Convex Sets

A set C is a *convex set* if $\lambda x + (1-\lambda)y \in C$ for each $x \in C$, $y \in C$ and $0 < \lambda < 1$. A sum $\lambda_1 x_1 + \ldots + \lambda_m x_m$ is called a *convex combination* of $x_1 \ldots x_m$ if the cofficients λ_i are non-negative and $\lambda_1 + \ldots + \lambda_m = 1$.

Properties

1. If A and B are convex sets then the following sets are convex:

$$A \cap B,$$

$$A + B,$$

$$\alpha A,$$

$$\bar{A},$$

$$Int(A).$$

2. A set C is convex if and only if it contains all the convex combinations of its elements.

The intersection of all the convex sets containing a given subset S is called the *convex hull* of S and is denoted by $Co(S)$.

The convex hull of a finite set is colled a *polytope*.

3. $Co(S)$ consists of all the convex combinations of the elements of S.

4. If A is a convex set then

$$Int(A) = Int(\bar{A}).$$

Cones

A set K is a *cone* if $\lambda x \in K$ for each $x \in K$ and $\lambda > 0$.

A *convex cone* is a cone which is a convex set.

A *finite cone* is the sum of a finite number of rays.

1. The intersection of an arbitrary collection of convex cones is a convex cone.

2. A finite cone is a closed convex cone.

Extreme points

Let C be a convex set. A point $x \in C$ is an *extreme point* of C iff there is no way to express x as a convex combination $\lambda y + (1 - \lambda z)$ where $y \neq z$.

Separation theorems

Let A and B be two sets. We say that the sets can be *separated* if there exists a hyperplane

$$H = \{x \in E^n : (a, x) = \alpha\}$$

such that

$$(a, x) \geq \alpha \text{ for all } x \in A$$

and

$$(a, x) \leq \alpha \text{ for all } x \in B.$$

We say that the sets can be *strongly separated* if there exists a hyperplane

$$H = \{x \in E^n : (a, x) = \alpha\}$$

such that

$$(a, x) > \alpha \text{ for all } x \in A$$

and

$$(a, x) < \alpha \text{ for all } x \in B.$$

Lemma 1. If A is a closed convex set and $0 \bar{\in} A$ then the sets $\{0\}$ and A can be strongly separated.

Lemma 2. If A is a convex set and $0 \bar{\in} A$ then the sets $\{0\}$ and A can be separated.

Theorem 1. Let A and B be convex sets and $A \cap B = \emptyset$. Then A and B can be separated.

Theorem 2. Let A be a convex set, K be a convex cone and $A \cap K = \emptyset$. Then A and K can be separated by hyperplane:

$$H = \{x \in E^n : (a, x) = 0\}.$$