Sketchy solutions of Moed.B.2, Hedva2.ME, 20119721

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(1) a. Denote $a_n = \frac{n(2sin\ x)^n}{n^2+300}$. Then $\lim\limits_{n\to\infty} \sqrt[n]{|a_n|} = |2sin(x)|$. Thus the series converges absolutely for $|2sin(x)| < 1$ and diverges for $|2sin(x)| > 1$. The border points:

• $2sin(x) = 1$. The series $\sum\limits_{n=1}^{\infty} \frac{n}{n^2+300}$ diverges by comparison to $\sum \frac{1}{n}$

• $2sin(x) = -1$. The series $\sum\limits_{n=1}^{\infty} \frac{(-1)^n n}{n^2+300}$ is the alternating series, the sequence $\frac{n}{n^2+300}$ converges monotonically (for n large enough) to 0. Thus by Leibnitz's criterion the series converges. The convergence is conditional (as for $2sin(x) = 1$ the series diverges).

Finally, the series converges absolutely for $-\frac{\pi}{6}+\pi n < x < \frac{\pi}{6}+\pi n$, $n\in\mathbb{Z}$ and converges conditionally for $x = \frac{\pi}{6}+2\pi n$, $x = \frac{5\pi}{6}+2\pi n$, $n\in\mathbb{Z}$.

b. We should compute $\oint\limits_C \vec{F}d\vec{C}$. Use the Stokes formula to transform this integral into the surface integral $\iint\limits_S rot(\vec{F})d\vec{S}$, over the surface $S = \{xsin(\alpha) + ysin(\beta) + zsin(\gamma) = 0,\ x^2+y^2+z^2 \le R^2\}$. The normal to this surface is taken upstairs, as the curve is oriented counterclockwise. The unit normal is: $\hat{\mathcal{N}} = \frac{(sin(\alpha),sin(\beta),sin(\gamma))}{\sqrt{sin^2(\alpha)+sin^2(\beta)+sin^2(\gamma)}}$. And $rot(\vec{F})\hat{\mathcal{N}} = -2\frac{sin(\alpha)+sin(\beta)+sin(\gamma)}{\sqrt{sin^2(\alpha)+sin^2(\beta)+sin^2(\gamma)}}$. Thus $\iint\limits_S rot(\vec{F})d\vec{S} = \iint\limits_S(-2\frac{sin(\alpha)+sin(\beta)+sin(\gamma)}{\sqrt{sin^2(\alpha)+sin^2(\beta)+sin^2(\gamma)}})dS$. Finally, $\iint\limits_S dS = Area(S) = \pi R^2$. Therefore: $\oint\limits_C \vec{F}d\vec{C} = -2\frac{sin(\alpha)+sin(\beta)+sin(\gamma)}{\sqrt{sin^2(\alpha)+sin^2(\beta)+sin^2(\gamma)}}\pi R^2$.

(2) a. Let $F(x,y,z) = e^{x+z} - (x+y^2)(x+z) - 1$. Then $\partial_z F|_{(-1,1,1)} \neq 0$. Thus, by the implicit function theorem, the equation $F(x,y,z) = 0$ has the unique solution $z(x,y)$ defined in some neighborhood of $(-1,1)$ and satisfying: $z(-1,1) = 1$.

The derivatives of this implicit function: $z'_x|_{(-1,1)} = -1$, $z'_y|_{(-1,1)} = 0$. The directional derivative in the direction $\vec{v} = (1,2) - (-1,1) = (2,1)$ is: $grad(z)\cdot\vec{v} = -2 < 0$. Therefore the function decreases in the direction from $(-1,1)$ to $(1,2)$.

b. We are looking for the minimum of the function $f(x,y) = x^2+y^2$ under the condition $g(x,y) = 7x^2+8xy+y^2-45 = 0$. The condition $f_x g_y = f_y g_x$ gives: $2x^2 - 2y^2 = 3xy$. From here we get: either $x = 2y$ or $x = -\frac{y}{2}$. Substitute these conditions to $g(x,y) = 0$. We get: if $x = 2y$ then $y = \pm 1$; if $2x = -y$ then there are no solutions. Thus there are just two points to consider: $(2,1)$ and $(-2,-1)$. In both cases $f(x,y) = 5$.

Finally, we note that $f(x,y)$ is a continuous function, while the curve $\{g(x,y) = 0\} \subset \mathbb{R}^2$ is a hyperbola. Thus $f(x,y)$ necessarily achieves its total minimum at least at one point. Therefore the points $(2,1)$ and $(-2,-1)$ are the closest points of this curve to the origin.

(3) a. The curve begins (as $t\to-\infty$) from (∞,∞) and ends (as $t\to\infty$) at $(\infty,-\infty)$. We check where y changes the sign: $y(t) > 0$ for $t\in(-\infty,-2)\cup(0,2)$ and $y(t) < 0$ for $t\in(-2,0)\cup(2,\infty)$.

If the curve has a loop, then $(x(t_1),y(t_1)) = (x(t_2),y(t_2))$ for some $t_1 \neq t_2$. This gives $t_1^2 - 1 = t_2^2 - 1$ and $4t_1 - t_1^3 = 4t_2 - t_2^3$. With the solution: $t_1 = -t_2 = 2$. Therefore the loop corresponds to the interval $t\in[-2,2]$.

To compute the area of this loop we apply Green's formula: $\oint\limits_{\partial S}(-ydx) = \iint\limits_S dxdy$. Note that in our case the orientation of the loop is clockwise, thus we should add minus. Therefore the area of this loop equals

$$-\oint\limits_{\gamma_{t\in[-2,2]}}(-ydx) = \int\limits_{-2}^{2}(4t-t^3)2tdt = \frac{8\cdot 16}{15}$$

b. Note that $|arctan(...)| < \frac{\pi}{2}$, thus the continuity is immediate. Note that $f'_x|_{(0,0)} = 0 = f'_y|_{(0,0)}$, as $f|_{xy=0} = 0$. We check the differentiability:

$$\lim\limits_{(x,y)\to(0,0)}\frac{f(x,y)-f(0,0)-f'_x|_{(0,0)}x-f'_y|_{(0,0)}y}{\sqrt{x^2+y^2}} = \lim\limits_{(x,y)\to(0,0)}\frac{x^2 arctan\frac{y}{x} - y^2 arctan\frac{x}{y}}{\sqrt{x^2+y^2}} = \lim\limits_{r\to 0} r\Big(cos^2(\phi)arctan\frac{y}{x} - sin^2(\phi)arctan\frac{x}{y}\Big) = 0.$$

(Again, using that $|arctan(...)|$ is bounded.) Thus the function is differentiable.

As we did not check that the function is continuously twice-differentiable, we cannot use the standard $f''_{xy} = f''_{yx}$. Thus we compute the 2'nd order derivatives directly. We record the derivatives of the first order:

$$f'_x = \begin{cases} 2x\cdot arctan\frac{y}{x} - y,\ xy\neq 0 \\ 0,\ xy = 0\end{cases},\quad f'_y = \begin{cases} -2y\cdot arctan\frac{y}{x} + x,\ xy\neq 0 \\ 0,\ xy = 0\end{cases}$$

Therefore $f''_{xy}|_{(0,0)} = 0$ and $f''_{yx}|_{(0,0)} = 0$. In particular $f''_{xy}|_{(0,0)} = f''_{yx}|_{(0,0)}$.

(4) a. Note that the body does not contain the origin. Thus the field is differentiable in the body. Thus we can use Gauss' formula: $\iint\limits_{\partial V}\vec{F}d\vec{S} = \iiint\limits_V div(\vec{F})dV$. By the direct computation: $div(\vec{F}) = 0$. Thus $\iint\limits_{\partial V}\vec{F}d\vec{S} = \iiint\limits_V div(\vec{F})dV = 0$.

b. The total mass is $\int\limits_{\gamma}|y|d\gamma$, where $\gamma = \left\{ \begin{array}{l} r^2 = a^2cos(2\phi), \\ \phi \in [-\frac{\pi}{4}, \frac{\pi}{4}] \cup [\frac{3\pi}{4}, \frac{5\pi}{4}] \end{array} \right\}$. Use the formula $\int\limits_{\gamma} f d\gamma = \int\limits_{t} f\sqrt{r^2 + (\frac{dr}{d\phi})^2}d\phi$ to get:

$$\int\limits_{\gamma}|y|d\gamma = 2\int\limits_{-\frac{\pi}{4}}^{\frac{\pi}{4}}|y|\sqrt{r^2 + (\frac{a^2sin(2\phi)}{r})^2}d\phi = 2\int\limits_{-\frac{\pi}{4}}^{\frac{\pi}{4}}|sin(\phi)|\sqrt{r^4 + (a^2sin(2\phi))^2}d\phi = 2a^2\int\limits_{-\frac{\pi}{4}}^{\frac{\pi}{4}}|sin(\phi)|d\phi = 4a^4(1 - \frac{1}{\sqrt{2}})$$

(5) The normal to the surface is $\vec{\mathcal{N}} = (4x - y - z, 4y - x - z, 4z - x - y)$. The surface is tangent to the given plane at the points where $\vec{\mathcal{N}} \sim (1, 2, 1)$. In particular this gives $\mathcal{N}_x = \mathcal{N}_z$, which means $x = z$. Then $\mathcal{N}_y = 2\mathcal{N}_x$ gives $y = \frac{4}{3}x$. Substitute this to the equation of the surface to get the points: $(3, 4, 3)$ and $(-3, -4, -3)$.