Sketchy solutions of Moed.A, Hedva1.EE, 201.1.981

05.07.2015 Ben Gurion University

1.a. Solution 1. We will use Cauchy criterion for convergence. For this we estimate:

$$|a_{n+1}-a_n| = |2sin\frac{a_n-a_{n-1}}{2}cos\frac{a_n+a_{n-1}}{2}| \le |(a_n-a_{n-1})cos\frac{a_n+a_{n-1}}{2}|.$$

Note that $|a_{n>0}| \le 1$, thus $|a_{n>1}| = |sin(a_{n-1})| \le |sin(1)|$. Thus we get: $|a_{n+1}-a_n| \le sin(1)|a_n-a_{n-1}|$. Note that $0 < sin(1) < 1$. Thus, as has been proved in the class, a_n converges to a finite limit.

Denote $x = \lim\limits_{n\to\infty} a_n$. Then $x = sin(x)$. And this equation has the unique solution: $x = 0$. (Indeed, for $x \ne 0$: $|sin(x)| < |x|$.)

Solution 2. Note that $|a_1| \le 1 < \frac{\pi}{2}$. Thus, if $a_1 \ge 0$ then $a_{n>1} \ge 0$, while if $a_1 \le 0$ then $a_{n>1} \le 0$. Thus, if $a_1 \ge 0$ then $0 \le a_{n+1} = sin(a_n) \le a_n$, while if $a_1 \le 0$ then $0 \ge a_{n+1} = sin(a_n) \ge a_n$. In both cases we get that the sequence is monotonic (and bounded) thus converges. Now the limit is obtained as in the solution above.

1.b. Present the expression for a_n in the form $\frac{1}{n} + \sum\limits_{k=1}^{n} \frac{1}{n}\frac{1}{\sqrt{1+\frac{k}{n}}}$. The second part is the Riemann sum for $\int\limits_0^1 \frac{dx}{\sqrt{1+x}}$. Therefore $\lim\limits_{n\to\infty} a_n = 2(\sqrt{2}-\sqrt{1})$.

2.a. Using Taylor expansions $sin(\alpha x) = \alpha x - \frac{(\alpha x)^3}{3!} + O(x^5)$ we get:

$$sin(\alpha sin(\beta x)) = \alpha(\beta x - \frac{(\beta x)^3}{3!}) - \frac{(\alpha\beta x)^3}{3!} + O(x^5) \text{ and } sin(\alpha sin(\beta x)) - sin(\beta sin(\alpha x)) = \frac{\beta\alpha^3 - \alpha\beta^3}{3!}x^3 + O(x^5).$$

By the initial assumption $\beta\alpha^3 - \alpha\beta^3 \ne 0$. Therefore $n = 3$ and $\lim\limits_{x\to 0} \frac{sin(\alpha sin(\beta x)) - sin(\beta sin(\alpha x))}{x^3} = \frac{\beta\alpha^3-\alpha\beta^3}{3!}$.

2.b. $\int\limits_2^3 x \cdot ln(x^2-1)dx = \frac{x^2}{2}ln(x^2-1)\Big|_2^3 - \int\limits_2^3 \frac{x^2 2xdx}{2(x^2-1)} = \frac{27ln(2)-4ln(3)}{2} - \int\limits_4^9 \frac{tdt}{2(t-1)} = \frac{24ln(2)-3ln(3)-5}{2}$.

3.a. Denote $f(x) = arcsin(e^{-x})$. Using Lagrange's theorem: $\left|\frac{arcsin(e^{-x})-arcsin(e^{-y})}{x-y}\right| = |f'(c)| = \frac{e^{-c}}{\sqrt{1-e^{-2c}}} = \frac{1}{\sqrt{e^{2c}-1}}$ for some $c \in (x,y)$. By the assumption $c > 2$ thus $e^{2c} > e^4 > 16$. Thus $\frac{1}{\sqrt{e^{2c}-1}} < \frac{1}{3}$. Therefore

$$|arcsin(e^{-x}) - arcsin(e^{-y})| \le \lambda\frac{|x-y|}{3}.$$

3.b. Let T be a period of $f(x)$. By Cantor's theorem $f(x)$ is uniformly continuous on say $[0,2T]$. Thus for any $\epsilon > 0$ there exists $\delta > 0$ (which does not depend on x) such that $|x_1-x_2| < \delta$ implies $|f(x_1)-f(x_2)| < \epsilon$ for any $x_1, x_2 \in [0,2T]$.

Now choose any $x_1, x_2 \in \mathbb{R}$ such that $|x_1-x_2| < \delta$. Present them in the form: $x_2 = \tilde{x}_2 + mT$, $x_1 = \tilde{x}_1 + mT$, where $\tilde{x}_1, \tilde{x}_2 \in [0,2T]$ and $|\tilde{x}_1 - \tilde{x}_2| < \delta$. Thus: $|f(x_1)-f(x_2)| = |f(\tilde{x}_1)-f(\tilde{x}_2)| < \epsilon$. Which is precisely the uniform continuity.

Question: why did we consider here the interval $[0,2T]$ and not just $[0,T]$?

4.a. The integrand is bounded and continuous. We should check the convergence of $\int\limits_0^\infty (**)dt$ and $\int\limits_{-\infty}^0 (**)dt$.

Note that $0 < \frac{e^{2t}}{1+e^{(1+\sqrt{2})t}} = \frac{1}{e^{-2t}+e^{(\sqrt{2}-1)t}} < e^{(1-\sqrt{2})t}$. Thus the integral $\int\limits_0^\infty (**)dt$ converges by comparison to $\int\limits_0^\infty e^{(1-\sqrt{2})t}dt$. Similarly, $0 < \frac{e^{2t}}{1+e^{(1+\sqrt{2})t}} < e^{2t}$, thus $\int\limits_{-\infty}^0 (**)dt$ converges by comparison to $\int\limits_{-\infty}^0 e^{2t}dt$.

4.b.i. The function $f(x)$ is differentiable as the integrand is continuous. In particular $f(x)$ is bounded on any finite interval. Moreover, it is bounded on $(0,\infty)$, by part a.

Note that $f(1) = 0$. As the integrand is positive, $f(x) > 0$ for $x > 1$ and $f(x) < 0$ for $x \in (0,1)$.

As the function is continuous and bounded, and $\lim\limits_{x\to 0^+} f(x)$, $\lim\limits_{x\to\infty} f(x)$ exist, $f(x)$ uniformly continuous. (As was proven in the class.)

As $f(x)$ is continuous everywhere in $(0,\infty)$ and $\lim\limits_{x\to 0^+} f(x)$ is finite, there are no vertical asymptotes at finite points.

There is horizontal asymptote, as $x \to \infty$, and it is: $y = \int\limits_0^\infty \frac{e^{2t}}{1+e^{(1+\sqrt{2})t}}dt < \infty$.

4.b.ii. $f'(x) = \frac{x}{1+x^{1+\sqrt{2}}} > 0$, so the function grows monotonically on $(0,\infty)$. Thus there are no local/global minima/maxima. $sup(f) = \lim\limits_{x\to+\infty} f(x)$ and $inf(f) = \lim\limits_{x\to 0^+} f(x)$, both are finite.

$f''(x) = \frac{1-\sqrt{2}x^{1+\sqrt{2}}}{(1+x^{1+\sqrt{2}})^2}$. Thus for $0 < x < 2^{-\frac{1/2}{1+\sqrt{2}}}$, $f''(x) > 0$ (the function is convex down) and for $x > 2^{-\frac{1/2}{1+\sqrt{2}}}$, $f''(x) < 0$ (the function is convex up). The point $x = 2^{-\frac{1/2}{1+\sqrt{2}}}$ is the only inflection point.