Sketchy solutions of Moed.B.1, Hedva2.ME, 20119721
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(1) a. Denote $a_n = (\frac{n+2}{n+1})^n \frac{(2x-1)^n}{2^n(x+3)^n}$, then $\lim_{n\to\infty} \sqrt[n]{|a_n|} = |\frac{2x-1}{2(x+3)}|$. Thus the series converges absolutely for $|\frac{(2x-1)}{(x+3)}| < 2$ and diverges for $|\frac{(2x-1)}{(x+3)}| > 2$. It remains to check the boundary points:

$\star$ If $\frac{2x-1}{x+3} = -2$, i.e. $x = -\frac{4}{5}$, then the series is $\sum\limits_{n\geq 1} (\frac{n+2}{n+1})^n (-1)^n$. It diverges, as $a_n \not\to 0$.

$\star$ The case $\frac{2x-1}{x+3} = 2$ is not realized as this equation has no solutions.

In total: the series converges absolutely for $|\frac{2x-1}{x+3}| < 2$ and diverges otherwise. The condition on x reads: $x > -\frac{5}{4}$.

b. We should compute $\oint_C \vec{F} d\vec{C}$. Use the Stokes formula to transform this integral into the surface integral $\iint rot(\vec{F}) d\vec{S}$, over the surface $S = \{x^2+y^2+z^2 = 2Rx,\ z > 0,\ x^2+y^2 \leq 2rx\}$. The normal to this surface is taken upstairs, as the curve is oriented counterclockwise. The normal is: $\vec{\mathcal{N}} = \partial_x \vec{r} \times \partial_y \vec{r} = (-\partial_x z, -\partial_y z, 1)$.

The natural parametrization of the surface is: $z = \sqrt{2Rx - x^2 - y^2}$ therefore

$$\iint rot(\vec{F}) d\vec{S} = \iint (2y-2z, 2z-2x, 2x-2y) \cdot \vec{\mathcal{N}} dxdy = \iint\limits_{x^2+y^2\leq 2rx} 2\Big(x - y + (z-y)\partial_x z + (x-z)\partial_y z\Big) dxdy$$

Note that $\partial_x z = \frac{R-x}{z}$, $\partial_y z = -\frac{y}{z}$. Therefore the integral to compute is: $\iint\limits_{x^2+y^2\leq 2rx} 2R(1-\frac{y}{z}) dxdy$.

The domain of the integration, the disc $\{(x-r)^2 + y^2 \leq r^2\}$, is symmetric with respect to the $\hat{x}$ axis (i.e. with respect to $y \leftrightarrow -y$). The function $\frac{y}{z}$ is anti-symmetric with respect to $y \leftrightarrow -y$. Therefore $\iint\limits_{x^2+y^2\leq 2rx} 2R\frac{y}{z} dxdy = 0$.

Thus the initial integral equals to:

$$\oint\limits_C \vec{F} d\vec{C} = \iint rot(\vec{F}) d\vec{S} = \iint\limits_{x^2+y^2\leq 2rx} 2R(1-\frac{y}{z}) dxdy = \iint\limits_{x^2+y^2\leq 2rx} 2R dxdy = 2R \cdot Area\Big\{(x-r)^2 + y^2 \leq r^2\Big\} = 2R\pi r^2$$

(2) a. First we expand the function x^y around the point $(1,1)$ up to the second order:

$$x^y = 1 + (yx^{y-1})|_{(1,1)}\Delta x + (ln(x)x^y)|_{(1,1)}\Delta y + \frac{(y(y-1)x^{y-2})|_{(1,1)}(\Delta x)^2 + 2(x^{y-1} + yln(x)x^{y-1})|_{(1,1)}\Delta x \Delta y + (ln^2(x)x^y)|_{(1,1)}(\Delta y)^2}{2} + \cdots$$
$$= 1 + \Delta x + \Delta x \Delta y + \cdots$$

Therefore $x^y y^x = (1 + \Delta x + \Delta x\Delta y + \cdots)(1 + \Delta y + \Delta x \Delta y + \cdots) = 1 + \Delta x + \Delta y + 3\Delta x \Delta y + \cdots$.
Thus approximately: $1.02^{1.03} 1.03^{1.02} \approx 1 + 0.02 + 0.03 + 3 \cdot 0.02 \cdot 0.03 = 1.0518$.

b. In the polar coordinates the equation of the curve is: $r^2(1 - \frac{sin^2(2\phi)}{2}) = 1$. As ϕ changes from 0 to 2π the point on the curve evolves once around the centre of coordinates. The curve lies between the circle of radius 1 and the circle of radius $\sqrt{2}$. The curve intersects the coordinates axes at the points $(\pm 1, 0)$, $(0, \pm 1)$.

The area:

$$\iint 1 dS = \int_0^{2\pi} d\phi \int_0^{\sqrt{\frac{1}{1-\frac{sin^2(2\phi)}{2}}}} r dr = \frac{1}{2}\int_0^{2\pi} \frac{d\phi}{1 - \frac{sin^2(2\phi)}{2}} = \int_0^{2\pi} \frac{d\phi}{1 + cos^2(2\phi)} = 8\int_0^{\frac{\pi}{4}} \frac{d\phi}{1 + cos^2(2\phi)}$$

Change the variable $t = 2\phi$, then the needed integral is

$$4\int_0^{\frac{\pi}{2}} \frac{dt}{1 + cos^2(t)} = 4\int_0^{\frac{\pi}{2}} \frac{\frac{1}{cos^2 t}dt}{\frac{1}{cos^2 t} + 1} \overset{x=tan(t)}{=} 4\int_0^{\infty} \frac{dx}{1 + x^2 + 1} = 4\int_0^{\infty} \frac{1}{2}\frac{dx}{1 + (\frac{x}{\sqrt{2}})^2} = 4\frac{1}{2}\sqrt{2}\frac{\pi}{2}$$

(3) a. Rename the axes, $y \leftrightarrow z$, then we should find the area of the surface $x^2+y^2+z^2 = a^2$ inside $x^2+y^2 \leq b^2$.

It is enough to compute the area of the part $z = \sqrt{a^2 - x^2 - y^2} \geq 0$ with $x^2 + y^2 \leq b^2$. The surface area is: $\iint 1 ds$.

The natural parametrization of the surface is in terms of x, y. Thus the normal is $\vec{\mathcal{N}} = \partial_x \vec{r} \times \partial_y \vec{r} = \det\begin{pmatrix} \hat{x} & \hat{y} & \hat{z} \\ 1 & 0 & \partial_x z \\ 0 & 1 & \partial_y z \end{pmatrix}$.

Therefore $||\vec{\mathcal{N}}|| = \sqrt{1 + (\partial_x z)^2 + (\partial_y z)^2}$. We convert the surface integral to the double integral:

$$\iint 1 ds = \iint\limits_{x^2+y^2\leq b^2} \sqrt{1 + (\partial_x z)^2 + (\partial_y z)^2} dxdy = \iint\limits_{x^2+y^2\leq b^2} \sqrt{1 + \frac{x^2}{a^2-x^2-y^2} + \frac{y^2}{a^2-x^2-y^2}} dxdy =$$
$$= \iint\limits_{x^2+y^2\leq b^2} \frac{adxdy}{\sqrt{a^2-x^2-y^2}} = a2\pi\int_0^b \frac{rdr}{\sqrt{a^2-r^2}} = 2\pi a(a - \sqrt{a^2-b^2})$$

Thus the total area is $2 \cdot 2\pi a(a - \sqrt{a^2 - b^2})$

b. First consider the function $g(t) = \begin{cases} \frac{ln(1+t)}{t}, & t \neq 0 \\ 1, & t = 0 \end{cases}$. The Taylor expansion of this function is readily obtained from the expansion of ln: $g(t) = 1 - \frac{t}{2} + \frac{t^2}{3} + \cdots$. Therefore the Taylor expansion of the initial function is: $f(x,y) = 1 - \frac{x^2+y^2}{2} + \frac{(x^2+y^2)^2}{3} + \cdots$. In particular the function is differentiable at the origin.

(4) a. Use Green's formula, $\oint\limits_{\partial\mathcal{D}} \vec{F} d\vec{\gamma} = \iint\limits_{\mathcal{D}} (\partial_x F_y - \partial_y F_x) dxdy$, where $\mathcal{D} = \{1 \leq x^2 + y^2 \leq 4,\ 0 < x \leq y \leq \sqrt{3}x\}$. By the direct check: $\partial_x F_y - \partial_y F_x = \frac{1}{x^2+y^2}$. Thus

$$\oint\limits_{\partial\mathcal{D}} \vec{F} d\vec{\gamma} = \iint\limits_{\mathcal{D}} \frac{dxdy}{x^2+y^2} = \int\limits_{\frac{\pi}{4}}^{\frac{\pi}{3}} d\phi \int\limits_{1}^{2} \frac{rdr}{r^2} = \frac{\pi}{12} ln(2)$$

b. The mass equals $\iiint\limits_{V} 1dV = \iint\limits_{S} |cos(x)cos(y)| dxdy$, where $S = \{|x+y| \leq \frac{\pi}{2}, |x-y| \leq \frac{\pi}{2}\}$. Note that for $(x,y) \in S$: $cos(x)cos(y) \geq 0$.

Change the variables: $s = x + y$, $t = x - y$. The Jacobian factor is $\frac{1}{2}$. Thus:

$$\iint\limits_{S} |cos(x)cos(y)| dxdy = \iint\limits_{\substack{|s| \leq \frac{\pi}{2} \\ |t| \leq \frac{\pi}{2}}} \frac{1}{2} cos\frac{s+t}{2} cos\frac{s-t}{2} dsdt = \frac{1}{4} \iint\limits_{\substack{|s| \leq \frac{\pi}{2} \\ |t| \leq \frac{\pi}{2}}} \Big(cos(s) + cos(t)\Big) dsdt = \pi$$

(5) Parameterize L_1 by x, i.e. $z = 3x$, $y = 2x$. Parameterize L_2 by y, i.e. $z = -2 - 3y$, $x = -1 - 2y$. Fix a point on L_1 and a point on L_2. The square-of-distance is: $f(x,y) = (-1 - 2y - x)^2 + (y - 2x)^2 + (-2 - 3y - 3x)^2 = 5 + 14y^2 + 14x^2 + 16y + 14x + 18xy$.

We are looking for the (global) minimum of this function. $grad(f) = 0$ gives the only suspicious point: $x = -\frac{26}{115}$, $y = -\frac{49}{115}$. Note that the minimum must be obtained at some point (because $f(x,y)$ is continuous, bounded from below and tends to ∞ as x or y go to ∞). Therefore $(x,y) = (-\frac{26}{115}, -\frac{49}{115})$ is precisely the point of the global minimum.