The Simplex Method. Problems with upper boundaries

Let us consider a linear program (problem C)

$$\min c[N] \times x[N]$$

subject to conditions:

$$A[M, N] \times x[N] = b[M],$$

$$l[N] \leq x[N] \leq u[N].$$

It is possible: $l_j = -\infty$ and $u_j = \infty$.

A basis N_B is *not degenerate* if the corresponding basic solution satisfy the following condition: $l_j < x_j < u_j$ for $j \in N_B$.

Assumptions:
(i) Each basis contains m components;
(ii) absence of degeneracy: each feasible basis is not degenerate.

Let N_B be a feasible basis and let x be the corresponding basic solution. Let the set $N_F = N \setminus N_B$ be divided into two sets:

$$N_F^- = \{j \in N_F : \ x_j = l_j\},$$

$$N_F^+ = \{j \in N_F : \ x_j = u_j\}.$$

Vector x is an optimal solution iff there exists a vector $y[M]$ such that

$$y[M] \times A[M, N_B] = c[N_B],$$

$$y[M] \times A[M, N_F^-] \leq c[N_F^-],$$

$$y[M] \times A[M, N_F^+] \geq c[N_F^+].$$

An improving algorithm (one step of the simplex method)

(1) Calculate the vector of the dual variables $y[M]$ from the following system of linear equations:

$$y[M] \times A[M, N_B] = c[N_B].$$

(2) If $y[M] \times A[M,j] \leq c[j]$ for all $j \in N_F^-$ and $y[M] \times A[M,j] \geq c[j]$ for all $j \in N_F^+$ then x is an optimal solution of problem (C) and y is an optimal solution of the dual problem.

(3) Find an element to be introduced into the basis: $j_0 \in N_F^-$ such that $y[M] \times A[M,j_0] > c[j_0]$ or $j_0 \in N_F^+$ such that $y[M] \times A[M,j_0] < c[j_0]$.

(4) Find a vector $\lambda[N_B]$ from the following system of linear equations:

$$A[M,N_B] \times \lambda[N_B] = A[M,j_0].$$

(5) Find an element to be removed from the basis: find $t^* = \min(t_1,t_2,t_3)$ where

if $j_0 \in N_F^-$

$$t_1 = \frac{x_{j_1} - l_{j_1}}{\lambda_{j_1}} = \min_{j \in N_B : \lambda[j] > 0} \frac{x_j - l_j}{\lambda_j};$$

$$t_2 = \frac{u_{j_2} - x_{j_2}}{-\lambda_{j_1}} = \min_{j \in N_B : \lambda[j] < 0} \frac{u_j - x_j}{-\lambda_j};$$

$$t_3 = u_{j_0} - l_{j_0};$$

if $j_0 \in N_F^+$

$$t_1 = \frac{x_{j_1} - l_{j_1}}{-\lambda_{j_1}} = \min_{j \in N_B : \lambda[j] < 0} \frac{x_j - l_j}{-\lambda_j};$$

$$t_2 = \frac{u_{j_2} - x_{j_2}}{\lambda_{j_1}} = \min_{j \in N_B : \lambda[j] > 0} \frac{u_j - x_j}{\lambda_j};$$

$$t_3 = u_{j_0} - l_{j_0};$$

If $t = \infty$ then problem (C) has no solution.

(6) Calculate a new basis N_B, new sets N_F^- and N_F^+ and new basic solution:

if $t^* = t_1$ and $j_0 \in N_F^-$:

$$N_B := N_B \cup j_0 \setminus j_1,$$

$$N_F^- := N_F^- \setminus j_0 \cup j_1,$$

N_F^+ is not changed,

$$x_{j_0} := l_{j_0} + t^*,$$

$$x_j = x_j - t^* \lambda_j \ (j \in N_B);$$

if $t^* = t_2$ and $j_0 \in N_F^-$:

$$N_B := N_B \cup j_0 \setminus j_2,$$

$$N_F^- := N_F^- \setminus j_0 \cup j_1,$$

$$N_F^+ := N_F^+ \cup j_2,$$

$$x_{j_0} := l_{j_0} + t^*,$$

$$x_j = x_j - t^*\lambda_j \ (j \in N_B);$$

if $t^* = t_1$ and $j_0 \in N_F^+$:

$$N_B := N_B \cup j_0 \setminus j_1,$$

$$N_F^- := N_F^- \cup j_1,$$

$$N_F^+ := N_F^+ \setminus j_0,$$

$$x_{j_0} := u_{j_0} - t^*,$$

$$x_j = x_j + t^*\lambda_j \ (j \in N_B);$$

if $t^* = t_2$ and $j_0 \in N_F^+$:

$$N_B := N_B \cup j_0 \setminus j_2,$$

$$N_F^+ := N_F^+ \setminus j_0 \cup j_2,$$

$$N_F^- \text{ is not changed},$$

$$x_{j_0} := u_{j_0} - t^*,$$

$$x_j = x_j + t^*\lambda_j \ (j \in N_B);$$

if $t^* = t_3$ and $j_0 \in N_F^-$:

$$N_B \text{ is not changed},$$

$$N_F^- := N_F^- \setminus j_0,$$

$$N_F^+ := N_F^+ \cup j_0,$$

$$x_{j_0} := u_{j_0},$$

$$x_j = x_j - t^*\lambda_j \ (j \in N_B);$$

if $t^* = t_3$ and $j_0 \in N_F^+$:

N_B is not changed,

$$N_F^+ := N_F^+ \setminus j_0,$$

$$N_F^- := N_F^- \cup j_0,$$

$$x_{j_0} := l_{j_0},$$

$$x_j = x_j + t^* \lambda_j \ (j \in N_B);$$