

$$\text{Proof of convergence} \quad \frac{1}{n} \sum_{k=1}^n f(x_k) \rightarrow \int f(x) dx$$

$$\begin{aligned} \hat{g}(n) &= \frac{1}{2\pi} \int_{-\pi}^{\pi} f(2x) e^{-2ikx} dx = \frac{1}{4\pi} \int_{-2\pi}^{2\pi} f(t) e^{-ikt} dt = \\ &= \frac{1}{2} \cdot \left( \underbrace{\frac{1}{2\pi} \int_{-\pi}^0 f(t) e^{-ikt} dt}_{f(k)} + \underbrace{\frac{1}{2\pi} \int_0^{2\pi} f(t) e^{-ikt} dt}_{\hat{f}(k)} \right) = \hat{f}(k) = \hat{f}\left(\frac{n}{2}\right) \end{aligned}$$

$$\hat{g}(n) = \frac{1}{2\pi} \int_{-\pi}^{\pi} f(2x) e^{-2ikx} \cdot e^{-ix} dx =$$

$$= \frac{1}{4\pi} \left( \underbrace{\int_{-2\pi}^0 f(t) e^{-ikt} \cdot e^{\frac{-it}{2}} dt}_{u=t+2\pi} + \underbrace{\int_0^{2\pi} f(t) e^{-ikt} e^{\frac{-it}{2}} dt}_{u=t} \right)$$

$$e^{-\frac{it}{2}} = e^{-\frac{i(u-2\pi)}{2}} = -e^{-\frac{iu}{2}}$$

$$f(u) = f(t), \quad e^{-iu} = e^{it}$$

$$\hookrightarrow = \frac{1}{4\pi} \left( \int_0^{2\pi} f(u) e^{-iku} \cdot (-e^{\frac{iu}{2}}) du + \int_0^{2\pi} f(u) e^{-iku} e^{\frac{iu}{2}} du \right) = 0$$

$\Rightarrow$   $a_n \rightarrow a_0$  as  $n \rightarrow \infty$ , i.e.  $c = \lim a_n$  (5)

$$\begin{aligned} \|a_n - c\| &= \left\| \frac{1}{n} (a_1 + \dots + a_n) - c \right\| = \left\| \frac{1}{n} [(a_1 - c) + (a_2 - c) + \dots + (a_n - c)] \right\| \leq \\ &\leq \frac{1}{n} (\|a_1 - c\| + \dots + \|a_n - c\|) \end{aligned}$$

$$\frac{M}{N} < \frac{\varepsilon}{2} \Rightarrow N \geq M \Rightarrow M = \|a_1 - c\| + \dots + \|a_{N_0} - c\|$$

$$\|a_n - c\| \leq \frac{M}{n} + \frac{\|a_{N_0+1} - c\| + \dots + \|a_n - c\|}{n} \leq \frac{\varepsilon}{2} + \frac{\varepsilon}{n} \leq \frac{\varepsilon}{2} + \frac{\varepsilon}{N_0} = \varepsilon$$

$$\text{and } a_n \rightarrow a_0 \text{ as } n \rightarrow \infty, \quad a_n = \begin{cases} 1 & \text{if } n \leq N_0 \\ 0 & \text{if } n > N_0 \end{cases}$$

$$\therefore \lim_{n \rightarrow \infty} \frac{a_n}{n} = 0$$

$$f(b) \neq 0 \Rightarrow b = 0 \text{ or } \rho, f(b) = f(0+b) = f(0) \cdot f(b) \Rightarrow \text{BP} \quad (6)$$

$$\cdot f(0) = 1 \Rightarrow \rho = 1, f(b) = e^{kb}, \text{ and } f(x) = e^{kx}$$

$f(x) \rightarrow \text{even } x \in [-k_0, k_0]$ , if  $x > 0$  even,  $f(x) = e^{kx}$

.  $\Re f(x) > 0$  even, even values

$$-k_0 < \operatorname{Im} k < k_0 \quad \Rightarrow \quad f(k_0) = e^{k_0} \text{ for } k_0$$

(abs min  $x \neq 0$  even values are  $\text{BP}$ )

$$e^{kx} = f(x_0) = f\left(\frac{x_0}{2} + \frac{x_0}{2}\right) = f\left(\frac{x_0}{2}\right) \cdot f\left(\frac{x_0}{2}\right)$$

$$\cdot (f(k_0) \text{ is even values } \Rightarrow \rho) \quad f\left(\frac{x_0}{2}\right) = \pm e^{k_0/2}$$

/ even values even  $e^{k_0/2}$   $\Rightarrow$ ,  $\operatorname{Im} k_0/2 \in \{\frac{\pi}{4}, \frac{3\pi}{4}\}$ , and

$$[-k_0, k_0] \ni \frac{x_0}{2} \text{ even, even values even } -e^{k_0/2} \text{ odd}$$

. ( $-e^{k_0/2}$  odd)  $f\left(\frac{x_0}{2}\right) = e^{k_0/2}$ , even,  $x$  even

$$e^{\frac{x_0}{2^n}} = f\left(\frac{x_0}{2^n}\right) \rightarrow \text{even, even values } \text{BP}, \text{ and } \text{odd}$$

$$\cdot f\left(\frac{m x_0}{2^n}\right) = \left(f\left(\frac{x_0}{2^n}\right)\right)^m = \left(e^{\frac{x_0}{2^n}}\right)^m = e^{\frac{mx_0}{2^n}} \text{ even}$$

$$\cdot [0, \infty) \rightarrow \text{BP} \quad \left\{ \frac{m}{2^n} \cdot x_0 \mid m \in \mathbb{N} \right\} \text{ even values}$$

$$\left( \begin{aligned} & (= f(0) = f(x-x) = \\ & = f(x) - f(-x)) \end{aligned} \right) \quad f(-x) = (f(x))^{-1} \quad [0, \infty) \Rightarrow \text{odd}$$

$$\left( \begin{aligned} & (e^{kx})^{-1} = e^{-kx} \end{aligned} \right) \quad x < 0 \text{ even } \Rightarrow f(x) = e^{kx} \quad -e^{-kx}$$

- even