SINGLE-VALUED MASSEY PRODUCTS

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Abstract. Given a differential graded algebra $\mathcal{C}$, we show that if the m-fold Massey product map $H^1(\mathcal{C})^m \to H^2(\mathcal{C})$ is non-empty on every m-tuple, then it is single-valued on every m-tuple.

1. Introduction

Massey products are higher order cohomological operations, which generalize the more familiar cup product in cohomology. They are widely used in algebraic topology (see e.g., [Mas58], [Fen83], [FS84]), and in recent years also became increasingly important in Galois cohomology and arithmetic geometry ([HW12], [Mor04], [Mor12], [Sha07], [Vog05], [Wic09], [Wic12], [Efr13]). Roughly speaking, Massey products consist of sets of solutions of certain systems of "differential equations" in a differential graded algebra. In general, these sets may contain more than one solution, and also might be empty. Thus, unlike cup products, Massey products are in general multi-valued maps. One is therefore led to search for situations where the Massey products are single-valued maps. A sufficient condition for this to hold was given by Kraines [Kra66, Lemma 20] (see also [May69, Prop. 2.4] in the more general context of matric Massey products). In this note we give a simple, and quite natural, necessary and sufficient condition for the Massey product to be single-valued.

To explain our main result in more detail, we briefly recall the definition of the m-fold Massey product in the special case of degree 1 cohomology elements. Let $(\mathcal{C}, d)$ be a differential graded algebra with cohomology graded ring $\mathcal{H}$ (see §2). Given cohomology classes $h_1, \ldots, h_m \in H^1$, $m \geq 2$, we consider systems of 1-cochains a_{ij} in the punctured triangle $1 \leq i < j \leq m+1$, $(i,j) \neq (1, m+1)$, such that

(1) $a_{i,i+1}$ is a cocycle with cohomology class h_i, $i = 1, 2, \ldots, m$;
(2) $d(a_{ij}) = \sum_{r=i+1}^{j-1} a_{ir} a_{rj}$ for every $1 \leq i < j \leq m+1$ with $(i,j) \neq (1, m+1)$.

One can show (see Lemma 3.3) that then $\sum_{r=2}^{m} a_{1r} a_{r,m+1}$ is a 2-cocycle. The **m-fold Massey product** $\langle h_1, \ldots, h_m \rangle$ consists of the cohomology classes of all 2-cocycles obtained in this manner. Our main result is:

Main Theorem. *Suppose that for all $h_1, \ldots, h_m \in H^1$ the set $\langle h_1, \ldots, h_m \rangle$ is non-empty. Then for all $h_1, \ldots, h_m \in H^1$ this set consists of exactly one element.*

The Main Theorem is deduced in §6 from the more general Corollary 5.3.

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Our proof of the Main Theorem is inspired by ideas from [Kra66] combined with the approach, by Babenko and Taĭmanov [BT00], to Massey products using the formal Maurer–Cartan operator on $(m+1)\times(m+1)$ matrices (see also [Mil06]). We apply it for so-called *full sets*, and not only for the punctured triangle $1 \le i < j \le m$, $(i,j) \ne (1, m+1)$. This generalization gives the flexibility needed for our main arguments. A key construction in the proof of the Main Theorem almost immediately implies the multi-linearity of the Massey product in our situation (Proposition 6.4).

This note is motivated by the study of absolute Galois groups of fields by means of their various canonical decreasing filtrations; see [EM11], [CEM12], [Efr13]. In particular, [Efr13] is based on the fact that certain m-fold Massey products in Galois cohomology are single-valued - see Example 6.5 for details.

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2. Matrices over a DGA

Let $\mathcal{C} = \bigoplus_{r=0}^{\infty} \mathcal{C}^r$ be a differential graded algebra (abbreviation: DGA) over a ring R. Thus $\mathcal{C}$ is a graded R-module with associative R-bilinear multiplication maps $\mathcal{C}^r \times \mathcal{C}^l \to \mathcal{C}^{r+l}$ and R-linear maps $d = d^r \colon \mathcal{C}^r \to \mathcal{C}^{r+1}$ such that

(i) $ab = (-1)^{rl} ba$ for $a \in \mathcal{C}^r$ and $b \in \mathcal{C}^l$;
(ii) $d \circ d = 0$;
(iii) $d(ab) = d(a)b + (-1)^r a d(b)$ for $a \in \mathcal{C}^r$ and $b \in \mathcal{C}^l$.

For $a \in \mathcal{C}^r$ we set $\bar{a} = (-1)^{r+1} a$. The multiplication maps uniquely extend to an R-bilinear map $\mathcal{C} \times \mathcal{C} \to \mathcal{C}$. Similarly, the differentials d^r (resp., the map $a \mapsto \bar{a}$) uniquely extend to an R-linear map $d \colon \mathcal{C} \to \mathcal{C}$ (resp., an R-linear involution $\mathcal{C} \to \mathcal{C}$, $a \mapsto \bar{a}$).

We fix a positive integer n and let $M_n(\mathcal{C})$ be the R-algebra of all $n \times n$ matrices with entries from $\mathcal{C}$. We define an R-linear map $d \colon M_n(\mathcal{C}) \to M_n(\mathcal{C})$ by $d((a_{ij})) = (d(a_{ij}))$, and an R-linear involution on $M_n(\mathcal{C})$ by $\overline{(a_{ij})} = (\bar{a}_{ij})$. As in [BT00], we define the **formal Maurer–Cartan operator** $\mu \colon M_n(\mathcal{C}) \to M_n(\mathcal{C})$ by

$$\mu(A) = d(A) - \bar{A}A.$$

One has the **Bianchi identity**

$$d(\mu(A)) = \overline{\mu(A)}A - A\mu(A)$$

[BT00, Lemma 1] (While [BT00] restrict to upper-triangular matrices, the proof of this fact applies for arbitrary matrices).

Given an R-submodule $\mathcal{C}_0$ of $\mathcal{C}$, we write $N(\mathcal{C}_0)$ for the set of all matrices $A = (a_{ij}) \in M_n(\mathcal{C})$ such that $a_{ij} \in \mathcal{C}_0$ for every i, j, and $a_{ij} = 0$ when $i \ge j$.

3. Full sets

Let $\mathcal{U}_n$ be the set of all pairs (i,j) of integers such that $1 \le i < j \le n$. We say that a subset T of $\mathcal{U}_n$ is **full** if for every $(i,j) \in T$ and integers i', j' such that $i \le i' < j' \le j$

also $(i', j') \in T$, i.e., T has no "holes" towards the main diagonal (see Figure 1). We write T^* for the set of all $(i, j) \in \mathcal{U}_n$ such that $(i', j') \in T$ whenever $i \le i' < j' \le j$ and $(i, j) \ne (i', j')$. In Figure 1 the grey region illustrates a full subset T of the triangle $\mathcal{U}_n$, and the dark regions are $T^* \setminus T$.

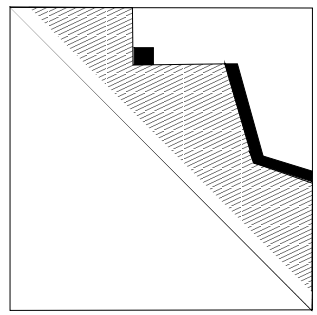

FIGURE 1

Lemma 3.1. *Let $T' \subseteq T \subseteq \mathcal{U}_n$ with T', T full. Then $T' = T$ if and only if $T \cap (T')^* \subseteq T'$.*

Proof. The "only if" part is trivial. For the "if" part, assume that $T' \ne T$. Choose $(i, j) \in T \setminus T'$ with $j - i$ minimal. Then $(i, j) \in (T')^*$. Therefore $T \cap (T')^* \not\subseteq T'$. □

Given a full subset T of $\mathcal{U}_n$, let I_T be the R-submodule of $N(\mathcal{C})$ consisting of all matrices $A = (a_{ij})$ with $a_{ij} = 0$ for all $(i, j) \in T$. It is closed under the operations d and $A \mapsto \overline{A}$.

Lemma 3.2. (a) *For $A \in I_T$ and $B \in N(\mathcal{C})$ one has $AB, BA \in I_{T^*}$.*
(b) *I_T is an ideal of $N(\mathcal{C})$.*
(c) *If $\mathcal{C}_0$ is an R-submodule of $\mathcal{C}$ and $B, B' \in N(\mathcal{C}_0)$ satisfy $B \equiv B' \pmod{I_T}$, then $\overline{B}B \equiv \overline{B'}B' \pmod{I_{T^*}}$ and $\mu(B) - \mu(B') \in I_T \cap (I_{T^*} + d(N(\mathcal{C}_0)))$.*

Proof. (a) is straightforward. (b) follows from (a).

For (c) let $C = B - B' \in I_T$. By (a),

$$\overline{B}B - \overline{B'}B' = \overline{B'}C + \overline{C}B' + \overline{C}C \in I_{T^*} \subseteq I_T.$$

Further, $\mu(B) - \mu(B') = d(C) - (\overline{B}B - \overline{B'}B')$ and $d(C) \in I_T \cap d(N(\mathcal{C}_0))$. □

Lemma 3.3 ([BT00, Cor. 1])**.** *If $A \in N(\mathcal{C})$ and $\mu(A) \in I_T$, then $d(\mu(A)) \in I_{T^*}$.*

Proof. As $\mu(A) \in I_T$ also $\overline{\mu(A)} \in I_T$. By Lemma 3.2(a), $A\mu(A), \overline{\mu(A)}A \in I_{T^*}$, and we use the Bianchi identity. □

4. The main construction

We now fix an R-submodule $\mathcal{C}_0$ of $\mathcal{C}$. For a full set T, let

$$N_T(\mathcal{C}_0) = \Big\{ A \in N(\mathcal{C}_0) \;\Big|\; \mu(A) \in I_T \Big\}.$$

We say that $\mathcal{C}_0$ has the T**-extension property** if for every full subset T' of T and every $A' \in N_{T'}(\mathcal{C}_0)$ there exists $A \in N_T(\mathcal{C}_0)$ with $A \equiv A' \pmod{I_{T'}}$.

Given $(s, s') \in \mathcal{U}_n$, we will decompose matrices $A = (a_{ij}) \in M_n(\mathcal{C})$ into nine blocks A_{kl}, $1 \le k, l \le 3$, according to the cases

$$1 \le i \le s,\ s < i < s',\ s' \le i \le n\ ; \quad 1 \le j \le s,\ s < j < s',\ s' \le j \le n :$$

$$\begin{array}{c|c|c|c|}
 & 1 \ \cdots \ s & s+1 \ \cdots \ s'-1 & s' \ \cdots \ n \\
\hline
\begin{array}{c} 1 \\ \vdots \\ s \end{array} & A_{11} & A_{12} & A_{13} \\
\hline
\begin{array}{c} s+1 \\ \vdots \\ s'-1 \end{array} & A_{21} & A_{22} & A_{23} \\
\hline
\begin{array}{c} s' \\ \vdots \\ n \end{array} & A_{31} & A_{32} & A_{33} \\
\hline
\end{array}$$

Proposition 4.1. *Let T be full and suppose that $\mathcal{C}_0$ has the T-extension property. Let $A = (a_{ij}) \in N_T(\mathcal{C}_0)$, let $(s, s') \in T$, let $b \in \mathcal{C}_0$ be a cocycle, and let $\lambda \in R$. Then there is $C = (c_{ij}) \in N_T(\mathcal{C}_0)$ such that*

(a) *A and C coincide except possibly in the $(1,3)$-block;*
(b) *$c_{ss'} = a_{ss'} - \lambda b$;*
(c) *if $s' = s+1$, then there exists $B = (b_{ij}) \in N_T(\mathcal{C}_0)$ such that $b_{s,s+1} = b$, $b_{i,i+1} = a_{i,i+1}$ for every $(i, i+1) \in T \setminus (s, s+1)$, and in the $(1,3)$-block, $\mu(C) = \mu(A) - \lambda\mu(B)$;*
(d) *if $s+1 < s'$, then $\mu(C) - \mu(A) \in I_{T^*} + d(N(\mathcal{C}_0))$.*

Proof. Let T' be the full subset of T obtained by deleting all entries in the $(1,3)$-block, but keeping (s, s'). We define

$$B' = \begin{bmatrix} A_{11} & 0 & B'_{13} \\ 0 & 0 & 0 \\ 0 & 0 & A_{33} \end{bmatrix},$$

where B'_{13} is zero except for the value b in its lower-left corner (i.e., at entry (s, s') of B'). Since A is zero on and below the main diagonal,

$$\overline{B'}B' \equiv \begin{bmatrix} \overline{A}_{11}A_{11} & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & \overline{A}_{33}A_{33} \end{bmatrix} \equiv \begin{bmatrix} d(A_{11}) & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & d(A_{33}) \end{bmatrix} = d(B') \pmod{I_{T'}},$$

so $B' \in N_{T'}(\mathcal{C}_0)$. The T-extension property yields $B \in N_T(\mathcal{C}_0)$ with $B \equiv B' \pmod{I_{T'}}$. Without loss of generality

$$B = \begin{bmatrix} A_{11} & 0 & B_{13} \\ 0 & 0 & 0 \\ 0 & 0 & A_{33} \end{bmatrix}, \quad \overline{B}B = \begin{bmatrix} \overline{A}_{11}A_{11} & 0 & \overline{A}_{11}B_{13} + \overline{B}_{13}A_{33} \\ 0 & 0 & 0 \\ 0 & 0 & \overline{A}_{33}A_{33} \end{bmatrix}$$

and the (s, s')-entry of B is b.

Next we set $C = A - \lambda \begin{bmatrix} 0 & 0 & B_{13} \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$. Then $C \in N(\mathcal{C}_0)$ and

$$\mu(C) = \mu(A) - \lambda \begin{bmatrix} 0 & 0 & d(B_{13}) - \overline{A}_{11}B_{13} - \overline{B}_{13}A_{33} \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}.$$

As $\mu(A), \mu(B) \in I_T$, we have $\mu(C) \equiv -\lambda\mu(B) \equiv 0 \pmod{I_T}$, so $C \in N_T(\mathcal{C}_0)$. Furthermore, in the $(1,3)$-block $\mu(C) = \mu(A) - \lambda\mu(B)$.

Assertions (a), (b) and (c) are now immediate.

For (d) we need to show that when $q = s' - s - 1 \geq 1$ the entries of $\mu(B)$ in the $(1,3)$-block corresponding to T^* are coboundaries. To this end we define a map $f\colon \mathcal{U}_{n-q} \to \mathcal{U}_n$ by

$$f(k,l) = \begin{cases} (k,l), & l \leq s \\ (k, l+q), & k \leq s < l \\ (k+q, l+q), & s < k \end{cases}$$

(see Figure 2). We consider the block matrix

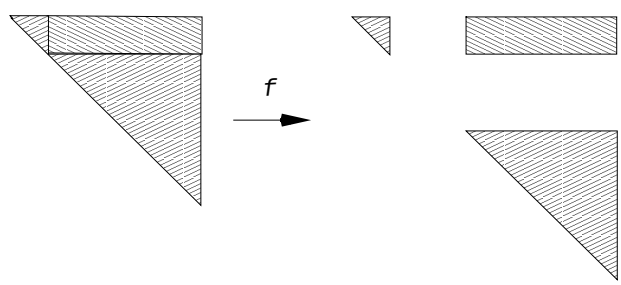

FIGURE 2

$$B_f = \begin{bmatrix} A_{11} & B_{13} & 0 \\ 0 & A_{33} & 0 \\ 0 & 0 & 0 \end{bmatrix} \in N(\mathcal{C}_0),$$

where this time the blocks are divided according to the cases

$$1 \leq i \leq s,\ s < i \leq n-q,\ n-q < i \leq n,$$
$$1 \leq j \leq s,\ s < j \leq n-q,\ n-q < j \leq n.$$

Then

$$\overline{B}_f B_f = \begin{bmatrix} \overline{A}_{11}A_{11} & \overline{A}_{11}B_{13} + \overline{B}_{13}A_{33} & 0 \\ 0 & \overline{A}_{33}A_{33} & 0 \\ 0 & 0 & 0 \end{bmatrix}.$$

Thus, if $(k,l) \in \mathcal{U}_n$ is in either the $(1,1)$-, $(1,2)$-, or $(2,2)$-block, then $b_{f(k,l)} = (B_f)_{kl}$ and $\mu(B)_{f(k,l)} = \mu(B_f)_{kl}$.

Next let (k,l) be an entry in the $(1,2)$-block with $(k, l+q) = f(k,l) \in T^*$. Since $q \geq 1$ we have $(k,l) \in T$. Let

$$T_0 = \{(k',l') \in \mathcal{U}_n \mid k \leq k' < l' \leq l,\ (k,l) \neq (k',l')\}$$

Then T_0 is also full and $(k,l) \in T_0^*$. Since T is full, $T_0 \subseteq f^{-1}[T] \cap T$. As $\mu(B) \in I_T$ this implies that $\mu(B_f) \in I_{T_0}$, so $B_f \in N_{T_0}(\mathcal{C}_0)$. The T-extension property now yields $B'_f \in N_T(\mathcal{C}_0)$ with $B_f \equiv B'_f \pmod{I_{T_0}}$. By Lemma 3.2(c), $\overline{B_f}B_f \equiv \overline{B'_f}B'_f \pmod{I_{T_0^*}}$. As $(k,l) \in T \cap T_0^*$ and $\mu(B'_f) \in I_T$, this gives

$$\mu(B)_{f(k,l)} = \mu(B_f)_{kl} = \mu(B_f)_{kl} - \mu(B'_f)_{kl} = d((B_f - B'_f)_{kl}) \in d(\mathcal{C}_0). \qquad \square$$

5. Uniqueness

We totally order $\mathcal{U}_n$ by setting

$$(i,j) \leq (i',j') \iff (j-i, i) \leq_{\text{lex}} (j'-i', i')$$

where $\leq_{\text{lex}}$ is the left-to-right lexicographical order. Thus, (i,j) either lies on a diagonal which is closer than (i',j') to the main diagonal, or else lies on the same diagonal, but $i \leq i'$.

Throughout this section we fix a full subset T of $\mathcal{U}_n$ and set

$$S = T \cap \{(i, i+1) \mid 1 \leq i \leq n-1\}.$$

We now prove our first uniqueness criterion.

Theorem 5.1. *Assume that the submodule $\mathcal{C}_0$ of $\mathcal{C}$ has the T-extension property. Let $A, A' \in N_T(\mathcal{C}_0)$ and suppose that $A \equiv A' \pmod{I_S}$. Then $\mu(A) - \mu(A') \in I_{T^*} + d(N(\mathcal{C}_0))$.*

Proof. We write $A = (a_{ij})$, $A' = (a'_{ij})$. When $A \equiv A' \pmod{I_T}$ this is contained in Lemma 3.2(c).

Next suppose $A \not\equiv A' \pmod{I_T}$. Let (s,s') be minimal in $(T, \leq)$ with respect to the property that $a_{ss'} \neq a'_{ss'}$. Thus $s+1 < s'$. Further, $(\overline{A}A)_{ss'} = (\overline{A'}A')_{ss'}$. We assume inductively that the assertion holds for matrices in $N_T(\mathcal{C}_0)$ for which the corresponding minimal pair (s,s') is larger.

Let $b = a_{ss'} - a'_{ss'} \in \mathcal{C}_0$. Since $\mu(A)_{ss'} = \mu(A')_{ss'} = 0$ we have

$$d(b) = d(A - A')_{ss'} = (\overline{A}A - \overline{A'}A')_{ss'} = 0.$$

Let $C = (c_{ij}) \in N_T(\mathcal{C}_0)$ be as in Proposition 4.1 with $\lambda = 1$. It coincides with A, and hence with A', strictly below entry (s,s') (in the sense of the total ordering $\leq$). Furthermore, $c_{ss'} = a_{ss'} - b = a'_{ss'}$. Thus C and A' coincide on and below (s,s'). By the

induction hypothesis, $\mu(C) - \mu(A') \in I_{T^*} + d(N(\mathcal{C}_0))$. Also, by (d) of Proposition 4.1, $\mu(A) - \mu(C) \in I_{T^*} + d(N(\mathcal{C}_0))$, and the assertion follows. □

Theorem 5.2. *The following conditions on the submodule $\mathcal{C}_0$ of the DGA $\mathcal{C}$ are equivalent:*

(a) *$\mathcal{C}_0$ has the T-extension property;*
(b) *For every $A \in N_S(\mathcal{C}_0)$ there exists $A' \in N_T(\mathcal{C}_0)$ with $A \equiv A' \pmod{I_S}$.*

Proof. (a)⇒(b): Immediate.

(b)⇒(a): Let T' be a full subset of T, and let $A \in N_{T'}(\mathcal{C}_0)$. We need to find $A' \in N_T(\mathcal{C}_0)$ with $A \equiv A' \pmod{I_{T'}}$. We may assume that T' is maximal, in the sense that there is no full set T'' with $T' \subset T'' \subseteq T$ and $A'' \in N_{T''}(\mathcal{C}_0)$ such that $A \equiv A'' \pmod{I_{T'}}$. We need to show that $T' = T$. In view of Lemma 3.1, it is enough to show that $T \cap (T')^* \subseteq T'$.

To this end take $(k,l) \in T \cap (T')^*$. (b) yields $A' \in N_T(\mathcal{C}_0)$ with $A \equiv A' \pmod{I_S}$. Since $T' \subseteq T$, the submodule $\mathcal{C}_0$ has the extension property also with respect to T', and $A' \in N_{T'}(\mathcal{C}_0)$. Hence we may use Theorem 5.1, with T replaced by T', to obtain that $\mu(A) - \mu(A') \in I_{(T')^*} + d(N(\mathcal{C}_0))$. As $A' \in N_T(\mathcal{C}_0)$ we further have $\mu(A')_{kl} = 0$. Therefore $\mu(A)_{kl} \in d(\mathcal{C}_0)$, whence $(\overline{A}A)_{kl} = d(a)$ for some $a \in \mathcal{C}_0$.

Now if $(k,l) \notin T'$, then we set $T'' = T' \cup \{(k,l)\}$ and let A'' be the matrix A but with a as its (k,l)-entry. Then $A'' \in N_{T''}(\mathcal{C}_0)$ and $A \equiv A'' \pmod{I_{T'}}$. This contradicts the maximality of T'. □

Combining Theorem 5.1 and Theorem 5.2, we obtain

Corollary 5.3. *Suppose that for every $A \in N_S(\mathcal{C}_0)$ there exists $A' \in N_T(\mathcal{C}_0)$ with $A \equiv A' \pmod{I_S}$. Let $A, A' \in N_T(\mathcal{C}_0)$ satisfy $A \equiv A' \pmod{I_S}$. Then $\mu(A) - \mu(A') \in I_{T^*} + d(N(\mathcal{C}_0))$.*

6. Massey products

Let $m \geq 2$. To define the m-fold Massey product, we take $n = m+1$ and fix throughout this section the full subset $T = \mathcal{U}_n \setminus \{(1,n)\}$ of $\mathcal{U}_n$. Then, in our previous notation, $T^* = \mathcal{U}_n$ and $S = \{(i,i+1) \mid i = 1,2,\ldots,m\}$.

Given $A = (a_{ij}) \in N(\mathcal{C}_0)$, the condition $\mu(A) \in I_T$ means that $d(a_{ij}) = \sum_{r=i+1}^{j-1} \bar{a}_{ir} a_{rj}$ for all $1 \leq i < j \leq n$ with $(i,j) \neq (1,n)$. We then say that A is a **T-defining system on $\mathcal{C}_0$**. By Lemma 3.3, $\mu(A)_{1n}$ is a cocycle. It is cohomologous to $\sum_{r=2}^{n-1} a_{1r}\bar{a}_{rn}$. Thus, when $\mathcal{C}_0$ is the degree 1 component $\mathcal{C}^1$ of $\mathcal{C}$, we recover the setup of the Introduction. In particular, the Main Theorem is a special case of Corollary 5.3.

Let $\mathcal{H} = \bigoplus_{r=0}^{\infty} H^r$ be the cohomology of $\mathcal{C}$, where $H^r = \operatorname{Ker}(d^r)/\operatorname{Im}(d^{r-1})$. We first define the Massey product at the level of cocycles:

Definition 6.1. Consider cocycles $a_1, \ldots, a_m \in \mathcal{C}_0$ and $A = (a_{ij}) \in N_T(\mathcal{C}_0)$ such that $\mu(A) \in I_T$ and $a_{i,i+1} = a_i$ for $i = 1,2,\ldots,m$. We define $\langle a_1, \ldots, a_m \rangle$ to be the cohomology class of the cocycle $\mu(A)_{1n}$.

Theorem 6.2. *Suppose that for every cocycles $a_1, \dots, a_m \in \mathcal{C}_0$ there is a T-defining system $A = (a_{ij})$ on $\mathcal{C}_0$ such that $a_i = a_{i,i+1}$, $i = 1, 2, \dots, m$. Then $\langle \cdot, \dots, \cdot \rangle \colon \mathcal{C}_0^m \to \mathcal{H}$ is a well-defined single-valued map.*

Proof. By Theorem 5.2, $\mathcal{C}_0$ has the T-extension property. Hence there exists A as in Definition 6.1. Corollary 5.3 shows that $\langle a_1, \dots, a_m \rangle$ is independent of the choice of A. □

It is well-known that, in some important situations, this map is already defined on the level of the cohomology. This is based on the following fact:

Lemma 6.3 ([Kra66, Th. 3], [BT00, Prop. 1])**.** *Let $A = (a_{ij}) \in N_T(\mathcal{C}_0)$, $1 \le s \le n-1$, and $b \in \mathcal{C}$. Suppose that $d(b) \in \mathcal{C}_0$, $a_{is}b, \bar{b}a_{s+1,j} \in \mathcal{C}_0$ for $i < s$ and $s+1 < j$. Then there exists $A' = (a'_{ij}) \in N_T(\mathcal{C}_0)$ such that $\mu(A) \equiv \mu(A') \pmod{d(N(\mathcal{C}))}$, $a'_{i,i+1} = a_{i,i+1}$ for all $i \neq s$, and $a'_{s,s+1} = a_{s,s+1} + d(b)$.*

Proof. Let $E_{ij}(c)$ be the matrix in $M_n(\mathcal{C})$ which is c at entry (i,j), and zero elsewhere. The assertion follows by a straightforward calculation with

$$A' = A + E_{s,s+1}(d(b)) + AE_{s,s+1}(b) - E_{s,s+1}(\bar{b})A.$$

Note that $A' = (a'_{ij})$, where

$$a'_{ij} = \begin{cases} a_{ij}, & i \neq s, j \neq s+1 \\ a_{s,s+1} + d(b), & i = s, j = s+1 \\ a_{i,s+1} + a_{is}b, & i < s, j = s+1 \\ a_{sj} + \bar{b}a_{s+1,j}, & i = s, s+1 < j, \end{cases}$$

so by assumption, $A' \in N(\mathcal{C}_0)$. □

Let $\mathcal{H}_0$ be the submodule of $\mathcal{H}$ consisting of all cohomology classes represented by cocycles in $\mathcal{C}_0$. It follows from Lemma 6.3 that $\langle \cdot, \dots, \cdot \rangle \colon \mathcal{C}_0^m \to \mathcal{H}$ induces a well-defined map $\langle \cdot, \dots, \cdot \rangle \colon \mathcal{H}_0^m \to \mathcal{H}$ in each of the cases (i) $\mathcal{C}_0 = \mathcal{C}$, and (ii) $\mathcal{C}_0 = \mathcal{C}^1$ (where (ii) is the case considered in the Introduction).

The construction in Proposition 4.1 also gives as a by-product a new proof in our case of the following well-known fact (see [May69, Prop. 2.7], [Fen83, Lemma 6.2.4]):

Proposition 6.4. *Assuming the T-extension property, the Massey product is R-multilinear.*

Proof. Let $h_1, \dots, h_m, g, g' \in \mathcal{H}_0$ and $\lambda \in R$ satisfy $h_s = \lambda g + g'$, where $1 \le s \le m$. Take $A = (a_{ij}) \in N_T(\mathcal{C}_0)$ such that $a_{i,i+1}$ is a cocycle with cohomology class h_i, $i = 1, 2, \dots, m$. Also choose a cocycle $b \in \mathcal{C}_0$ with cohomology class g. Then $a_{s,s+1} - \lambda b$ is a cocycle with cohomology class g'. Proposition 4.1 (with $s' = s+1$) yields $B = (b_{ij}), C = (c_{ij}) \in N_T(\mathcal{C}_0)$ such that

$$b_{i,i+1} = c_{i,i+1} = a_{i,i+1}, \quad i = 1, 2, \dots, s-1, s, s+2, s+3, \dots, m,$$
$$b_{s,s+1} = b, \quad c_{s,s+1} = a_{s,s+1} - \lambda b$$

and $\mu(C) = \mu(A) - \lambda\mu(B)$ in the $(1,3)$-block. In particular, $\mu(A)_{1n} = \lambda\mu(B)_{1n} + \mu(C)_{1n}$. Consequently,

$$\langle h_1, \dots, h_m \rangle = \lambda \langle h_1, \dots, h_{s-1}, g, h_{s+1}, \dots, h_m \rangle + \langle h_1, \dots, h_{s-1}, g', h_{s+1}, \dots, h_m \rangle. \quad \square$$

Example 6.5. Let p be a prime number and G a pro-p group. Consider the DGA $C^\bullet(G, \mathbb{Z}/p) = \oplus_{r=0}^{\infty} C^r(G, \mathbb{Z}/p)$ over $\mathbb{Z}/p$ of all continuous inhomogenous cochains $G^r \to \mathbb{Z}/p$ with the cup product, and $\bigoplus_{r=0}^{\infty} H^r(G)$ its cohomology $\mathbb{Z}/p$-algebra [NSW08, Ch. I]. Suppose that G has a presentation $G = S/N$, where S is a free pro-p group and N is a closed normal subgroup of S contained in the m-th term $S_{(m,p)}$ of the p-Zassenhaus filtration of S (see [NSW08]). It was proved by Vogel [Vog05, Th. A3] that then the m-fold Massey product $H^1(G)^m \to H^2(G)$ is single valued. Indeed, it is shown in [Efr13, Prop. 8.4] that the assumption of Theorem 6.2 is satisfied.

References

[BT00] I. K. Babenko and I. A. Taĭmanov, *Massey products in symplectic manifolds*, Mat. Sb. **191** (2000), 3–44; English transl., Sb. Math. **191** (2000), 1107–1146.

[CEM12] S. K. Chebolu, I. Efrat, and J. Mináč, *Quotients of absolute Galois groups which determine the entire Galois cohomology*, Math. Ann. **352** (2012), 205–221.

[EM11] I. Efrat and J. Mináč, *On the descending central sequence of absolute Galois groups*, Amer. J. Math. **133** (2011), 1503–1532.

[Efr13] I. Efrat, *The Zassenhaus filtration, Massey products, and representations of profinite groups* (2013), to appear, available at `arXiv:1301.0896v2`.

[Fen83] R. A. Fenn, *Techniques of Geometric Topology*, London Math. Society Lect. Note Series, vol. 57, Cambridge Univ. Press, Cambridge, 1983.

[FS84] R. Fenn and D. Sjerve, *Basic commutators and minimal Massey products*, Canad. J. Math. **36** (1984), 1119–1146.

[HW12] M. Hopkins and K. Wickelgren, *Splitting varieties for triple Massey products* (2012), preprint, available at `arXiv:1210.4964`.

[Kra66] D. Kraines, *Massey higher products*, Trans. Amer. Math. Soc. **124** (1966), 431–449.

[Mas58] W. S. Massey, *Some higher order cohomology operations*, Symposium internacional de topología algebraica International symposium on algebraic topology, Universidad Nacional Autónoma de México and UNESCO, Mexico City, 1958, pp. 145–154.

[May69] J. P. May, *Matric Massey products*, J. Algebra **12** (1969), 533–568.

[Mil06] D. Millionschikov, *Massey products in graded Lie algebra cohomology*, Contemporary geometry and related topics, Univ. Belgrade Fac. Math., Belgrade, 2006, pp. 353–377.

[Mor04] M. Morishita, *Milnor invariants and Massey products for prime numbers*, Compositio Math. **140** (2004), 69–83.

[Mor12] M. Morishita, *Knots and primes*, Universitext, Springer, London, 2012.

[NSW08] J. Neukirch, A. Schmidt, and K. Wingberg, *Cohomology of Number Fields, Second edition*, Springer, Berlin, 2008.

[Sha07] R. Sharifi, *Massey products and ideal class groups*, J. reine angew. Math. **603** (2007), 1–33.

[Vog05] D. Vogel, *On the Galois group of 2-extensions with restricted ramification*, J. reine angew. Math. **581** (2005), 117–150.

[Wic09] K. Wicklegren, *Lower central series obstructions to homotopy sections of curves over number fields*, Ph.D. Thesis, Stanford University, 2009.

[Wic12] K. Wickelgren, *n-Nilpotent obstructions to π_1-sections of $\mathbb{P}^1 \setminus \{0, 1, \infty\}$ and Massey products*, Conference on Galois-Teichmüller Theory and Arithmetic Geometry (H. Nakamura et al, eds.) (Kyoto, 2010), Advanced studies in pure mathematics, vol. 63, Mathematical Society of Japan, 2012, pp. 579–600.

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