Sketchy solutions of Moed.B, Hedva1.EE, 201.1.981

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1.a. Suppose $a \le b$, present the sequence in the form $b\sqrt[n]{(\frac{a_n}{b})^n + (\frac{b_n}{b})^n}$. Note that $b_n \to b > 0$, thus for any $\epsilon > 0$: $1-\epsilon < \frac{b_n}{b} < 1+\epsilon$, for n large enough. Similarly, $\frac{a_n}{b} < 1+\epsilon$. Therefore $(1-\epsilon)^n < (\frac{a_n}{b})^n + (\frac{b_n}{b})^n < 2(1+\epsilon)^n$. We have proved: for any $\epsilon > 0$ and n large enough there holds $(1-\epsilon) < \sqrt[n]{(\frac{a_n}{b})^n + (\frac{b_n}{b})^n} < (1+\epsilon)\sqrt[n]{2}$. Therefore $\lim\limits_{n\to\infty} \sqrt[n]{(\frac{a_n}{b})^n + (\frac{b_n}{b})^n} = 1$. Thus $\lim\limits_{n\to\infty} \sqrt[n]{(a_n)^n + (b_n)^n} = b$.

Similarly in the case $b \le a$. In total: $\lim\limits_{n\to\infty} \sqrt[n]{(a_n)^n + (b_n)^n} = max(a,b)$.

1.b. Note that $cos(x)$ is negative for $x \in [\frac{\pi}{2}, \frac{2\pi}{3}]$. Therefore:

$$area = \int\limits_{\frac{\pi}{4}}^{\frac{\pi}{2}} cos(x)|cos(2x)|dx - \int\limits_{\frac{\pi}{2}}^{\frac{2\pi}{3}} cos(x)|cos(2x)|dx \overset{\substack{cos(2x)=1-2sin^2(x)\\ t:=sin(x)}}{==} \int\limits_{\frac{1}{\sqrt{2}}}^{1} |1-2t^2|dt - \int\limits_{1}^{\frac{\sqrt{3}}{2}} |1-2t^2|dt =$$

$$= \int\limits_{\frac{1}{\sqrt{2}}}^{1} (2t^2-1)dt + \int\limits_{\frac{\sqrt{3}}{2}}^{1} (2t^2-1)dt = \frac{\sqrt{2}}{3} + \frac{\sqrt{3}}{4} - \frac{2}{3}$$

2.a. The function $f(x)$ is defined at all the points where the denominator does not vanish. If the denominator, $\frac{1}{f(x)} = e^x - e^{-x} + \sqrt{2}sin(x\sqrt{3}) + ln(5)$, vanishes at some point x_0, then $\lim\limits_{x\to x_0} f(x) = \pm\infty$. In particular $f(x)$ cannot be uniformly continuous in any neighborhood of x_0.

Note that $\lim\limits_{x\to+\infty} \frac{1}{f(x)} = +\infty$, $\lim\limits_{x\to-\infty} \frac{1}{f(x)} = -\infty$. Therefore the equation $e^x - e^{-x} + \sqrt{2}sin(x\sqrt{3}) + ln(5) = 0$ has at least one root. At this point $f(x)$ has a vertical asymptote. Thus $f(x)$ is not uniformly continuous in its domain of definition.

2.b. It is enough to check the convergence of $\lim\limits_{N\to\infty} \int\limits_1^N sin(x^2)dx$. The change of variable $t := x^2$ converts this limit into $\lim\limits_{N\to\infty} \int\limits_1^{N^2} \frac{sin(t)dt}{2\sqrt{t}}$. Now the integration by parts leads to: $\lim\limits_{N\to\infty} \Big(-\frac{cos(t)}{2\sqrt{t}}\Big|_1^{N^2} - \int\limits_1^{N^2} \frac{cos(t)dt}{4\sqrt{t^3}}\Big)$. Here the second summand converges absolutely, e.g. by comparison to $\int\limits_1^\infty \frac{dt}{4\sqrt{t^3}}$.

3.a. Consider the function $f(x) = xsin\frac{x}{2} - cos(x) - ln(17)$. Note that $f(2\pi n) < 0$, while $f(\pi + 4\pi n) > 0$. Therefore on each interval $(4\pi n, \pi + 4\pi n)$ and $(\pi + 4\pi n, 2\pi + 4\pi n)$ there is at least one root of the equation.

3.b. Note that $\lim\limits_{n\to+\infty} x_n = +\infty$. (For example, on each interval $[\pi n, \pi(n+1)]$ there is at most a finite number of roots.) Note that $cos(x_n) + ln(17)$ is bounded. Thus $x_n sin\frac{x_n}{2} = cos(x_n) + ln(17)$ forces: $sin\frac{x_n}{2} \to 0$. Therefore the points $\frac{x_n}{2}$ approach the points of the sequence $\{\pi m\}_{m\in\mathbb{N}}$. More precisely, there is a sequence of natural numbers $\{m_n\}_n$, satisfying: $\Big(\frac{x_n}{2} - \pi m_n\Big) \to 0$.

By part a. there are two subsequences of x_n, for one subsequence $\Big(\frac{x_n}{2} - \pi m_n\Big) \to 0^+$, for the other $\Big(\frac{x_n}{2} - \pi m_n\Big) \to 0^-$. Therefore, for the first subsequence $\{\frac{x_n}{2\pi}\} \to 0$, while for the second subsequence $\{\frac{x_n}{2\pi}\} \to 1$. Thus, the sequence $\{\frac{x_n}{2\pi}\}$ does not converge.

4.a. As $f'(x) > 0$, the function is monotonically increasing. In particular f is 1:1 and is surjective onto its image. Therefore f is invertible. As f is monotonic, continuous and is defined on the whole $\mathbb{R}$, the inverse function is continuous as well.

As $\lim\limits_{x\to+\infty} f(x) = \pi$, we can assume: $f'(x) > 3$ for $x \ge x_0$, for some $x_0 \gg 0$. Then Lagrange's theorem on $[x_0, x_1]$ gives: $f(x_1) = f(x_0) + (x_1 - x_0)f'(c) > f(x_0) + 3(x_1 - x_0)$. Therefore $\lim\limits_{x_1\to\infty} f(x_1) = \infty$.

By the definition of inverse function: $g(f(x)) \equiv x$. Thus $\lim\limits_{y\to+\infty} g(y) = \lim\limits_{x\to\infty} g(f(x)) = +\infty$. (Here we use: g is continuous and increasing.)

4.b. Note that $\lim\limits_{y\to+\infty} g(y) = \infty$ and $\lim\limits_{x\to\infty} f(x) = \infty$, therefore $\lim\limits_{y\to\infty} \int\limits_0^{g(y)} f(t)dt = \infty$. Note that $g(y)$ is differentiable. Thus we can apply l'Hopital's rule: $\lim\limits_{y\to\infty} \frac{\int\limits_0^{g(y)} f(t)dt}{y^2} = \lim\limits_{y\to\infty} \frac{g'(y)f(g(y))}{2y} = \lim\limits_{y\to\infty} \frac{1}{f'|_{g(y)}} \frac{y}{2y} = \lim\limits_{x\to\infty} \frac{1}{f'(x)} \frac{1}{2} = \frac{1}{2\pi}$.