Sketchy solutions of Moed.D, Hedva2.ME, 20119721

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(1) a. Denote $a_n = (\frac{e^n}{n} + \frac{\pi^{\frac{n}{2}}}{n^2})x^n$. Note that for $n \gg 1$: $\frac{e^n}{n} \gg \frac{\pi^{\frac{n}{2}}}{n^2}$. (For example take the n'th root of both sides.) Thus $\sqrt[n]{|a_n|} \to e \cdot x$. Therefore the series converges absolutely for $|e \cdot x| < 1$ and diverges for $|e \cdot x| > 1$.

Now we check the border points. For $x = \frac{1}{e}$ the series is: $\sum\limits_{n=1}^{\infty} \left(\frac{1}{n} + \frac{\pi^{\frac{n}{2}}}{e^n n^2}\right)$. Here the second part converges (e.g. compare to $\sum \frac{1}{n^2}$), while the first part diverges. Thus the whole series diverges. For $x = -\frac{1}{e}$ the series is: $\sum\limits_{n=1}^{\infty} \left(\frac{(-1)^n}{n} + (-1)^n \frac{\pi^{\frac{n}{2}}}{e^n n^2}\right)$. Here the second part converges absolutely (by the previous check). The first part converges by Leibnitz criterion.

In total, the series converges for $x \in [-\frac{1}{e}, \frac{1}{e})$. The convergence for $x = -\frac{1}{e}$ is conditional.

b. We use Stokes' formula: $\oint\limits_{\partial S} \vec{F} d\vec{\gamma} = \iint\limits_{\substack{z=x^2-y^2\\ x^2+y^2\le 1}} rot(\vec{F}) d\vec{S}$. As $\vec{\gamma}$ runs counterclockwise, the normal to $\vec{S}$ is taken upstairs, $\vec{\mathcal{N}}_S = (-2x, 2y, 1)$. Therefore $\oint\limits_{\partial S} \vec{F} d\vec{\gamma} = \iint\limits_{\substack{z=x^2-y^2\\ x^2+y^2\le 1}} \det \begin{pmatrix} -2x & 2y & 1 \\ \partial_x & \partial_y & \partial_z \\ z & x & y \end{pmatrix} dxdy = \iint\limits_{x^2+y^2\le 1} (-2x + 2y + 1) dxdy = \pi$

(2) a. Denote $f(x, y, z) = z^3 - 3xyz - 4$, then $\partial_z f = 3z^2 - 3xy$ and $\partial_z f|_{(2,1,-2)} \neq 0$. Thus there exists a differentiable function $z(x, y)$. Then $\partial_x z = \frac{yz}{z^2-xy}$ and $\partial_y z = \frac{xz}{z^2-xy}$ and further:

$$\partial^2_x z = \frac{y\partial_x z}{z^2 - xy} - \frac{yz(2z\partial_x z - y)}{(z^2 - xy)^2}, \quad \partial^2_{xy} z = \frac{y\partial_y z + z}{z^2 - xy} - \frac{yz(2z\partial_y z - x)}{(z^2 - xy)^2}, \quad \partial^2_y z = \frac{x\partial_y z}{z^2 - xy} - \frac{xz(2z\partial_y z - x)}{(z^2 - xy)^2}$$

In total:

b. $\partial_x z = 2x - 12$, $\partial_y z = 2y + 16$, thus $grad(z) = (0, 0)$ only at the point $(x, y) = (6, -8)$, which is not in the disc. So, there are no critical points in the interior of the disc. It remains to check the boundary. Denote $g(x, y) = x^2 + y^2 - 25$. Then Lagrange's condition reads: $(2x - 12)2y = (2y + 16)2x$. Which means: $4x = -3y$. Together with $g(x, y) = 0$ we get the points $(3, -4)$ and $(-3, 4)$. As $f(3, -4) < 0$ while $f(-3, 4) > 0$ we get: $(3, -4)$ is the global minimum, $(-3, 4)$ is the global maximum.

(3) a. The curve bounds the diamond, in Cartesian coordinates the curve equation is $(\frac{x}{a})^{\frac{2}{3}} + (\frac{x}{b})^{\frac{2}{3}} = 1$. To compute its area we use the Green formula:

$$S = -\oint\limits_{\gamma} ydx = -\int\limits_{t=0}^{2\pi} asin^3(t)3bcos^2(t)(-sin(t)dt) = 3ab\int\limits_{0}^{2\pi} sin^4(t)cos^2(t)dt = \frac{3ab}{8}2\pi$$

b. $\lim\limits_{(x,y)\to(0,0)} f(x, y) = \lim\limits_{r\to 0} \frac{sin(\theta)sin(r^2cos(\theta))}{r} = 0$, thus the function is continuous.

$\partial_x f|_{(0,0)} = \lim\limits_{\Delta x\to 0} \frac{f(\Delta x,0)-f(0,0)}{\Delta x} = 0$ and similarly $\partial_y f|_{(0,0)} = 0$. To check the differentiability it remains to check that the remainder vanishes fast enough:

$$\lim_{(x,y)\to(0,0)} \frac{f(x,y) - x\partial_x f|_{(0,0)} - y\partial_y f|_{(0,0)}}{\sqrt{(x^2 + y^2)}} = \lim_{(x,y)\to(0,0)} \frac{ysin(x^2)}{(x^2 + y^2)^{\frac{3}{2}}} = \lim_{(x,y)\to(0,0)} \frac{sin(\theta)sin(r^2cos(\theta))}{r^2}$$

The limit does not exist. Thus the function is not differentiable.

(4) a. $mass = \iint\limits_{\substack{\sqrt{x}\le y\le\sqrt{x}+1\\ 2\le y+\sqrt{x}\le 5}} 2ydxdy \overset{u=y-\sqrt{x},\ v=y+\sqrt{x}}{=\!=\!=} \iint\limits_{\substack{0\le u\le 1\\ 2\le v\le 5}} (v+u)|\frac{\partial(x,y)}{\partial(u,v)}|dudv$. Note that $|\frac{\partial(u,v)}{\partial(x,y)}| = \frac{1}{\sqrt{x}}$. Thus

$$mass = \iint\limits_{\substack{0\le u\le 1\\ 2\le v\le 5}} (v+u)\frac{(v-u)}{2}dudv = 19$$

b. Note that $div(\vec{F}) = 0$. Therefore the fields is conservative outside the origin. Thus we can deform the surface of integration, as far as the origin stays inside:

$$\oiint\limits_{M} \vec{F} d\vec{S} = \oiint\limits_{x^2+y^2+z^2=1} \vec{F} d\vec{S} = \oiint\limits_{x^2+y^2+z^2=1} \frac{(x,y,z)}{1} d\vec{S} \overset{Gauss}{=} \iiint\limits_{x^2+y^2+z^2\le 1} 3dxdydz = 4\pi$$

(5) $T = \{\frac{x^2+y^2}{4} \le z \le 4,\ x \ge 0,\ y \ge 0\}$.

$\iiint\limits_{T} f(x, y, z)dxdydz = \int\limits_{0}^{4} dx \int\limits_{0}^{\sqrt{16-x^2}} dy \int\limits_{\frac{x^2+y^2}{4}}^{4} f(x, y, z)dz = \int\limits_{0}^{4} dy \int\limits_{\frac{y^2}{4}}^{4} dz \int\limits_{0}^{\sqrt{4z-y^2}} f(x, y, z)dx$