is a cup product

$$\mathbb{R}\Gamma_{\mathrm{rig},p}(Y/K) \times \mathbb{R}\Gamma_{\mathrm{rig},cp}(Y/K) \to \mathbb{R}\Gamma_{\mathrm{rig},cp}(Y/K)$$

which is compatible with Frobenius and with the cup product on the rigid complexes.

For purposes of defining syntomic homology, we are therefore going to change the rigid homology complex to the quasi-isomorphic complex

$$\mathbb{R}\Gamma^{\mathrm{rig}}(Y/K) \cong \mathcal{H}\mathrm{om}(\mathbb{R}\Gamma_{\mathrm{rig},cp}(Y/K), K)\ ,$$

on which there is a Frobenius map which is the dual of φ^*.

Note that the perfectified complex is more complicated than the non-perfectified complex, though quasi-isomorphic to it. So we use it only when we need a Frobenius action on the dual.

To complete the picture we need filtrations on the de Rham complexes with compact supports. This can be done as follows (as suggested by Katz): We may assume that our compactification $\overline{X}$ is such that the complement is a normal crossings divisor D. We may then choose as the coherent extension for the sheaves of differential forms the corresponding sheaves with logarithmic poles on $\overline{X}$. This leads to the formula

$$\mathbb{R}\Gamma_{\mathrm{dR},c}(X/K) = \mathbb{R}\Gamma(\overline{X}, \varprojlim_n \mathcal{J}^n \Omega^{\bullet}_{\overline{X}}(\log D))\ ,$$

but it is in fact sufficient to take $n = 1$,

$$\mathbb{R}\Gamma_{\mathrm{dR},c}(X/K) = \mathbb{R}\Gamma(\overline{X}, \mathcal{J}\Omega^{\bullet}_{\overline{X}}(\log D)) ,$$

and the filtration arises from giving $\Omega^{\bullet}_{\overline{X}}(\log D)$ the stupid filtration.

All complexes with compact supports are contravariant for proper morphisms, and consequently the corresponding homological complexes are covariant for proper morphisms.

Now, replacing each complex in the definition of the cohomology with the corresponding complex in homology we obtain the definitions of the various versions of syntomic and finite polynomial homologies, which are covariant for proper morphisms. It is obvious that if $\mathcal{X}$ is smooth over $\mathcal{O}_K$ of relative dimension d there is a quasi-isomorphism

$$\mathbb{R}\Gamma^{2d-\bullet}_{\mathrm{syn}}(\mathcal{X}, d-n) \to \mathbb{R}\Gamma^{\mathrm{syn}}_{\bullet}(\mathcal{X}, n) .$$

As an immediate consequence we obtain in the smooth case our Gysin maps

$$\mathbb{R}\Gamma_{\mathrm{syn}}(\mathcal{X}, n) \xrightarrow{f_*} \mathbb{R}\Gamma_{\mathrm{syn}}(\mathcal{Y}, n+d)[2d]$$

for a proper morphism $f : \mathcal{X} \to \mathcal{Y}$ with $\dim X + d = \dim Y$

The following result, though technical, is rather standard. We hope that details will appear in future work of Chiarellotto and his coauthors, or in future work of the author.

Conjecture 4.2. *The pushforward in syntomic cohomology commutes with pushforward in motivic cohomology.*

Our goal now is to extend the Poincaré duality construction of pushforwards in finite polynomial cohomology described in Section 2 to the non-proper case. For this we will need to define cohomology with compact support in the syntomic and finite polynomial settings. Our goal is to make such a definition so that we have a cup product with the corresponding cohomology without compact support and such that the pushforwards in cohomology satisfy a projection formula with respect to this cup product. We will discuss the case of modified syntomic cohomology to fix ideas, but the discussion applies to all cohomologies.

We begin by noting, as in [Bes00b], that there are better ways for writing the modified syntomic complexes. There we have used the notion of a quasi-fibered product, which is just a special case of a homotopy limit, but here we just use homotopy limits to begin with. The modified syntomic complex is better written as the homotopy limit (quasi-fibered product) of the following diagram of complexes

$$\begin{array}{ccc} F^n\mathbb{R}\Gamma_{\mathrm{dR}}(X/K) & & MF(1-\varphi^*/q^n) \\ & \searrow \quad \swarrow & \\ & \mathbb{R}\Gamma_{\mathrm{rig}}(\mathcal{X}_s/K) & \end{array}$$

In this diagram we have the mapping fiber $MF(1-\varphi^*/q^n)$ on $\mathbb{R}\Gamma_{\mathrm{rig}}(\mathcal{X}_s/K)$ with the obvious map to $\mathbb{R}\Gamma_{\mathrm{rig}}(\mathcal{X}_s/K)$ and the composed map

$$F^n\mathbb{R}\Gamma_{\mathrm{dR}}(X/K) \to \mathbb{R}\Gamma_{\mathrm{dR}}(X/K) \xrightarrow{\mathrm{sp}} \mathbb{R}\Gamma_{\mathrm{rig}}(\mathcal{X}_s/K) ,$$

where the rightmost map is the specialization map. If we try to write a similar diagram with cohomology with compact supports we are forced, due to the reversed direction of the cospecialization map, to make the following definition.

Definition 4.3. The modified syntomic complex with compact supports $\mathbb{R}\Gamma_{\mathrm{ms},c}(\mathcal{X}, n)$ is the homotopy limit of the following diagram

$$\begin{array}{ccc} F^n\mathbb{R}\Gamma_{\mathrm{dR},c}(X/K) & & MF(1-\varphi^*/q^n) \\ & \searrow \quad \swarrow & \\ & \mathbb{R}\Gamma_{\mathrm{dR},c}(X/K) & \end{array}$$

where now we have the composed map

$$MF(1-\varphi^*/q^n) \to \mathbb{R}\Gamma_{\mathrm{rig},c}(\mathcal{X}_s/K) \to \mathbb{R}\Gamma_{\mathrm{dR},c}(X/K)$$

with the second map being the cospecialization map (4.2) (lifted to the level of complexes).

We have similarly finite polynomial complexes $\mathbb{R}\Gamma_{\mathrm{fp}}(\mathcal{X}, n, m)$. Similarly to (2.7) we have the following result.

Proposition 4.4. *We have maps*

$$\pi : H^i_{\mathrm{fp},c}(\mathcal{X}, n, m) \to F^n H^i_{\mathrm{dR},c}(X/K) \cap H^i_{\mathrm{rig},c}(\mathcal{X}_s/K)^{w=m} \tag{4.3}$$

and

$$\iota : H^{i-1}_{\mathrm{dR},c}(X/K)/F^n \to H^i_{\mathrm{fp},c}(\mathcal{X}, n, m) \tag{4.4}$$

such that when $H^{i-1}_{\mathrm{rig},c}(\mathcal{X}_s/K)^{w=m} = 0$ they fit into a short exact sequence

$$0 \to H^{i-1}_{\mathrm{dR},c}(X/K)/F^n \xrightarrow{\iota} H^i_{\mathrm{fp},c}(\mathcal{X}, n, m) \xrightarrow{\pi} F^n H^i_{\mathrm{dR},c}(X/K) \cap H^i_{\mathrm{rig},c}(\mathcal{X}_s/K)^{w=m}$$

Proof. The map $\iota : H^{i-1}_{\mathrm{dR},c}(X/K) \to H^i_{\mathrm{fp},c}(\mathcal{X}, n, m)$ is just the obvious map from the mapping fiber construction, and the fact that $F^n H^{i-1}_{\mathrm{dR},c}(X/K)$ is in its kernel is obvious. Unfolding the mapping fiber construction for a fixed polynomial P we see that we can write the complex as the mapping fiber of

$$\begin{aligned} F^n\mathbb{R}\Gamma_{\mathrm{dR},c}(X/K) \oplus \mathbb{R}\Gamma_{\mathrm{rig},c}(\mathcal{X}_s/K) &\to \mathbb{R}\Gamma_{\mathrm{dR},c}(X/K) \oplus \mathbb{R}\Gamma_{\mathrm{rig},c}(\mathcal{X}_s/K) \\ (x, y) &\mapsto (x - \operatorname{cosp} y, P(\varphi^*)y)\ , \end{aligned} \tag{4.5}$$

and the resulting long exact sequence easily yields the following short exact sequence

$$\begin{aligned} 0 &\to (\operatorname{coker} F^n H^{i-1}_{\mathrm{dR},c}(X/K) \oplus H^{i-1}_{\mathrm{rig},c}(\mathcal{X}_s/K) \to H^{i-1}_{\mathrm{dR},c}(X/K) \oplus H^{i-1}_{\mathrm{rig},c}(\mathcal{X}_s/K)) \\ &\to H^i_{\mathrm{fp},c}(\mathcal{X}, n, m) \to F^n H^i_{\mathrm{dR},c}(X/K) \cap H^i_{\mathrm{rig},c}(\mathcal{X}_s/K)^{P(\varphi^*)=0} \end{aligned}$$

with the maps in the cokernel induced from the maps in (4.5). Under the conditions of the proposition the map $P(\varphi^*)$ is invertible and therefore the left most object in the short exact sequence is isomorphic to $H^{i-1}_{\mathrm{dR},c}(X/K)/F^n$. □

The problem with these definitions is that the cup product is not obvious from them. If we try to use the formulas for the cup product on quasi-fibered products we see that the individual terms in the diagram do not have products between them. For this purpose it is better to replace the two homotopy limits with quasi-isomorphic ones which do have products between them. Thus we will write the modified syntomic complex as the homotopy limit of

(4.6)

$$\begin{array}{ccccccc} F^n\mathbb{R}\Gamma_{\mathrm{dR}}(X/K) & & & \mathbb{R}\Gamma_{\mathrm{dR}}(X/K) & & & MF(1-\varphi^*/q^n) \\ & \searrow & \swarrow & & \searrow & \swarrow & \\ & & \mathbb{R}\Gamma_{\mathrm{dR}}(X/K) & & \mathbb{R}\Gamma_{\mathrm{rig}}(\mathcal{X}_s/K) & & \end{array}$$

while the complex with compact supports is the homotopy limit of

$$\begin{array}{ccccccc} F^n\mathbb{R}\Gamma_{\mathrm{dR},c}(X/K) & & & \mathbb{R}\Gamma_{\mathrm{rig},c}(\mathcal{X}_s/K) & & & MF(1-\varphi^*/q^n) \\ & \searrow & \swarrow & & \searrow & \swarrow & \\ & & \mathbb{R}\Gamma_{\mathrm{dR},c}(X/K) & & \mathbb{R}\Gamma_{\mathrm{rig},c}(\mathcal{X}_s/K) & & \end{array}$$

Now there are products for the corresponding terms, the only non-obvious one is

(4.7)

$$\mathbb{R}\Gamma_{\mathrm{dR}}(X/K) \times \mathbb{R}\Gamma_{\mathrm{rig},c}(\mathcal{X}_s/K) \to \mathbb{R}\Gamma_{\mathrm{rig}}(\mathcal{X}_s/K) \times \mathbb{R}\Gamma_{\mathrm{rig},c}(\mathcal{X}_s/K) \to \mathbb{R}\Gamma_{\mathrm{rig},c}(\mathcal{X}_s/K)$$

That all the cup products are compatible is clear, except for one case, whose commutativity follows from the commutativity of the following diagram

(4.8)

$$\begin{array}{ccc} & \mathbb{R}\Gamma_{\mathrm{dR}}(X/K) \times \mathbb{R}\Gamma_{\mathrm{rig},c}(\mathcal{X}_s/K) & \\ \swarrow & & \searrow \\ \mathbb{R}\Gamma_{\mathrm{rig}}(\mathcal{X}_s/K) \times \mathbb{R}\Gamma_{\mathrm{rig},c}(\mathcal{X}_s/K) & & \mathbb{R}\Gamma_{\mathrm{dR}}(X/K) \times \mathbb{R}\Gamma_{\mathrm{dR},c}(X/K) \\ \downarrow & & \downarrow \\ \mathbb{R}\Gamma_{\mathrm{rig},c}(\mathcal{X}_s/K) & \longrightarrow & \mathbb{R}\Gamma_{\mathrm{dR},c}(X/K) \end{array}$$

(this commutativity is established in [CCM10]). Products on such homotopy limits have been written down, e.g., by Beilinson [Bei86]. Below we will give a general recipe for such things that will have the advantage of making the projection formula obvious.

Remark 4.5. One way to construct the cup product is to note that the homotopy limit of a diagram of the form

$$\begin{array}{ccccc} A & & B & & C \\ & \searrow \swarrow & & \searrow \swarrow & \\ & D & & E & \end{array}$$

can be constructed from the quasi-fibered product ([Bes00b, Section 3] and (5.3) below) as $A \tilde{\times}_D (B \tilde{\times}_E C)$ and then using the product on quasi-fibered products constructed in [Bes00b, Lemma 3.2].

In any way, we now have at our disposal the cup product, e.g., in finite polynomial cohomology

$$\mathbb{R}\Gamma_{\mathrm{fp}}(\mathcal{X}, n_1, m_1) \times \mathbb{R}\Gamma_{\mathrm{fp},c}(\mathcal{X}, n_2, m_2) \to \mathbb{R}\Gamma_{\mathrm{fp},c}(\mathcal{X}, n_1+n_2, m_1+m_2) \tag{4.9}$$

Just like in Lemma 2.9 we have projection formulas.

Proposition 4.6. *The projection formulas*

$$\iota(x) \cup y_c = \iota \operatorname{cosp}(x \cup \pi(y_c)) \ .$$

where $x \in H_{\mathrm{rig}}$, $y_c \in H_{\mathrm{fp},c}$, $\pi(y_c) \in H_{\mathrm{rig},c}$ *(we neglect indices here) and where* ι *and* π *are the maps defined in* (2.5) *and* (4.3) *respectively.*

Proof. Such formulas (there are some similar formulas but we isolated the one that is required later on) follow easily from the description of the cup product in Remark 4.5 and from [Bes00c, Lemma 3.2.2]). Applying the above, one should actually have $\pi(y_c)$ in the cohomology of $MF(P(\varphi^*))$ but then one only uses its image in H_{rig}. □

Definition 4.7. Suppose $\dim X = d$, $m \neq 2d$, $n > d$. Then, $H^{2d}_{\mathrm{rig},c}(\mathcal{X}_s/K)$ is pure of weight $2d$. Thus we may apply Proposition 4.4 to conclude that we have an isomorphism

$$\iota : H^{2d}_{\mathrm{dR},c}(X/K) \xrightarrow{\sim} H^{2d+1}_{\mathrm{fp},c}(\mathcal{X}, n, m) \ .$$

The trace map is then the isomorphism

$$\operatorname{tr} : H^{2d+1}_{\mathrm{fp},c}(\mathcal{X}, n, m) \xrightarrow{\iota^{-1}} H^{2d}_{\mathrm{dR},c}(X/K) \xrightarrow{\operatorname{tr}} K \ .$$

5. The projection formula

The goal of this section is to prove a projection formula in syntomic and finite polynomial cohomology, Theorem 5.2 for the pushforward map in finite polynomial cohomology.

Note first that if $f : X \to Y$ is a morphism of varieties over K such that $\dim X + d = \dim Y$, then, applying Poincaré duality we have a pushforward map in de Rham cohomology with compact supports

$$f_* : H^i_{\mathrm{dR},c}(X/K) \to H^{i+2d}_{\mathrm{dR},c}(Y/K) \ , \tag{5.1}$$

shifting the filtration by d.

Remark 5.1. It is easy to see that in the particular case $i = 2\dim X$ this map commutes with the trace map, because via the trace map it is dual to the pullback map on H^0.

Theorem 5.2. *Let* $\mathcal{X}, \mathcal{Y}$ *be smooth integral* $\mathcal{O}_K$*-schemes and let* $d = \dim Y - \dim X$*. Then there exists a pushforward maps*

$$f_* : H^i_{\mathrm{fp},c}(\mathcal{X}, n, m) \to H^{i+2d}_{\mathrm{fp},c}(\mathcal{Y}, n+d, m+2d)$$

for an arbitrary morphism $f : \mathcal{X} \to \mathcal{Y}$*, and*

$$f_* : H^i_{\mathrm{fp}}(\mathcal{X}, n, m) \to H^{i+2d}_{\mathrm{fp}}(\mathcal{Y}, n+d, m+2d) \ ,$$

for a proper f*, such that for a proper* f *one has the projection formula*

$$\beta \cup f_*\alpha = f_*(f^*\beta \cup \alpha)$$

while for arbitrary f *one has the projection formula*

$$\alpha \cup f_*\beta = f_*(f^*\alpha \cup \beta)$$

for $\alpha \in H_{\mathrm{fp}}(\mathcal{X})$ *and* $\beta \in H_{\mathrm{fp},c}(\mathcal{Y})$*,with any choice of indices. The pushforward maps are compatible with pushforward maps in rigid and de Rham cohomologies, with or without compact support, induced from Poincaré duality, via the maps* π *and* ι*.*

Corollary 5.3. *Identifying $H_{\mathrm{fp},c}^{2\dim X+1}(\mathcal{X},?,?)$ with K via the trace map, for appropriate indices $?$, to obtain K-valued cup products, and doing the same for $\mathcal{Y}$, we have, with $d=\dim Y-\dim X$ the following equality for $\alpha\in H_{\mathrm{fp}}^i(\mathcal{X},n_1,m_1)$ and $\beta\in H_{\mathrm{fp},c}^j(\mathcal{Y},n_2,m_2)$, where $i+j=2\dim X+1$, $n_1+n_2>\dim X$, $m_1+m_2\neq 2\dim X$,*

$$\beta\cup f_*\alpha=f^*\beta\cup\alpha$$

for a proper f and

$$\alpha\cup f_*\beta=f^*\alpha\cup\beta$$

for an arbitrary f.

Proof. Immediate from the Theorem and from Remark 5.1. □

In order to prove Theorem 5.2 we will sketch a method for writing down cup products, which could be of interest in its own right, that would make this transparent.

The first instance of this method already appears in [Bes00a]. Suppose that $A^\bullet$ is a complex with an endomorphism ϕ and that P is a polynomial. Then we may form the mapping fiber $MF_{A,P}:=MF(A^\bullet\xrightarrow{P(\phi)}A^\bullet)$. One observes that the sequence

$$K[t]\xrightarrow{\cdot P(t)}K[t]\to V_P$$

defines a $K[t]$ module V_P and a free $K[t]$ resolution $C_P^\bullet$ of it and that $MF_{A,P}$ is just $\mathbb{R}\operatorname{Hom}$, in the category of $K[t]$ modules, between V_P and $A^\bullet$. Now one observes the equality

$$V_{P*Q}=V_P\otimes V_Q$$

from which the product

$$MF_{A,P}\times MF_{B,Q}\to MF_{A\otimes B,P*Q},$$

which is used in the definition of the product on fp cohomology, may be deduced. Explicitly, one can write down a map of resolutions $C_{P\otimes Q}^\bullet\to C_P^\bullet\otimes C_Q^\bullet$ and the product can be written explicitly as the composition

$$\operatorname{Hom}(C_P^\bullet,A^\bullet)\times\operatorname{Hom}(C_Q^\bullet,B^\bullet)\to\operatorname{Hom}(C_P^\bullet\otimes C_Q^\bullet,A^\bullet\otimes B^\bullet)\to\operatorname{Hom}(C_{P*Q}^\bullet,A^\bullet\otimes B^\bullet)\ .$$

Let us now discuss pushforwards. For clarify we formalize the situation as follows.

Definition 5.4. A product triple is a triple of complexes $A^\bullet$, $A_c^\bullet$ and $A_{cc}^\bullet$ together with a bilinear product $\cup: A^\bullet\times A_c^\bullet\to A_{cc}^\bullet$, denoted collectively as A. A map of triples is a map between all the components commuting with the cup product. A *pf-map* $A\to B$ consists of a pushforward maps $f_*:A^\bullet\to B^\bullet$ and $f_*:A_{cc}^\bullet\to B_{cc}^\bullet$, of possibly non-zero degree, and a pullback maps $f^*:B_c^\bullet\to A_c^\bullet$ such that the projection formula $f_*(a)\cup b=f_*(a\cup f^*b)$ holds in $B_{cc}^\bullet$ up to homotopy for $a\in A^\bullet$ and $b\in B_c^\bullet$.

The above description of the product makes the following Lemma obvious.

Lemma 5.5. *If ϕ is an endomorphism of A and P and Q are polynomials, then a new product triple $MF_{A,P,Q}$ is obtained by $(MF_{A,P},MF_{A_c,Q},MF_{A_{cc},P*Q})$. If $f:A\to B$ is a pf-map of product triples commuting with ϕ in the obvious way, then it induces a pf-map of product triples $MF_{A,P,Q}\to MF_{B,P,Q}$.*

We now make a similar construction of the product on homotopy limits of complexes, which would make a similar projection formula obvious as well. We only do it for the quasi-fibered product but it is clear how to proceed in general.

Let

$$A_1^\bullet \to A_3^\bullet \leftarrow A_2^\bullet \tag{5.2}$$

be a diagram of complexes. Recall [Bes00b, Section3] that these have a quasi-fibered product

$$A_1^\bullet \underset{A_3^\bullet}{\tilde{\times}} A_2^\bullet := MF(A_1^\bullet \oplus A_2^\bullet \to A_3^\bullet) \tag{5.3}$$

and that there is a cup product on quasi-fibered products given, e.g., by the formulas in [Bes00b, Lemma 3.2].

Let K be any field and let C be the category of diagrams of K-vector spaces $V \xrightarrow{\gamma} W \xleftarrow{\delta} U$. This is an abelian category. We will write an object of such a category as $[V;\gamma,W,\delta;U]$ or simply as $[V,W,U]$ if the maps γ and δ are understood and a morphism in this category as $[v,w,u]$. We may isolate two projective objects of C: $O_1 := [K;\mathrm{id}\oplus 0, K\oplus K, 0\oplus \mathrm{id}; K]$ and $O_2 := [0,K,0]$. Now, the complex

$$D^\bullet := \quad O_2 \xrightarrow{[0,\mathrm{id}\oplus(-\mathrm{id}),0]} O_1$$

is easily seen to be a projective resolution of $O_3 := [K;\mathrm{id},K,\mathrm{id};K]$.

We may view the diagram of complexes (5.2) as a complex $A^\bullet$ in C and then it is easy to see that the Hom complex $\mathcal{H}\mathrm{om}(D^\bullet, A^\bullet)$ is nothing but the quasi-fibered product (5.3)

The object O_3 is neutral with respect to the pointwise tensor product on the category C. it follows that $D^\bullet \otimes D^\bullet$ is again a projective resolution of O_3. There therefore exists a unique up to homotopy map of resolutions $r : D^\bullet \to D^\bullet \otimes D^\bullet$. Using the pullback via r we may define the cup product on quasi-fibered products as in the following Lemma, which further shows that this is the same product mentioned above.

Lemma 5.6. *Suppose that $B^\bullet$ is another diagram of complexes. Then, the map*

$$\mathcal{H}\mathrm{om}(D^\bullet, A^\bullet) \times \mathcal{H}\mathrm{om}(D^\bullet, B^\bullet) \xrightarrow{\otimes} \mathcal{H}\mathrm{om}(D^\bullet \otimes D^\bullet, A^\bullet \otimes B^\bullet) \xrightarrow{or} \mathcal{H}\mathrm{om}(D^\bullet, A^\bullet \otimes B^\bullet)$$

gives the cup product of [Bes00b, Lemma 3.2].

Proof. We consider the following commutative diagram, where the rows are the resolutions $D^\bullet$ and $D^\bullet \otimes D^\bullet$ and the vertical maps make up the morphism of resolutions r.

$$\begin{array}{ccccccc}
[K,K,K] & \xleftarrow{[\mathrm{id},+,\mathrm{id}]} & [K;\mathrm{id}\oplus 0, K\oplus K, 0\oplus\mathrm{id};K] & \longleftarrow & & & [0,K,0] \\
\Big\downarrow = & & \Big\downarrow r_1=[\mathrm{id},?,\mathrm{id}] & & & & \Big\downarrow r_2=[0,?',0] \\
[K,K,K] & \xleftarrow[{[\mathrm{id},+\otimes+,\mathrm{id}]}]{} & [K,(K\oplus K)^{\otimes 2},K] & \xleftarrow[{[0,\alpha_1,0]}]{} & [0,(K\oplus K)^2,0] & \xleftarrow[{[0,\alpha_2,0]}]{} & [0,K,0]
\end{array}$$

Here, the map $+$ adds the two coordinates. The map α_1 from $(K\oplus K)\otimes K \oplus K\otimes(K\oplus K) = (K\oplus K)^2$ to $(K\oplus K)^{\otimes 2}$ is given by $\mathbf{x}\oplus\mathbf{y} \mapsto \mathbf{x}\otimes(1,-1) - (1,-1)\otimes\mathbf{y}$. In the object on the bottom row $(K\oplus K)^{\otimes 2}$ the two maps $K \to (K\oplus K)^{\otimes 2}$ send 1 to $(1,0)\otimes(1,0)$ and $(0,1)\otimes(0,1)$ respectively. Finally, the map on the bottom line $\alpha_2 : K \to (K\oplus K)^2$ is obtained by sending 1 to $((1,-1),(1,-1))$. Now we can begin to compute the possible maps of resolutions r_1 and r_2. In r_1 the two side maps are clearly forced by commutativity of the diagram to be the identity maps. To make r_1 a map in C, the map ? has no choice but to be $(x,y)\mapsto x(1,0)\otimes(1,0)+y(0,1)\otimes(0,1)$. Turning to r_2, the commutativity of the diagram implies that the map $?'$ should map 1 to an element $(\mathbf{x},\mathbf{y})\in(K\oplus K)^{\times 2}$ with the property that

$$\mathbf{x}\otimes(1,-1)+(1,-1)\otimes\mathbf{y}=(1,0)\otimes(1,0)-(0,1)\otimes(0,1)$$

This is an explicit linear equation whose solution may be computed to depend on one parameter α as follows

$$\mathbf{x}=(\alpha,1-\alpha)\ ,\ \mathbf{y}=(1-\alpha,\alpha)\ .$$

The map in the Lemma can now be easily computed and checked to agree with the map defined in [Bes00b, Lemma 3.2]. □

Corollary 5.7. *Given a diagram of product triples*

$$A_1\to A_3\leftarrow A_2$$

the quasi-fibered product $A_1\tilde{\times}_{A_3}A_2$ *has a structure of a product triple. Given a similar diagram with* B_i*'s instead of* A_i*'s and pf-maps* $f_i: A_i\to B_i$ *for* $i=1,2,3$ *commuting with the maps in the diagrams, the obvious induced map*

$$A_1\underset{A_3}{\tilde{\times}}A_2\to B_1\underset{B_3}{\tilde{\times}}B_2$$

is a pf-map

Proof of Theorem 5.2. For a scheme $\mathcal{X}$ and we construct product triples corresponding to the de Rham, filtered de Rham and rigid complexes. Let us discuss the rigid case as the other two cases do not have any additional complications. Suppose $\dim X=d'$. Then we set

$$\begin{aligned}A^i&=\operatorname{Hom}(\mathbb{R}\Gamma_{\mathrm{rig},c}^{2d'-i}(\mathcal{X}_s/K),K)\\ A_c^\bullet&=\mathbb{R}\Gamma_{\mathrm{rig},c}\\ A_{cc}^i&=\operatorname{Hom}(\mathbb{R}\Gamma_{\mathrm{rig}}^{2d'-i}(\mathcal{X}_s/K),K)\end{aligned}\tag{5.4}$$

with obvious differentials. Poincaré duality implies that these complexes are quasi-isomorphic to the rigid and rigid with compact support complexes. The cup product is defined by the formula

$$(a\cup a_c)(x)=a(a_c\cup x)\ .$$

for $a\in A^i, a_c\in A_c^j$ and $x\in\mathbb{R}\Gamma_{\mathrm{rig}}^{2d'-i-j}(\mathcal{X}_s/K)$. Let us call the resulting product triple A_{rig}. We have similar product triples A_{dR} and F^nA_{dR}

Consider now a proper morphism $f:\mathcal{X}\to\mathcal{Y}$, with Y of dimension $d+d'$, and denote by B the corresponding complexes for $\mathcal{Y}$. Then one immediately obtains the required pushforward maps,

$$f_*: A^i\to B^{i+d}\ ,\ f_*: A_{cc}^i\to B_{cc}^{i+d}$$

as the duals of the pullback maps in the complexes with and without compact supports, and the projections formula is trivially satisfied:

$$(f_*(a)\cup b_c)(y)=f_*(a)(b_c\cup y)=a(f^*(b_c\cup y)=a(f^*(b_c)\cup f^*(y))=(a\cup f^*(b_c))(f^*(y))=f_*(a\cup f^*(b_c))(y)\ .$$

Thus, we obtain a pf-map of product triples $f: A_{\mathrm{rig}}\to B_{\mathrm{rig}}$ and we have similar pf-maps in the de Rham and filtered de Rham context.

If we need complexes with an action of Frobenius we replace in (5.4) the complexes $\mathbb{R}\Gamma_{\mathrm{rig}}$ and $\mathbb{R}\Gamma_{\mathrm{rig},c}$ with $\mathbb{R}\Gamma_{\mathrm{rig},p}$ and $\mathbb{R}\Gamma_{\mathrm{rig},cp}$ from Definition 4.1 respectively, and define φ^*, when it is not already defined, as $q^{d'}$ times the dual of $(\varphi^*)^{-1}$. This gives us a product triple $A_{\mathrm{rig},p}$. In the situation of pushforwards described above, pushforwards commute with Frobenius, except that we we need to multiply the Frobenius on $\mathcal{Y}$ by q^d (this will account for the fact that pushforwards change the twist.

Now we may finally write the diagram of product triples, analogous to (4.6)

$$\begin{array}{ccccccc} F^n A_{\mathrm{dR}} & & & A' & & & A_{\mathrm{rig},p} \\ & \searrow & \swarrow & & \searrow & \swarrow & \\ & & A_{\mathrm{dR}} & & A_{\mathrm{rig}} & & \end{array} \tag{5.5}$$

where A' is the triple $A_{\mathrm{dR}}, (A_{\mathrm{rig}})_c, (A_{\mathrm{rig}})_{cc}$, where the product is obtained as in (4.7) by applying the map $A_{\mathrm{dR}} \to A_{\mathrm{rig}}$ (obtained by duality from the cospecialization map), with the cup product on the triple A_{rig}. As in the corresponding discussion in Section 4, the fact that this is indeed a diagram of triples (i.e., commutes with cup products) is obvious everywhere except one place, where it follows again formally from the commutativity of (4.8).

Now we may apply Lemma 5.5 and Corollary 5.7 repeatedly to deduce that for each $\mathcal{X}$ as above and for polynomials P and Q there are product triples whose terms are quasi-isomorphic to $\mathbb{R}\Gamma_{\mathrm{f},P}(\mathcal{X},n)$, $\mathbb{R}\Gamma_{\mathrm{f},Q,c}(\mathcal{X},m)$ and $\mathbb{R}\Gamma_{\mathrm{f},P*Q,c}(\mathcal{X},m+n)$ respectively, such that the product induces the same map on cohomology as the cup product between these complexes, and such that a morphism $f:\mathcal{X}\to\mathcal{Y}$ induces a pf-map of the triples, whose components induce the maps f^* and f_* on cohomology. Taking cohomology one gets the first projection formula. To get the second projection formula, simply switch the roles of $A^\bullet$ and $A_c^\bullet$. $\square$

We end this section by considering the following situation, that will be used in the next section. Suppose $f:\mathcal{Y}\to\mathcal{X}$ is open immersion with $\mathcal{X}$ proper. Then we have

$$H^i_{\mathrm{fp},c}(\mathcal{Y},n,m) \xrightarrow{f_*} H^i_{\mathrm{fp},c}(\mathcal{X},n,m) = H^i_{\mathrm{fp}}(\mathcal{X},n,m)\ . \tag{5.6}$$

Lemma 5.8. *Suppose $\mathcal{X}-\mathcal{Y}$ has codimension at least 2 in $\mathcal{X}$, then the map*

$$f_*: H^2_{\mathrm{fp},c}(\mathcal{Y},n,m)\to H^2_{\mathrm{fp}}(\mathcal{X},n,m)$$

is surjective.

Proof. This is a consequence of purity. The obstruction for surjectivity lies in the third cohomology of $MF(\mathbb{R}\Gamma_{\mathrm{fp},c}(\mathcal{Y},n,m)\to\mathbb{R}\Gamma_{\mathrm{fp},c}(\mathcal{X},n,m))$. This in turn may be rewritten as a quasi-fibered product of the mapping fibers corresponding to the rigid and de Rham complexes, and this sits in a short exact sequence involving the second and third cohomologies of these mapping fibers. It therefore suffices to show that the second and third cohomologies of $MF(\mathbb{R}\Gamma_{\mathrm{dR},c}(Y)\to\mathbb{R}\Gamma_{\mathrm{dR},c}(X))$ are trivial, and similar results in the rigid and filtered de Rham setting. Poincaré duality makes these cohomologies dual to $H^i_{\mathrm{dR},X-Y}(X/K)$ (cohomology with supports in $X-Y$), with $i=1,2$, and these are 0 by purity and the codimension 2 assumption. $\square$

6. Computation of the regulator

In this section we will use the tools described in previous sections to give a proof of Theorem 1.1. We begin with a more general setup: We are given smooth $\mathcal{O}_K$ curves $\mathcal{X}_i$ and morphisms $g_i : \mathcal{X}_i \to \mathcal{S}$. In each $\mathcal{X}_i$ we have an open subscheme $\mathcal{Y}_i$, the complement of a finite number of sections, and a cohomology class $\alpha_i \in H^1_{\mathrm{ms}}(\mathcal{Y}_i, 1)$. Removing the images of all of the above sections under the g_i's we obtain an open subscheme $\mathcal{Y}$ of $\mathcal{S}$, the complement of the union $\mathcal{D}$ of a finite number of sections, and we have a cohomology class

$$\alpha = \sum (g_i)_* \alpha_i \in H^3_{\mathrm{ms}}(\mathcal{Y}, 2)$$

We assume further that there exists

$$\tilde{\alpha} \in H^3_{\mathrm{ms}}(\mathcal{S}, 2) \xleftarrow{\sim \text{ Prop. 2.6}} H^2_{\mathrm{dR}}(S/K)/F^2$$

with $\tilde{\alpha}|_{\mathcal{Y}} = \alpha$.

Suppose now that $\mu \in F^1 H^2_{\mathrm{dR}}(S/K)$ and we wish to compute the cup product $\tilde{\alpha} \cup \mu$. According to Proposition 2.10 we may lift μ to an element $\beta \in H^2_{\mathrm{fp}}(\mathcal{S}, 1, 2)$ and compute instead the cup product in H_{fp} cohomology,

$$\tilde{\alpha} \cup \beta = \tilde{\alpha} \cup \mu \ , \tag{6.1}$$

where $\tilde{\alpha}$ is now considered an element of $H^3_{\mathrm{fp}}(\mathcal{S}, 2, 4)$ via (2.6).

The next step is to rewrite this cup product in terms of α. To do this, we use Lemma 5.8 to find $\tilde{\beta} \in H^2_{\mathrm{fp},c}(\mathcal{Y}, 1, 2)$ mapping to β via (5.6). By using the projection formulas Corollary 5.3 for the proper maps g_i and for the open immersion $\mathcal{Y} \subset \mathcal{S}$, we have

$$\beta \cup \tilde{\alpha} = \tilde{\beta} \cup \alpha = \tilde{\beta} \cup (\sum (g_i)_* \alpha_i) = \sum (g_i^* \tilde{\beta}) \cup \alpha_i \ . \tag{6.2}$$

Remark 6.1. Let us note the following regarding the choice of $\tilde{\alpha}$ and $\tilde{\beta}$. We have the long exact sequence

$$\cdots \to H^3_{\mathrm{fp},\mathcal{D}}(\mathcal{S}, 2, 4) \to H^3_{\mathrm{fp}}(\mathcal{S}, 2, 4) \to H^3_{\mathrm{fp}}(\mathcal{Y}, 2, 4) \to H^4_{\mathrm{fp},\mathcal{D}}(\mathcal{S}, 2, 4) \to \cdots$$

We have by purity and the short exact sequence (2.5) that $H^3_{\mathrm{fp},\mathcal{D}}(S, 2, 4) = 0$ while $H^4_{\mathrm{fp},\mathcal{D}}(S, 2)$ may not equal 0. This shows that a lift $\tilde{\alpha}$ to α may not exist but that it is always unique if it exists, which we are assuming. Consider now the corresponding situation for cohomology with compact supports. There we know by Lemma 5.8 that β may always be lifted to a $\tilde{\beta}$ but this lift may not be unique. If, however α comes from an $\tilde{\alpha}$, then the choice of this lift will not matter for the cup product.

We now impose the assumption that μ is the cup product of two cohomology classes, $\mu = \omega \cup \eta$, with $\omega \in F^1 H^1_{\mathrm{dR}}(S/K)$ and $\eta \in H^1_{\mathrm{dR}}(S/K)$. Under this assumption we may distinguish a particular $\tilde{\beta}$ as follows.

Lemma 6.2. *For any open immersion of an $\mathcal{O}_K$-schemes $\mathcal{Y}$ into a proper connected $\mathcal{O}_K$-scheme $\mathcal{X}$ with complement which is the image of a finite number of sections the natural maps*

$$H^1_{\mathrm{fp},c}(\mathcal{Y}, 0, 1) \to H^1_{\mathrm{fp}}(\mathcal{X}, 0, 1) \to H^1_{\mathrm{dR}}(X/K)$$

are isomorphisms.

Proof. Suppose that the complement $\mathcal{D}$ is smooth over $\mathcal{O}_K$. Then we can interpret cohomology with compact support as relative cohomology

$$H^1_{\mathrm{fp},c}(\mathcal{X}\ rel\ \mathcal{D},0,1)\ .$$

We then have a long exact sequence

$$\cdots \to H^0_{\mathrm{fp}}(\mathcal{D},0,1) \to H^1_{\mathrm{fp},c}(\mathcal{Y},0,1) \to H^1_{\mathrm{fp}}(\mathcal{X},0,1) \to H^1_{\mathrm{fp}}(D,0,1) \to \cdots$$

and we easily see from (2.5) that $H^0_{\mathrm{fp}}(\mathcal{D},0,1) \cong H^0_{\mathrm{dR}}(D/K)^{\mathcal{P}_1(\varphi)=0} = 0$ and $H^1_{\mathrm{fp}}(\mathcal{D},0,1) \cong H^0_{\mathrm{dR}}(D/K)/F^0 = 0$ because all polynomial in $\mathcal{P}_1$ applied to φ are invertible on $H^0_{\mathrm{dR}}(D/K)$. This shows that the first map is an isomorphism. In general, this argument needs to be slightly modified in a similar way to the proof of Lemma 5.8. That the second is an isomorphism is an easy consequence of the short exact sequence (2.8). □

Let $\tilde{\eta} \in H^1_{\mathrm{fp},c}(\mathcal{Y},0,1)$ be the unique element mapping to η under the isomorphisms of Lemma 6.2. We further fix $\tilde{\omega} \in H^1_{\mathrm{fp}}(\mathcal{S},1,1)$ lifting ω in (2.8). We recall from Remark 2.14 that we may interpret $\tilde{\omega}$ as a Coleman integral F_ω of ω. It is clear that we may take $\tilde{\beta} = \tilde{\omega} \cup \tilde{\eta} \in H^2_{\mathrm{fp},c}(\mathcal{S},1,2)$. From (6.1) and (6.2) it follows that we have

$$\tilde{\alpha} \cup \omega \cup \eta = \sum g_i^*(\tilde{\omega}) \cup g_i^*(\tilde{\eta}) \cup \alpha_i\ , \tag{6.3}$$

where we have that $g_i^*(\tilde{\eta})$ maps to $g_i^*\eta$ under the isomorphism of Lemma 6.2 while $g_i^*(\tilde{\omega})$ may be interpreted as a Coleman integral of ω given by $g_i^*F_\omega$.

At this point we are left with a computation that may be carried out for each of the i's individually. We therefore remove the subscript i and replace $g_i^*(\tilde{\omega})$ and $g_i^*(\tilde{\eta})$ by $\tilde{\omega}$ and $\tilde{\eta}$ respectively. Suppose then that $\mathcal{X}$ is a smooth complete curve over $\mathcal{O}_K$ and that $\mathcal{Y} \subset \mathcal{X}$ is an open subscheme obtained by removing a finite number of sections. Let X and Y be the generic fibers of $\mathcal{X}$ and $\mathcal{Y}$ respectively. Let $\tilde{\omega} \in H^1_{\mathrm{fp}}(\mathcal{X},1,1)$, or equivalently, by Remark 2.14, a Coleman integral F_ω to a holomorphic form ω on X. We are furthermore given $[\eta] \in H^1_{\mathrm{dR}}(X/K)$, which is represented by a form η of the second kind on X. Via Lemma 6.2 we may also think of $[\eta]$ as an element $\tilde{\eta} \in H^1_{\mathrm{fp},c}(\mathcal{Y},0,1)$. Finally, we are given $f \in \mathcal{O}_K^\times(\mathcal{Y})$ and consider $\mathrm{reg}(f) \in H^1_{\mathrm{ms}}(\mathcal{Y},1) \subset H^1_{\mathrm{fp}}(\mathcal{Y},1,2)$.

We may thus consider the cup product

$$\mathrm{reg}(f) \cup \tilde{\omega} \cup \tilde{\eta} \in H^3_{\mathrm{fp},c}(\mathcal{Y},2,4) \xrightarrow{\sim} H^2_{\mathrm{dR},c}(Y/K) \xrightarrow{\mathrm{tr}} K$$

We identify this cup product with an element of K. The next Proposition, together with (6.3), completes the proof of Theorem 1.1.

Proposition 6.3. *We have*

$$\mathrm{reg}(f) \cup \tilde{\omega} \cup \tilde{\eta} = \langle F_\eta, \log(f); F_\omega \rangle_{\mathrm{gl}}\ .$$

To begin proving this, we break down the cup product above as the cup product of $\mathrm{reg}(f) \cup \tilde{\omega} \in H^2_{\mathrm{fp}}(\mathcal{Y},2,3)$ with $\tilde{\eta}$.

From (2.7) (see also Remark 2.12) we easily obtain an isomorphism

$$H^1_{\mathrm{rig}}(\mathcal{Y}_s/K) \xrightarrow{\sim} H^2_{\mathrm{fp}}(\mathcal{Y},2,3)\ . \tag{6.4}$$

Let $[\theta]$ be the image of $\operatorname{reg}(f)\cup\tilde{\omega}$ under the inverse of (6.4). Let $[\eta]' \in H^1_{\mathrm{rig},c}(\mathcal{Y}_s/K)$ be the image of $\tilde{\eta} \in H^1_{\mathrm{fp},c}(\mathcal{Y},0,1)$ under the map (4.3). By Proposition 4.6 we have

$$\operatorname{reg}(f)\cup\tilde{\omega}\cup[\eta] = [\theta]\cup[\eta]' \,.$$

We next give an explicit description of de Rham and rigid cohomologies with compact supports and of the cup product in the one dimensional case. From the description of cohomology with compact supports as relative cohomology it is easy to derive the following description of $H^1_{\mathrm{dR},c}(Y/K)$. Elements consist of pairs $(\eta,\{g_x\}_{x\in X-Y})$ where η is a form of the second kind on X and the g_x are explicit formal integrals around the points x. We quotient out by the relation that identifies to 0 the pair $(dg,\{g|_x\})$ for a rational function on X, where the $g|_x$ are the local expansions of g near x.

Recall now that the cup product on de Rham cohomology of curves may be given by the following simple recipe: Suppose $[\omega]$ and $[\eta]$ are two cohomology classes on X represented by the forms of second kind ω and η respectively. The cup product is then given by

$$\sum_x \operatorname{Res}_x(\int \eta)\omega$$

where the sum is over all closed points $x\in X$, the integral $\int\eta$ is a formal integral at x, which exists because the form is of the second kind, and the residue is independent of the constant of integration because ω is of the second kind.

This formula easily extends to a formula for the cup product $H^1_{\mathrm{dR},c}(Y/K)\times H^1_{\mathrm{dR}}(Y/K)\to K$, where Y is open in X. Now, ω may have residues at the points of $X-Y$, while $[\eta]$ now is a cohomology class with compact supports, so that we are given local integrals of η at the points of $X-Y$ as above. Clearly, this additional information compensates for the possibility of residues of ω in the above formula.

The situation with rigid cohomology is identical. One merely has to replace the local integrals and residues around the x's in $X-Y$ with local integrals and residues along the annuli ends of the wide open space U corresponding to $\mathcal{Y}_s$. Thus, if $\omega\in\Omega^1(U)$ represents an element $[\omega]\in H^1_{\mathrm{rig}}(\mathcal{Y}_s/K)$ while $(\eta,\{(g_e)|_e\})$ represent an element $[\eta]\in H^1_{\mathrm{rig},c}(\mathcal{Y}_s/K)$. Their cup product is

$$[\omega]\cup[\eta] = \sum_x \operatorname{Res}_x(\int\eta)\omega + \sum_e \operatorname{Res}_e g_e\omega \,,$$

where the x's range over the underlying affinoid to U.

Going back to the situation at hand, recall that the class $[\eta]'\in H^1_{\mathrm{rig},c}(\mathcal{Y}_s/K)$ is derived from $[\eta]\in H^1_{\mathrm{dR}}(X/K)$, represented by the form of the second kind η, by composing the inverse of the isomorphism provided by Lemma 6.2 with the map coming from (4.3). It can be described explicitly as follows. We choose a Coleman integral F_η to η and we send $[\eta]$ to the class $[\eta]'=(\eta,\{(F_\eta)|_e\})$ where the last term denotes the local expansion on the annuli end e. It is easy to see that this is well defined. This description immediately leads to the formula

$$[\theta]\cup[\eta]' = \langle F_\eta, F_\theta\rangle_{\mathrm{gl}} \tag{6.5}$$

Proof of Proposition 6.3. We begin by explicitly computing the cup product $\operatorname{reg}(f)\cup\tilde{\omega}\in H^2_{\mathrm{fp}}(\mathcal{Y},2,3)$. First we switch to the tilded version of cohomology, which looses nothing by Remark 2.12. Thus, let U again be the wide open space corresponding to $\mathcal{Y}$. Let $P(t)=1-t/q$ and let $Q(s)\in\mathcal{P}_1\cdot\mathcal{P}_2$ be a polynomial with the property that $Q(\phi^*)F_\omega \in A(U)$. It follows easily from the description of regulators of functions [Bes00b, Proposition 10.3] that, in the representation (2.13), $\mathrm{reg}(f)$ is represented by $(d\log f, P(\phi^*)\log(f))$ (this last function is indeed in $A(U)$!) while $\tilde{\omega}$ is represented by $(\omega, Q(\phi^*)F_\omega)$. Let $a(t,s)$ and $b(t,s)$ be polynomials satisfying (2.10).

Recall that

$$\mathrm{reg}(f) \cup \tilde{\omega} = [\theta] \in H^1_{\mathrm{rig}}(\mathcal{Y}_s/K) \cong H^2_{\mathrm{fp}}(\mathcal{Y}, 2, 3) ,$$

where the last isomorphism is (6.4). Recall now that $H^1_{\mathrm{rig}}(\mathcal{Y}_s/K)$ may be identified with the de Rham cohomology $H^1_{\mathrm{dR}}(U/K)$ and thus $[\theta]$ may be identified with the cohomology class of a one form $\theta \in \Omega^1(U)$. Using the formula (2.11) for the cup product, which is also valid as we remarked already in the tilded version, and taking into account the map in (2.5) inducing the isomorphism above, we find that the relation satisfied by θ is the following.

$$P*Q(\phi^*)\theta = \cdot\,(b(t,s)(d\log(f)\otimes Q(\phi^*)F_\omega) + a(t,s)(P(\phi^*)\log(f)\otimes\omega)) + dh \tag{6.6}$$

where here the rule is that t acts on the first term in the tensor product as ϕ^* while s acts on the second term, and the product sign at the front corresponds to the multiplication map on the tensor product.

Recall now the bilinear pairing $\langle\!\langle\ \rangle\!\rangle$ from (2.14). By (6.5) and Proposition 2.17 we need to compute $\langle\!\langle F_\eta, \theta\rangle\!\rangle$ and the proposition can be reformulated as

$$\langle\!\langle F_\eta, \theta\rangle\!\rangle = \langle\!\langle F_\eta, \log(f)\omega\rangle\!\rangle .$$

By Lemma 2.16 both sides, viewed as functions of η, are functionals on the cohomology $H^1_{\mathrm{dR}}(U/K)$. If $P*Q$ has degree n, then we observe that $t^n P*Q(q/t)$ has only negative weights, and its evaluation at ϕ^* is therefore invertible on $H^1_{\mathrm{dR}}(U/K)$. It now follows from (2.16) that the above formula follows from

$$\langle\!\langle F_\eta, P*Q(\phi^*)(\theta)\rangle\!\rangle = \langle\!\langle F_\eta, P*Q(\phi^*)(\log(f)\omega)\rangle\!\rangle \tag{6.7}$$

By (6.6) and by the vanishing of $\langle\!\langle F_\eta, dh\rangle\!\rangle$ (Proposition 2.17), the left hand side expands to

$$b(s,t)\langle F_\eta, Q(\phi^*)F_\omega; \log(f)\rangle_{\mathrm{gl}} + a(t,s)\langle F_\eta, P(\phi^*)\log(f); F_\omega\rangle_{\mathrm{gl}} ,$$

where the notation means that for each monomials $t^m s^n$ one is to apply $(\phi^*)^m$ to the second term in the triple index and $(\phi^*)^n$ to the third term before computing the index (notice that in b the roles of s and t are switched). The right hand side of (6.7) expands, using (2.10), to the rather similarly looking

$$b(t,s)\langle F_\eta, \log(f); Q(\phi^*)F_\omega\rangle_{\mathrm{gl}} + a(t,s)\langle F_\eta, P(\phi^*)\log(f); F_\omega\rangle_{\mathrm{gl}} .$$

The second term (multiplied by $a(t,s)$) is identical. The following lemma shows that the first terms also agree and finishes the proof. □

Lemma 6.4. *We have for each m and n*

$$\langle F_\eta, (\phi^*)^n Q(\phi^*)F_\omega; (\phi^*)^m\log(f)\rangle_{\mathrm{gl}} + \langle F_\eta, (\phi^*)^m\log(f); (\phi^*)^n Q(\phi^*)F_\omega\rangle_{\mathrm{gl}} = 0 .$$

Proof. Indeed, by the triple identity (2) the sum above equals the negative of

$$\langle (\phi^*)^n Q(\phi^*)F_\omega, (\phi^*)^m\log(f); F_\eta\rangle_{\mathrm{gl}} ,$$

by this last term is zero because $Q(\phi^*)F_\omega \in A(U)$ and by Lemma 2.16. □

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