The syntomic regulator for $K_1$ of surfaces

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Plan

- The problem and results over $\mathbb{C}$
- Double and triple indices
- Statement of the main result
- fp-cohomology and pushforwards in syntomic cohomology
- Syntomic cohomology with compact supports
- Sketch of the proof
- Applications
The elements we consider

\( S/L \) - smooth projective surface over a field \( L \)

\[
H^1(S, \mathcal{K}_2) = H^1 \text{ of }
\]

\[
K_2(L(S)) \to \bigoplus_{\text{cod} C=1} L(C)^\times \to \bigoplus_x \mathbb{Z}
\]

\[
\theta = \sum (Z_i, f_i) \in H^1(S, \mathcal{K}_2)
\]

\[
H^1(S, \mathcal{K}_2) \otimes \mathbb{Q} \cong H^3_M(S, \mathbb{Q}(2))
\]
The regulator

$$\text{reg}_{\text{ét}}(\theta) \in H^1(L, H^2_{\text{ét}}(\bar{S}, \mathbb{Q}_p(2)))$$

Assume: $$[L : \mathbb{Q}_p] < \infty$$

**Bloch and Kato**

$$\exp : H^{2i-j-1}_{dR}(S/L)/F^i \rightarrow H^1(L, H^{2i-j-1}_{\text{ét}}(\bar{S}, \mathbb{Q}_p(i)))$$

$$i = 2, j = 1$$

$$\exp : H^2_{dR}(S/L)/F^2 \rightarrow H^1(L, H^2_{\text{ét}}(\bar{S}, \mathbb{Q}_p(2)))$$
Goal

1. find \( \text{reg}_{\text{syn}}(\theta) \in H^2_{dR}(S/L)/F^2 \) s.t. \( \exp(\text{reg}_{\text{syn}}(\theta)) = \text{reg}_{\text{ét}}(\theta) \)

2. describe explicitly
   \[
   \text{reg}_{\text{syn}}(\theta) \in H^2_{dR}(S/L)/F^2 = \text{Hom}(F^1H^2_{dR}(S/L), L)
   \]

Goal 1 ← integrality assumption

\( S \) - smooth integral model of \( S \)

\[
\begin{array}{c}
K_j(S) \quad \begin{array}{c}
\downarrow \text{reg}_{\text{syn}} \\
H^{2i-j-1}_{dR}(S/L)/F^i \quad \exp \quad H^1(L, H^2_{\text{ét}}-j-1(\bar{S}, \mathbb{Q}_p(i))))
\end{array} \\
\end{array} \\
\end{array}
\]

Goal 1 is achieved if \( \theta \) comes from \( S \)

Goal 2 ← Strong integrality: each \( Z_i \) and \( f_i \) are integral
Classical case

$L = \mathbb{C}$

\[ \text{reg}(\theta) \in H_D^3(S, \mathbb{R}(2)) \]

**Theorem (Beilinson)**

\[ \omega \in F^1 H^2_{dR}(S/\mathbb{C}). \]

\[
\text{reg}(\theta)(\omega) = \frac{1}{2\pi \sqrt{-1}} \sum \int_{Z_i - Z_{i}^{\text{sing}}} \omega \log |f_i| .
\]
Coleman integration

\[ \log \text{ - branch of the } p\text{-adic logarithm}\]
\[ C/L \text{ - proper curve}\]

**Theorem**

\[ \omega \in \Omega^1(L(C)) \Rightarrow F_\omega : C(\bar{L}) \to \bar{L} \text{ (depends on the choice of } \log)\]
unique up to constant.

**Remark**

Coleman integrals exist in higher dimensions

**Local integration:**
\[ A_{\log} = L((z))[\log(z)]\]
\[ 0 \to L \to A_{\log} \xrightarrow{d} A_{\log} dz \to 0 \text{ - integration by parts}\]
\[ \omega \in L((z))dz \subset A_{\log} dz \Rightarrow F_\omega \in A_{\log} \text{ up to constant}\]
Double indices

\( \omega, \eta \in L((z))dz \).

want a notion of \( \text{Res}_0 F_{\omega \eta} \)

**Problem**

\( \omega = a_0 dz/z + \cdots \Rightarrow F_\omega = a_0 \log(z) + \cdots, \text{Res}_0 \log(z)dz/z = ? \).

\( \text{Res}_0 \omega = 0 \Rightarrow \) no problem

\( \text{Res}_0 \eta = 0 \Rightarrow \) no problem: define as \( -\text{Res}_0 F_{\eta \omega} \)

**General solution**

double index \( \langle F_\omega, F_\eta \rangle \) depending on both \( F_\omega \) and \( F_\eta \), bilinear, antisymmetric and equal to above when defined.

Key idea: \( \langle \log(z), \log(z) \rangle = 0 \).

\[ \langle F_\omega + C, F_\eta \rangle = \langle F_\omega, F_\eta \rangle + C \text{Res}_0 \eta \]
\[ \omega, \eta \in \Omega^1(L(C)) \]
\[ \sum_{x \in C} \langle F_\omega, F_\eta \rangle_x := \langle F_\omega, F_\eta \rangle_{gl} \]
\[ F_\eta, F_\omega \text{ are Coleman integrals.} \]

**Interesting properties**

- depends only on \( \omega, \eta \)
- \( = \Psi(\omega) \cup \Psi(\eta) \)
- where \( \Psi : \Omega^1(L(C)) \to H^1_{dR}(C/L) \)
Triple index

Data: $F, G, H \in A_{\log}, \ dF, dG, dH \in L((z))dz$

We Want

Something like $\text{Res}_0(\int FGdH) = ?$

Obvious easy case: $G \in L((z)) \Rightarrow \langle F, \int GdH \rangle$
So: should know $\int GdH$

Auxiliary data

$\int RdS \in A_{\log}, \ R \neq S \in \{F, G, H\}$
s.t. $\int RdS + \int SdR = RS$
Theorem

 Exists and unique \((F, G, H) + \text{ auxiliary data} \rightarrow \langle F, G; H \rangle\) which is:

- trilinear
- symmetric in \(F, G\)
- Triple identity \(\langle F, G; H \rangle + \langle F, H; G \rangle + \langle H, G; F \rangle = 0\)
- Reduction to double index \(\langle F, G; H \rangle = \langle F, \int G dH \rangle\) when \(G \in L((z))\)

Global index

\[
\langle F, G; H \rangle_{\text{gl}} = \sum_{x \in C} \langle F, G; H \rangle_x
\]

independent of auxiliary data (chosen as Coleman integrals)
Main theorem setup

\[ g_i : X_i \to S \] - Normalizations of \( Z_i \)

**Integrality assumption**

- \( g_i \) extend to smooth integral models \( g_i : X_i \to S \)
- \( \text{div}(f_i) \) do not contain the special fiber

\[ \omega \in F^1 H^1_{\text{dR}}(S/L) \]
\[ \eta \in H^1_{\text{dR}}(S/L) \] represented by form of the second kind \( \eta \) on \( S \).
\[ \mu = \omega \cup [\eta] \in F^1 H^2_{\text{dR}}(S/L) \]

\( F_\omega, F_\eta \) Coleman integrals of \( \omega, \eta \) respectively
Statement of the main theorem

**Theorem (***

\[ \text{reg}_{\text{syn}}(\theta)(\mu) = \sum_i \langle g_i^* F_\eta, \log(f_i); g_i^* F_\omega \rangle_{gl,X_i} \]

**Remark**

(*** has to do with compatibility between pushforwards in syntomic and motivic cohomology
Syntomic cohomology

$L$ - a $p$-adic field

$\mathcal{X}/\mathcal{O}_L$ - smooth

$\mathcal{X}$ - generic fiber

$\mathcal{X}_s$ - special fiber

Rigid and de Rham complexes

$\mathbb{R}\Gamma_{dR}(\mathcal{X}/L)$ de Rham complex computing $H_{dR}(\mathcal{X}/L)$ with a filtration $F^i$

$\mathbb{R}\Gamma_{rig}(\mathcal{X}_s/L)$ rigid complex computing $H_{rig}(\mathcal{X}_s/L)$ with a Frobenius $\phi$ of degree $q$
**Definition**

The modified syntomic complex of $X$ is

$$\mathbb{R}\Gamma_{\text{ms}}(X, n) := \lim_{\to} MF(F^n \mathbb{R}\Gamma_{dR}(X/L) \xrightarrow{1-(\phi/q^n)^j} \mathbb{R}\Gamma_{\text{rig}}(X_s/L))$$

**Remark**

If $X$ proper and $2n \neq i - 1$ then $1 - (\phi/q^n)^j$ is invertible on $H^{i-1}_{\text{rig}}(X_s/L)$ hence

$$H^{i-1}_{dR}(X/L)/F^n \hookrightarrow H^i_{\text{ms}}(X, n) \twoheadrightarrow F^n H^i_{dR}(X/L) \cap H^i_{\text{rig}}(X/L)^{\phi^j = q^{nj}}$$
The Syntomic regulator

\[
\text{reg} : H^i_M(\mathcal{X}, \mathbb{Q}(n)) \to H^i_{\text{syn}}(\mathcal{X}, n)
\]

\[
H^{i-1}_{dR}(X/L)/F^n \xhookrightarrow{} H^{i}_{\text{ms}}(X, n) \rightarrow F^n H^i_{dR}(X/L) \cap H^{i}_{\text{rig}}(X/L) = q^{nj}
\]

Fact

For \( X \) proper and \( i \neq 2n \), \( H^{i}_{\text{ms}}(X, n) \cong H^{i-1}_{dR}(X/L)/F^n \)

Theorem (Niziol)

\[
\begin{align*}
H^i_M(\mathcal{X}, \mathbb{Q}(n)) &\xrightarrow{} H^i_M(X, \mathbb{Q}(n)) \\
H^i(\mathcal{X}, n) = H^{i-1}_{dR}(X/L)/F^n &\xrightarrow{\text{exp}} H^1(L, \mathbb{H}^{2i-1}_{\text{ét}}(\bar{X}, \mathbb{Q}_p(i)))
\end{align*}
\]
No Poincare duality

\[ H_{dR}^{i-1}(X/L)/F^n \leftrightarrow H_{ms}^i(X, n) \rightarrow F^n H_{dR}^i(X/L) \cap H_{rig}^i(X/L)_{\phi} = q^{nj} \]

Problem

The sequence is non-symmetric \( \Rightarrow \) No Poincare duality

Solution

Replace \( 1 - (\phi/q^n)^j \) by more general polynomials
Consider polynomials $P(t)$ “of weight $2i$”

\[ P|Q \Rightarrow \]

\[ F^n \mathbb{R} \Gamma_{dR}(X/L) \xrightarrow{P(\phi)} \mathbb{R} \Gamma_{\text{rig}}(\mathcal{X}_s/L) \]

\[ Q/P(\phi) \]

\[ F^n \mathbb{R} \Gamma_{dR}(X/L) \xrightarrow{Q(\phi)} \mathbb{R} \Gamma_{\text{rig}}(\mathcal{X}_s/L) \]

**Definition**

\[ H^i_{fp}(\mathcal{X}, n) := \lim_{P} H^i(MF(F^n \mathbb{R} \Gamma_{dR}(X/L) \xrightarrow{P(\phi)} \mathbb{R} \Gamma_{\text{rig}}(\mathcal{X}_s/L))) \]
cupid product and Poincare duality

**Proposition**

For $X$ proper
\[ H_{dR}^{i-1}(X/L)/F^n \leftrightarrow H_{fp}^i(X, n) \rightarrow F^n H_{dR}^i(X/L) \]

so there is a chance for Poincare duality

**Proposition**

There exists a cup product
\[ H_{fp}^i(X, n) \times H_{fp}^j(X, m) \rightarrow H_{fp}^{i+j}(X, n + m) \]

compatible with the short exact sequence
Proof of existence of cup products

Idea

- In the category of $L[t]$-modules

$$C_P := (L[t] \xrightarrow{P(t)} L[t])$$

is a free resolution of $V_P := L[t]/P(t)$

- $\mathcal{H}om(C_P, \mathbb{R}\Gamma_{\text{rig}}(\mathcal{X}_s/L)) = \text{MF}(\mathbb{R}\Gamma_{\text{rig}}(\mathcal{X}_s/L) \xrightarrow{P(\phi)} \mathbb{R}\Gamma_{\text{rig}}(\mathcal{X}_s/L))$

- $V_P \otimes V_Q = V_{P*Q} \Rightarrow C_P \otimes C_Q$ is a resolution of $V_{P*Q} \Rightarrow C_{P*Q} \rightarrow C_P \otimes C_Q$

cup product is induced by

$$\mathcal{H}om(C_P, \mathbb{R}\Gamma_{\text{rig}}(\mathcal{X}_s/L)) \times \mathcal{H}om(C_Q, \mathbb{R}\Gamma_{\text{rig}}(\mathcal{X}_s/L))$$

$$\rightarrow \mathcal{H}om(C_{P*Q}, \mathbb{R}\Gamma_{\text{rig}}(\mathcal{X}_s/L))$$
Poincare duality

Proposition

For $X$ proper of relative dimension $d$ the cup product and

$$\text{tr} : H_{fp}^{2d+1}(X, d + 1) = H_{dR}^{2d}(X/L) \cong L$$

gives Poincare duality

$$H_{fp}^i(X, n) = H_{fp}^j(X, m)^*, \ i + j = 2d + 1, \ m + n = d + 1$$
Pushforwards

For \( f : \mathcal{X} \to \mathcal{Y} \) of proper schemes, can define \( f_* \) on \( H_{\text{fp}} \) as the dual of \( f^* \).
This is reasonably computable

**Theorem (B)**

\( f_* \) commutes with \( f_* \) on Chow groups via

\[
CH^i(\mathcal{X}) \to H^{2i}_{\text{syn}}(\mathcal{X}, i) \to H^{2i}_{\text{fp}}(\mathcal{X}, i)
\]

The proof is Riemann-Roch, following Gillet and Messing
What about non-proper schemes

Can’t expect Poincare duality in the form above. Instead

1 Define cohomologies with compact supports and homologies, satisfying twisted Poincare duality (Bloch-Ogus) axioms
2 Define pushforwards based on homology
3 Define products between cohomologies and cohomologies with compact supports.
4 Prove a projection formula \( f_*(a \cup f^* b) = f_*(a) \cup b \)

1-3 are done by Chiarellotto, Ciccioni and Mazzari (for syntomic, fp is the same)
Berthelot defines rigid complexes with compact supports $\mathbb{R}\Gamma_{\text{rig},c}(\mathcal{X}_s/L)$ a cup product and Poincare duality.

Berthelot and Baldassarri define a map $\mathbb{R}\Gamma_{\text{rig},c} \to \mathbb{R}\Gamma_{\text{dR},c}$.

To get homology, dualize complexes with compact supports and write the same cone.

Poincare duality (in Bloch-Ogus sense) follows from Poincare duality in the rigid and de Rham cases.
Cohomology with compact supports

**Definition**

The (modified) syntomic complex with compact supports is the limit over $j$ of the homotopy limits of the following diagram

$$
F^n \mathbb{R} \Gamma_{dR,c} \quad \quad \mathbb{R} \Gamma_{dR,c} \quad \quad \mathbb{R} \Gamma_{rig,c} \quad \quad \mathbb{R} \Gamma_{rig,c} \quad \quad MF_c
$$

where $MF_c = MF(1 - (\phi/q^n)^i : \mathbb{R} \Gamma_{rig,c}(X_s/L) \rightarrow \mathbb{R} \Gamma_{rig,c}(X_s/L))$
Similarly one can write the modified syntomic complex as the limit of homotopy limits of

\[ F^n \mathbb{R} \Gamma_{dR} \]

and the terms match

\[ \mathbb{R} \Gamma_{dR} \times \mathbb{R} \Gamma_{rig,c} \rightarrow \mathbb{R} \Gamma_{rig} \times \mathbb{R} \Gamma_{rig,c} \rightarrow \mathbb{R} \Gamma_{rig,c} \]

\[ \mathbb{R} \Gamma_{fp} \times \mathbb{R} \Gamma_{fp,c} \rightarrow \mathbb{R} \Gamma_{fp,c} \]

The projection formula is essentially formal
Sketch of the proof

- $\sum(Z_i, f_i)$ gives an element in $H^3_{\mathcal{M}}(S, \mathbb{Q}(2))$ as follows:
  - It gives an element in $H^1_{\mathcal{M}}(\bigcup Z_i - \bigcup \text{div } f_i, \mathbb{Q}(1))$
  - Element is closed implies an extension to $H^1_{\mathcal{M}}(\bigcup Z_i, \mathbb{Q}(1))$
  - Pushforward to $S$

- In particular, the restriction to $S - \bigcup \text{div } f_i$ is the sum of the pushforward from $H^1_{\mathcal{M}}(X_i - \text{div } f_i, \mathbb{Q}(1))$ of the class of $f_i$.

- In syntomic cohomology, the analogue of the last statement holds and characterizes $\text{reg}(\theta)$.

- Lift $\mu$ to an element of $H^2_{\text{fp}}(S - \bigcup \text{div } f_i)$ (Coleman integration in hiding)

- Use projection formulas to transform the computation to a cup product computation on $X_i - \text{div } f_i$. 
Applications

- Langer - Example of $\theta \in H^1(E \times E, \mathcal{K}_2)$, $E$ a CM elliptic curve, with decomposable regulator
- $p$-adic analogue of Beilinson’s Theorem for the self product of a modular curve?