Sketchy solutions of Moed.C, Hedva1.EE, 201.1.981

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1.a. Note that $a_n > 0$ for any n. Therefore $a_{n+1} < \frac{1}{c}$. So, the sequence is bounded from both sides.

Note that $a_{n+1} - a_n = \frac{\frac{1}{a_{n-1}} - \frac{1}{a_n}}{(c+\frac{1}{a_n})(c+\frac{1}{a_{n-1}})}$. Thus the sign of $a_{n+1} - a_n$ is preserved along the sequence, i.e. a_n is monotonic. Therefore a_n converges.

The limit $x = \lim a_n$ satisfies: $x = \frac{x}{xc+1}$. From here: $cx^2 = 0$, i.e. $\lim a_n = 0$.

1.b. The sequence is $b_n = \frac{1}{n} + \sum\limits_{i=1}^{n} \frac{e^{\frac{i}{n}}}{n}$. Here the second term is the Riemann sum for $\int\limits_0^1 e^x dx$. Therefore $\lim b_n = 0 + \int\limits_0^1 e^x dx = e - 1$.

2.a. Note that the part $\frac{1}{[x+1]^2}$ is locally constant, for $n-1 \le x < n$: $\frac{1}{[x+1]^2} = \frac{1}{n^2}$. It is (uniformly) continuous except for the points $x \in \mathbb{N}$. At these points it is continuous from the right, but discontinuous from the left.

The part $sin(\pi x) \cdot sin\Big(\frac{1}{sin(\pi x)}\Big)$ is periodic and continuous for $x \notin \mathbb{N}$. Though it is not defined for $x \in \mathbb{N}$, there exists the finite limits: $\lim\limits_{x \to n} sin(\pi x) \cdot sin\Big(\frac{1}{sin(\pi x)}\Big) = 0$. Therefore this part is uniformly continuous on the whole $\mathbb{R} \setminus \mathbb{N}$.

In total, $f(x)$ is bounded and uniformly continuous on each interval $[n, n+1)$, for $n \in \mathbb{N}$. But $f(x)$ is discontinuous from the left at all the points $x \in \mathbb{N}$.

2.b. Note that $\int\limits_1^\infty \frac{dx}{[x+1]^2} < \int\limits_1^\infty \frac{dx}{x^2} < \infty$. Therefore the convergence/divergence depend only on $\int\limits_1^\infty sin(\pi x) \cdot sin\Big(\frac{1}{sin(\pi x)}\Big)dx$. But the integrand is periodic, and is not identically zero. Therefore the later integral diverges. Hence $\int\limits_1^\infty f(x)dx$ diverges.

3.a. We are studying the solutions of the equation $f(x) = \frac{1}{x^2}$. It is given that $f(\pi n) = 0 < \frac{1}{(\pi n)^2}$. Therefore (using the mean value theorem) it is enough to prove the existence of infinite amount of point where $f(x) > \frac{1}{x^2}$.

Suppose there are no such points for $x \gg 0$. Then (by the mean value theorem), for $x \gg 0$ holds: $f(x) \le \frac{1}{x^2}$. But then, by the comparison criterion for convergence, $\int\limits_1^\infty f(x)dx < \infty$. Contrary to the given data. Therefore, there exist a sequence of points $\tilde{x}_n \to \infty$ where $f(\tilde{x}_n) > \frac{1}{\tilde{x}_n^2}$.

Together with $f(\pi n) = 0$ we get (by the mean value theorem) a sequence of points $x_n \to \infty$ with $f(x_n) = \frac{1}{x_n^2}$.

3.b. Define $f(x) = sin(x) - x + \frac{x^3}{3!}$. Note that $f''(x) = -sin(x) + x$. Thus $f''(x) \ge 0$ for $x \ge 0$. Thus $f'(x)$ is non-decreasing for $x \ge 0$. As $f'(0) = 0$, we get: $f'(x) \ge 0$ for $x \ge 0$. Finally, as $f(0) = 0$, we get: $f(x) \ge 0$.

4.a. We use the formula $a^3 - b^3 = (a-b)(a^2 + ab + b^2)$. Thus the "conjugate" for $\sqrt[3]{1+x} - \sqrt[3]{1-x}$ is $(\sqrt[3]{1+x})^2 + \sqrt[3]{1+x}\sqrt[3]{1-x} + (\sqrt[3]{1-x})^2$. Therefore:

$$\lim_{x\to\infty} \frac{\sqrt[3]{1+x} - \sqrt[3]{1-x}}{\sqrt[3]{1+2x} - \sqrt[3]{1-2x}} = \lim_{x\to\infty} \frac{(1+x)-(1-x)}{(1+2x)-(1-2x)} \cdot \frac{\sqrt[3]{(1+x)^2} + \sqrt[3]{1-x^2} + \sqrt[3]{(1-x)^2}}{\sqrt[3]{(1+2x)^2} + \sqrt[3]{1-4x^2} + \sqrt[3]{(1-2x)^2}} = \frac{1}{2^{1+\frac{2}{3}}}.$$

4.b. $\int\limits_2^3 \frac{dx}{10-x^2} = \frac{1}{2\sqrt{10}} \int\limits_2^3 \Big(\frac{1}{\sqrt{10}-x} + \frac{1}{\sqrt{10}+x}\Big)dx = \frac{1}{2\sqrt{10}} ln\Big(\frac{\sqrt{10}+3}{\sqrt{10}-3} \cdot \frac{\sqrt{10}-2}{\sqrt{10}+2}\Big)$.