Calculus1.EE, BGU, Spring 2015.

Below are some solutions/answers to some of the questions. The work is in the progress, the file is getting longer (by regular updates).

Partial answers to some of the questions of HWK1

2. a. Subsets of S: S, $\{2,3,8\}$, $\{2,4,8\}$, $\{2,3,4\}$, $\{3,4,8\}$, $\{2,8\}$, $\{2,3\}$, $\{2,4\}$, $\{3,8\}$, $\{4,8\}$, $\{3,4\}$, $\{2\}$, $\{3\}$, $\{4\}$, $\{8\}$, $\varnothing$.

b. $S\cup\varnothing=S$, $S\cap\varnothing=\varnothing$, $S\cup\mathbb{R}=\mathbb{R}$, $S\cap\mathbb{R}=S$, $\mathbb{R}\setminus S=\{x\in\mathbb{R}|\ x\neq 2,3,4,8\}$, $S\cap(\mathbb{R}\setminus\mathbb{Q})=\varnothing$, $S\cap(\mathbb{Z}\setminus\mathbb{N})=\varnothing$.

4. a. If $x\in\mathbb{Z}$ then $n=(x+1)$ is the needed number, as $x<x+1<y$. Otherwise $x<n=\lceil x\rceil<y$.

b. The proof of $\sqrt{3}\notin\mathbb{Q}$ is the same as for $\sqrt{2}\notin\mathbb{Q}$, which is done in the class. From here: $1+\sqrt{3}\notin\mathbb{Q}$. (Otherwise $\sqrt{3}=(1+\sqrt{3})-1\in\mathbb{Q}$.)

The case of $\sqrt{2}+\sqrt{3}$. First one proves that $\sqrt{6}\notin\mathbb{Q}$. (Similarly to $\sqrt{3}\notin\mathbb{Q}$.) Now, suppose $x=\sqrt{2}+\sqrt{3}\in\mathbb{Q}$, then $\frac{x^2-5}{2}=\sqrt{6}\in\mathbb{Q}$. Contradiction.

c. yes. ($\frac{a}{b}\times\frac{c}{d}=\frac{ac}{bd}$, which is a fraction of integers. $\frac{a}{b}+\frac{c}{d}=\frac{ad+cb}{bd}$)

d. and e. $\pm\sqrt{2}\notin\mathbb{Q}$ but both $\sqrt{2}\cdot(-\sqrt{2})\in\mathbb{Q}$ and $\sqrt{2}+(-\sqrt{2})\in\mathbb{Q}$.

6. By the assumption $x\geq 0$. Suppose $x>0$. Take $c=x\in\mathbb{R}$, then by the assumption $x<x$. Contradiction.

9. Suppose $x=inf(A)$ and for some $\epsilon>0$: $[x,x+\epsilon)\cap A=\varnothing$. Then $x+\epsilon$ is also a lower bound of A, contrary to the maximality of x.

Suppose x is a lower bound of A and for any $\epsilon>0$ there exists $y\in A$ such that $y<x+\epsilon$. Suppose there exists a bigger lower bound, $x<x_1$, of A. Choose $\epsilon=\frac{x_1-x}{2}>0$ to get the contradiction.

11. Let $x=inf(T)$, then $T\subseteq[x,\infty)$. By the assumptions x is an upper bound of S, thus $S\subseteq(-\infty,x]$. As $S\cup T=\mathbb{R}$ we have: $x=sup(S)$. Hence the statement.

Partial answers to some of the questions of HWK2

1. Consider the subsequences: $a_{2n}=0$, $a_{4n+1}=\sqrt{4n+1}$, $a_{4n-1}=-\sqrt{4n-1}$. Note that a_{2n} converges (and is bounded), while a_{4n+1}, a_{4n-1} are monotonic and unbounded. Thus a_n is not monotonic, unbounded, not converging.

2. ii. Present $a_n=\frac{3}{\sqrt{n+3}+\sqrt{n}}\to 0$.

iii. $a_n=\frac{\sqrt{n}}{\sqrt{n+3}+\sqrt{n}}\to 1$.

vii. As $\sqrt[n]{n}\to 1$, for n large enough $0<\sqrt[n]{n}-1<0.5$. Thus by sandwich lemma $(\sqrt[n]{n}-1)^n\to 0$.

3. iv. $a_{2n}=\frac{2n}{2n+1}\to 1$, $a_{2n+1}=0\to 0$. Thus a_n does not converge.

v. By the definition of the sequence $a_{2n+1}-a_{2n}=1\not\to 0$, thus a_n cannot converge.

4. i. no

ii. yes

iii. $a_n=1,0,2,0,3,0,4,0,\ldots$

iv. yes

v. take $a_n=(-1)^n$, $b_n=(-1)^{n+1}$.

vi. Take $a_n=0$, $b_n=(-1)^n$.

vii. $a_n=\frac{1}{n}>0$ but $\lim a_n\not>0$.

viii. If $a_n\to L>0$ then for $\epsilon=L$ and big enough n one has: $|a_n-L|<\epsilon=L$, which implies: $0<a_n$.

ix. Take as a_n the decimal expansion of $\sqrt{2}$. Then $a_n\in\mathbb{Q}$ but converges to $\sqrt{2}\notin\mathbb{Q}$. Take $b_n=\frac{\sqrt{2}}{n}$. Then $b_n\notin\mathbb{Q}$ but $b_n\to 0\in\mathbb{Q}$.

x. Take $a_n=(-1)^n$

xi. Take $a_n=\left\{\begin{array}{ll}\frac{1}{n}: & n\in 2\mathbb{Z}\\ \frac{1}{n^2}: & n\notin 2\mathbb{Z}\end{array}\right.$

5. $\frac{n}{\sqrt{n^2+n}}=n\cdot min\{\frac{1}{\sqrt{n^2+k}}\}\leq a_n\leq n\cdot max\{\frac{1}{\sqrt{n^2+k}}\}=\frac{n}{\sqrt{n^2+1}}$. Now use this sandwich.

6. i. If $a_n\to a$ then $\forall\epsilon>0$: $|a-a_n|<\epsilon$ for $n>N$. Thus for $n>N$: $|b_n-a|\leq\frac{\sum\limits_{i=1}^{N}|a_i-a|}{n}+\frac{\sum\limits_{i=N+1}^{n}|a_i-a|}{n}$. Thus, as N is fixed, we can take n so large that $|\frac{\sum\limits_{i=1}^{N}|a_n-a|}{n}|<\epsilon$. Then $|b_n-a|<2\epsilon$.

ii. $a_n = (-1)^n$, $b_n \to 0$.

7. iii. $\frac{a_{n+1}}{a_n} = \frac{1}{cos(\frac{\pi}{2^{n+1}})} > 1$, thus a_n increases. On the other hand: for $x > 0$, $sin(x) < x$, thus $a_n \leq \pi$. Hence a_n converges.
iv. $\frac{a_{n+1}}{a_n} < 1$ thus a_n decreases. And $a_n > 0$, thus a_n converges.
v. $a_{n+1} - a_n = \frac{1}{2n+2} + \frac{1}{2n+1} - \frac{1}{n} < 0$, thus a_n decreases. And $a_n > 0$, thus a_n converges.

Partial answers to some of the questions of HWK3

1. From $a_{n+1} = a_n(a_n - 1)$ and a_1 =3 one gets: a_n increases and $a_n \geq 3$ (e.g. by induction). Thus $a_{n+1} \geq 2a_n \geq \cdots \geq 2^{n-1}a_1$. Thus $a_n \to \infty$.
Note, if one does not check the existence of the limit and just assumes that $lim(a_n) = a$ is finite then from $a = a^2 - a$ one gets wrong answers $a = 0, 2$.

6. The sequence is increasing. If it is bounded then it converges to a finite limit, contradicting $a_{n+1} - a_n \to L > 0$. Thus the sequence is unbounded (and increasing), thus $a_n \to \infty$

7.b. From the convergence of a_{2n+15} and $a^3_{2n+3} - a^3_{2n+4}$ one has: the subsequences a_{2n} and a_{2n+1} converge. Further, from the convergence of a_{n^3} one gets: $\lim\limits_{n\to\infty} a_{2n} = \lim\limits_{n\to\infty} a_{2n+1}$. Thus a_n converges.
(Note that the condition of convergence of $a^2_{2n+3} - a^2_{2n+4}$ is not necessary.)

9. i. $a_n = n$. ii. $a_n : 1, 2, 3, 4, 5, 6, 7, 1, 2, 3, 4, 5, 6, 7, \ldots$. iii. $a_n : 1, 0, 2, 0, 3, 0, \ldots$
iv. and v. $1, 2, 1, 2, 3, 1, 2, 3, 4, 1, 2, 3, 4, 5, \ldots,$

11. a. A counterexample: $a_n = ln(ln(n))$. b. A counterexample: $a_n = n + (-1)^n$, $b_n = -n$.
c. A counterexample: $a_n = n, b_n = \frac{(-1)^n}{n}$. d. A counterexample: $a_n = 0, 1, 0, 2, 0, 3, 0, 4, 0, \ldots, b_n = 1, 0, 2, 0, 3, 0, 4, 0, \ldots$.
Then 0 is a partial limit of both a_n and b_n but not of $a_n + b_n$.
e. If a_n is monotonic and has a converging subsequence, then a_n is bounded. Hence converging.

12. a. $|a_{n+m} - a_n| = |(a_{n+m} - a_{n+m-1}) + \cdots + (a_{n+1} - a_n)| \leq \frac{1}{(n+m)(n+m+1)} + \cdots + \frac{1}{n(n+1)} = \frac{1}{n} - \frac{1}{n+m+1} \to 0$. (Now use Cauchy criterion for convergence.)
b. Similarly.
c. $|a_{n+1} - a_n| = \frac{1}{2}|a_n - a_{n-1}|$, now use part ii.

13. a. Note that $a_1 > 0$, thus by induction $a_n > 0$. Thus $a_{n+1} - a_n > 0$, i.e. a_n increases.
Suppose $a_n \to L < \infty$, then $a_{n+1} - a_n \to 0$. In contradiction to $a_{n+1} - a_n = \frac{1}{a_n} \to \frac{1}{L} \neq 0$. Thus $a_n \to \infty$.
b. The induction: we assume $a_n < 3\sqrt{n}$ and prove $a_{n+1} < 3\sqrt{n+1}$.
Indeed: $a^2_{n+1} = a^2_n + 2 + \frac{1}{a^2_n} < 9n + 2 + 1 < 9(n + 1) = (3\sqrt{n+1})^2$. Here we use $a_n \geq 1$, as a_n increases.

14. b. Take e.g. $b_n = \frac{1}{\sqrt{|a_n|+1}}$.

Partial answers to some of the questions of HWK4

1. i. Has been proven in the class.
ii. A counterexample: $a_n = (-1)^n$, $b_n = (-1)^{n+1}$.
iii. and iv. Has been proven in the class.

2. bet. i. Let $g(x) := f(\frac{1+x}{x-1})$. As $\mathcal{D}_f = \{0 < x < 1\}$, $\mathcal{D}_g = \{0 < \frac{1+x}{x-1} < 1\} = \{x < -1\}$.
ii. and iii. are done similarly.

3. i. $Im(f) = \{y :$ the equation $y = \frac{1}{1+x^2}$ has a solution $\} = \{0 < y \leq 1\}$
Similarly for ii. and iii.
iv. Present $f(x) = \sqrt{2}sin(x + \frac{\pi}{4})$ to get: $Im(f) = \{|y| \leq \sqrt{2}\}$.
v. $Im([x]) = \mathbb{Z}$.
vi. $Im\{x\} = [0, 1)$.

5. a. i. Use $a^2 + b^2 \geq 2ab$ to get: $|\frac{x}{x^2+2}| \leq \frac{1}{2\sqrt{2}}$
5. b. i. A counterexample: $f(x) = sin(x) = -g(x)$.

6. ii. $f(-x) = -f(x)$, iv. $f(-x) = -f(x)$ (multiply by conjugate)

7. $f(x) = \frac{f(x)+f(-x)}{2} + \frac{f(x)-f(-x)}{2}$.

8. a. i. Note that $f(x+2\pi) = f(x)$. To find the (minimal) period present $f(x) = 2sin(x)cos(x)(2cos(x)+1)$. If $f(x) = f(x+T)$, then for $x = 0$ one has: $sin(T)cos(T)(2cos(T)+1) = 0$. Suppose $T \in (0, 2\pi)$. Then T is one of: $\frac{\pi}{2}$, π, $\frac{3\pi}{2}$, $\frac{2\pi}{3}$, $\frac{4\pi}{3}$. By the direct check: neither of these satisfies $f(x) = f(x+T)$ for $x \in \mathbb{R}$ arbitrary. Thus $T = 2\pi$.
iii. $f(x) = \sqrt{A^2+B^2}sin(\lambda x + \alpha)$, where $sin(\alpha) = \frac{A}{\sqrt{A^2+B^2}}$. Thus $T = \frac{2\pi}{\lambda}$.

8. b. Let $\chi(x) = \left\{ \begin{array}{l} 1: \ x \in \mathbb{Q} \\ 0: \ x \notin \mathbb{Q} \end{array} \right.$. The periods of $\chi(x)$ are precisely all the rational numbers. Note that no irrational number can be a period of $\chi(x)$.

8. c. Suppose $sin(x^2) = sin((x+T)^2)$ for any $x \in \mathbb{R}$. Then $sin(2xT+T^2)cos(2x^2+2xT+T^2) = 0$ for any x. If $T \neq 0$ then choose any x which is not a solution of $2xT + T^2 \in \pi Z$, neither of $2x^2+2xT+T^2 \in \frac{\pi}{2} + \pi\mathbb{Z}$.

9. b. A counterexample: $f(x) = 2x + sin(x)$, $g(x) = -2x$. To check that $f(x)$ is increasing note: $f(x_2) - f(x_1) = 2(x_2 - x_1) + 2sin\frac{x_2-x_1}{2}cos\frac{x_2+x_1}{2}$ and $|2sin\frac{x_2-x_1}{2}cos\frac{x_2+x_1}{2}| \leq |x_2 - x_1|$. Thus, for $x_2 > x_1$ we get: $f(x_2) > f(x_1)$.
c. A counterexample: $f(x) = 2x + sin(x)$, $g(x) = \frac{1}{x}$ on the interval $(0, a)$, for any $a > 2\pi$.

Partial answers to some of the questions of HWK5

2. a. ii. Present $f(x) = asin(x) + bcos(x) = \sqrt{a^2+b^2}sin(x+\phi)$, where $cos\phi = \frac{a}{\sqrt{a^2+b^2}}$, $sin\phi = \frac{b}{\sqrt{a^2+b^2}}$. Thus $max = \sqrt{a^2+b^2}$, $min = -\sqrt{a^2+b^2}$,.

6. First we pass from $\lim\limits_{x\to 0} f(\frac{1}{x})$ to $\lim\limits_{t\to\infty} f(t)$ by the change of variable, $x \to \frac{1}{t}$. If $f(t)$ is not constant, then $f(t_1) \neq f(t_2)$ for some $t_1, t_2 \in \mathcal{D}_f$. Thus consider the sequences: $t_1 + nT$ and $t_2 + nT$, where T is a period of f. Then

$$\lim_{n\to\infty} f(t_1 + nT) = f(t_1) \neq f(t_2) = \lim_{n\to\infty} f(t_2 + nT).$$

8. i. A counterexample: $f(x) = \chi(x)$, $g(x) = 1 - \chi(x)$, here $\chi(x)$ is the Dirichlet function.
ii. no.
iii. no.
iv. if $\lim\limits_{x\to a} f(x)$ exists then $f(x)$ is bounded in some neighborhood of $x = a$.
v. $f(x) = -x^2 \leq 0$ but $\lim\limits_{x\to 0} f(x) = 0$.
vi. yes
vii. $f(x) = x^2 \geq 0$, $\lim\limits_{x\to 0} f(x) = 0$.
viii. $f(x) = x^2$, $g(x) = -x^2$.
ix. A counterexample: $f(x) = \left\{ \begin{array}{l} 0, \ x \neq 0 \\ 1, \ x = 0 \end{array} \right.$ and $g(x) \equiv 0$. Then $\lim\limits_{x\to 0} f(g(x)) = 1 \neq \lim\limits_{x\to 0} f(x)$.

10. Proof of Weierstrass theorem. Consider the case of $\lim\limits_{x\to x_0^-} f(x)$ and $f(x)$ non-decreasing.

proof (using Cauchy's definition of the limit.) Let $L = \sup\limits_{x<x_0} f(x)$, as the function is bounded near x_0 this sup is finite. If L is actually achieved at some point $x_1 < x_0$, then (as $f(x)$ is non-decreasing) $f(x) \equiv L$ on (x_1, x_0). In particular $\lim\limits_{x\to x_0^-} f(x) = L$.

Suppose L is not achieved, then, by the properties of sup, for any $\epsilon > 0$ there exists some $L - \epsilon < L_1 < L$ which is a value of $f(x)$. Namely, $L = f(x_1)$, for some $x_1 < x_0$. But then, for $x \in (x_1, x_0)$: $L - \epsilon < f(x) < L$. As we wanted to prove.

proof (using Heine's definition of the limit.) Let $x_n \to x_0^-$. Though x_n is not necessarily monotonic, it has an increasing subsequence, $x_{n_k} \to x_0$. Then the sequence $f(x_{n_k})$ is also increasing, and bounded. Thus $f(x_{n_k})$ converges. As L defined as sup, $limf(x_{n_k}) \leq L$. If $limf(x_{n_k}) < L$, then for any $x < x_0$: $f(x) \leq limf(x_{n_k}) < L$, contradicting that $L = sup$. Thus $limf(x_{n_k}) = L$.

In particular, for any $\epsilon > 0$ for n_k large enough $f(x_{n_k}) > L - \epsilon$. But then (as $f(x)$ is increasing) there exists $\delta > 0$ such that $f(x) > L - \epsilon$, for $x > x_0 - \delta$. Thus $limf(x_n) = L$. As we wanted to prove.

Partial answers to some of the questions of HWK6

1. iii. $\lim\limits_{x\to 0^-} \frac{sin(x)}{|x|} = -1 \neq f(0)$.

iv. For $x \neq 0$ can present: $f(x) = \frac{sin^2(x)}{x^2(1+cos(x))}$, thus $\lim\limits_{x\to 0} f(x) = 1/2$.

vi. For $|x| > 1$: $f(x) = 0$, in particular continuous. For $\frac{1}{n+1}x \leq \frac{1}{n}$ $f(x) = nx$. Thus $f(x)$ is continuous in the intervals $(\frac{1}{n+1}, \frac{1}{n})$ but discontinuous at the points $\{\frac{1}{n}\}_{n\in\mathbb{N}}$.

vii. The function is periodic, $f(x+2) = f(x)$. Thus it's enough to check the continuity for $x \in [0,2]$. The function is piecewise linear and continuous.

2. i., iii., iv., v., yes.

ii. , vi. no

3. Consider $f(x) = sin\frac{1}{x}$ on $(0,1)$ and $x_n = \frac{8}{\pi n}$.

4. i. no. ii. $f(x) = \chi(x)$, $g(x) = 1 - \chi(x)$, where $\chi(x)$ is Dirichlet function.

iii. $f(x) = \chi(x)$.

iv. Consider two increasing sequences: $0 \leq x_1 < x_2 < \cdots < x_n < \cdots \leq 1$ and $0 \leq y_1 < y_2 < \cdots < y_n < \cdots \leq 1$. Define the function: $f|_{[x_{n-1},x_n]} = y_n$. The graph consists of infinity of segments, at increasing heights. Thus f is monotonic, with infinite number of discontinuities.

v., vi., vii. viii. ix. no

x. A counterexample: $\mathcal{D} = [-2,-1] \cup [1,2]$. $f|_{[-2,-1]} = -1$ $f|_{[1,2]} = 1$.

xi., xii. xiii. no.

7. As A is bounded, $inf(A)$, $sup(A)$ are finite numbers. If $max(A)$ exists then it is $sup(A)$ and $f|_{max(A)} > 1$. But then $f > 1$ in some neighborhood of $max(A)$, contradiction.

8. Look at $f(x + T/2) - f(x)$.

10. In i. ii. iii., v. vi. vii. x. not uniformly continuous.

iv., vi., vii., viii., ix uniformly continuous.

Partial answers to some of the questions of HWK7

1. i. yes.

ii. $f^2(x)$ and $f(x)g(x)$ are not necessarily uniformly cont.

iii., iv., viii., ix., x., no

v., vii., xii. yes

vi. yes for $f \pm g$. Nothing can be said about $f(x)g(x)$.

4. $f(x)$ is uniformly cont. on $(0,1)$ but not on $(1,\infty)$.

8. i. and iii. $f'(x)$ is continuous in $\mathbb{R}$.

ii. $f'(x)$ exists everywhere but is not continuous at $x = 0$.

iv. $f'(x)$ exists for $x \neq 1$ and is continuous.

v. By the direct check, $f(x) = \begin{cases} \frac{\pi}{2} - (x - 2\pi n), & x \in [2\pi n, 2\pi n + \pi) \\ -\frac{\pi}{2} + (x - \pi - 2\pi n), & x \in [2\pi n + \pi, 2\pi n + 2\pi) \end{cases}$. Thus $f(x)$ is continuous everywhere, differentiable for $x \notin \pi n$ and the derivative is continuous (at the points where it exists).

9. Note that $f(x) = \prod\limits_{i=1}^{2014}(i - x)$, for $x < 1$ all the brackets are positive. Thus $ln(f(x)) = \sum\limits_{i=1}^{2014} ln(i - x)$, thus, for $x < 1$, $\frac{f'(x)}{f(x)} = \sum\limits_{i=1}^{2014} \frac{1}{i-x}$. From here: $f'(0) = 2014! \sum\limits_{i=1}^{2014} \frac{1}{i}$.

Partial answers to some of the questions of HWK8

3. A counterexample: $f(x) = sin(e^x)$.

4. i. true for odd/even, periodic. Not true for bounded.

ii. and iv. A counterexample: $f(x) = |x|$.

iii. From the definition of derivative follows: $f'(x) \equiv 0$.

v. and x. a counterexample $f(x) = sin(e^{2x})$

vii. a counterexample $f(x) = x \cdot sin(\frac{1}{x})$.

viii. The functions is differentiable, thus it is enough to check that the derivative is bounded. For that it is enough to check that $\lim\limits_{x\to\infty} x(\frac{\pi}{2} - arctan(x))$ is finite.

Alternatively, one can check that for $x \to \infty$: $f(x) \to \infty$ in the form: $f(x) \sim x$. Thus it is natural to consider $f(x) - x$. Then one checks that $\lim\limits_{x\to\infty}(f(x) - x) = 0$, thus $f(x) - x$ is uniformly continuous. As x is uniformly continuous one gets: $f(x)$ is uniformly continuous.

ix. If $\lim\limits_{x\to\infty} f'(x) = 0$, then for $x > M$: $|f'(x)| < \epsilon$. Thus, by Lagrange's theorem on any interval $[M,N]$: $|\frac{f(x)-f(M)}{x-M}| < \epsilon$. From here follows $\lim\limits_{x\to\infty} \frac{f(x)}{x} = 0$.

(Note that $\lim\limits_{x\to\infty} f'(x) = 0$ does not imply that $f(x)$ is bounded, e.g. $f(x) = ln(x)$.)

Similarly for the case of $f'(x)$ bounded.

xi. xiii. Follows from Lagrange thm

xii. a counterexample: $f = \begin{cases} x^2 sin\frac{1}{x^3},\ x \neq 0 \\ 0,\ x = 0 \end{cases}$.

xiv. a counterexample: $f(x) = x^2\chi(x)$, where $\chi(x)$ is the Dirichlet function

xv. a counterexample: $f(x) = \frac{x}{2} + x^2 sin\frac{1}{x}$.

5. a. Check that the functions are continuous and monotonic, then use the mean value theorem.

b. Define the polynomial $p(x) := \sum\limits_{i=0}^{n} \frac{c_i x^{i+1}}{i+1}$. Then $p(0) = 0$. Further, the condition $\sum\limits_{i=0}^{n} \frac{c_i}{i+1} = 0$ gives $p(1) = 0$. So, the continuous function $p(x)$ must have a min or max in $(0,1)$. Thus $p'(x) = 0$ for some $x \in (0,1)$.

c. (First solution) We can assume that $f(x) > 0$ on (a,b). (If needed one can shrink the interval to a smaller, preserving $f(a) = 0 = f(b)$ and $f|_{(a,b)} > 0$. If $f(x) < 0$ then consider $-f(x)$). Consider the equation $\alpha x + ln(f(x)) = 0$. As $ln(f(x)) \to -\infty$ for $x \to a$ or $x \to b$, we get: $ln(f(f))$ is continuous and bounded from above, there exists a local maximum of $\alpha x + ln(f(x))$ at some point $c \in (a,b)$. At this point one has: $(\alpha x + ln(f(x)))' = 0$, i.e. $\alpha f(c) + f'(c) = 0$, as is to be proved.

(Second solution) Consider the function $g(x) = e^{\alpha x} f(x)$. It is continuous and $g(a) = 0 = g(b)$. Thus $g'(c) = 0$. Note that $g'(x) = e^{\alpha x}(\alpha f(x) + f'(x))$ and $e^{\alpha x}$ does not vanish.

Partial answers to some of the questions of HWK9

1. a. The proof is by induction, similar to the proof of $(a+b)^n = ...$

For $n = 0$, $n = 1$ the formula holds. Suppose it holds for some n. Then

$$(f(x)g(x))^{(n+1)} = \Big(\sum_{k=0}^{n} \binom{n}{k} f^{(k)}(x) g^{(n-k)}(x)\Big)' = \sum_{k=0}^{n} \binom{n}{k} \Big(f^{(k+1)}(x) g^{(n-k)}(x) + f^{(k)}(x) g^{(n+1-k)}(x)\Big) = \cdots$$

b. ii. $f(x)^{(n)} = sin(x + \frac{\pi n}{2})$.

iii. Use $sin^2(x) = \frac{1 - cos(2x)}{2}$.

iv. Using question 1 and 2.ii. we get: $(e^x sin(x))^{(n)} = e^x \sum\limits_{k=0}^{n} \binom{n}{k} sin(x + \frac{\pi k}{2})$

v. $f^{(n)} = x sin(x + \frac{\pi n}{2}) + n \cdot sin(x + \frac{\pi(n-1)}{2})$

vi. Use $\frac{1}{x^2 - a^2} = \frac{1}{2a}(\frac{1}{x-a} - \frac{1}{x+a})$ and $(\frac{1}{x+a})^{(n)} = \frac{(-1)^n n!}{(x+a)^{n+1}}$.

2. a. i. $ln(x+a) = ln(a) + ln(1 + \frac{x}{a}) \rightsquigarrow ln(a) + \sum\limits_{n\geq 1} \frac{(-1)^{n-1}(x/a)^n}{n}$. (Here we use the expansion of $ln(1+x)$, obtained in the class.)

ii. $\frac{1}{x^2 - a^2} = -\frac{1}{2a}(\frac{1}{a-x} + \frac{1}{a+x}) = -\frac{1}{2a^2}(\frac{1}{1-\frac{x}{a}} + \frac{1}{1+\frac{x}{a}}) \rightsquigarrow -\frac{1}{2a^2} \sum\limits_{n\geq 0}((\frac{x}{a})^n + (-\frac{x}{a})^n) = -\frac{1}{a^2} \sum\limits_{n\geq 0} (\frac{x}{a})^{2n}$.

(Here we use the expansion of $\frac{1}{1+x}$, obtained in the class.)

iii. $sin(x+a) = sin(x)cos(a) + sin(a)cos(x) = cos(a) \sum\limits_{n\geq 0} \frac{(-1)^n x^{2n+1}}{(2n+1)!} + sin(a) \sum\limits_{n\geq 0} \frac{(-1)^n x^{2n}}{(2n)!}$.

b. i. $e^{x^3} = \sum\limits_{n\geq 0} \frac{x^{3n}}{n!}$.

ii. Using part a.iii we get: $sin(x^2 + a) = cos(a) \sum\limits_{n\geq 0} \frac{(-1)^n x^{4n+2}}{(2n+1)!} + sin(a) \sum\limits_{n\geq 0} \frac{(-1)^n x^{4n}}{(2n)!}$.

iii. Note that $(arctan(x))' = \frac{1}{1+x^2} = \sum\limits_{n\geq 0} (-x^2)^n$. Thus $arctan(x) = \sum\limits_{n\geq 0} \frac{(-1)^n x^{2n+1}}{2n+1}$.

c. i. $ln\frac{sin(x)}{x} = ln(1 - \frac{x^2}{3!} + \frac{x^4}{5!} + o(x^5)) = -\frac{x^2}{3!} + \frac{x^4}{5!} - \frac{(-\frac{x^2}{3!} + \frac{x^4}{5!})^2}{2} + o(x^4) = -\frac{x^2}{3!} + \frac{x^4}{5!} - \frac{x^4}{(3!)^2 2} + o(x^4)$.

ii. Use $sin^2(x) = \frac{1 - cos(2x)}{2}$.

e. e.g. use l'Hopital's rule n times.

3. i. Multiply by the conjugate.

ii. present in the form $(\lim\limits_{x\to 0} \frac{x}{sin(x)})(\lim\limits_{x\to 0} x sin\frac{1}{x})$

vii. One approach:

$$\Big(\sqrt{x+1} - \sqrt{x}\Big) + \Big(\sqrt{x-1} - \sqrt{x}\Big) = \frac{1}{\sqrt{x+1} + \sqrt{x}} - \frac{1}{\sqrt{x-1} + \sqrt{x}} = \frac{\sqrt{x-1} - \sqrt{x+1}}{(\sqrt{x+1} + \sqrt{x})(\sqrt{x-1} + \sqrt{x})} =$$
$$= \frac{-2}{(\sqrt{x+1} + \sqrt{x})(\sqrt{x-1} + \sqrt{x})(\sqrt{x-1} + \sqrt{x+1})}$$

Another approach:

$$\sqrt{x+1} + \sqrt{x-1} - 2\sqrt{x} = \sqrt{x}\Big(\sqrt{1 + \frac{1}{x}} + \sqrt{1 - \frac{1}{x}} - 2\Big) = \sqrt{x}\Big(1 + \frac{1}{2x} - \frac{1}{8x^2} + 1 - \frac{1}{2x} - \frac{1}{8x^2} - 2 + o(\frac{1}{x^2})\Big)$$

4. b.iii. $cosh(x) = \sum\limits_{n\geq 0} \frac{x^{2n}}{(2n)!}$, $sinh(x) = \sum\limits_{n\geq 0} \frac{x^{2n+1}}{(2n+1)!}$

5. b. For $x \neq 0$: $f'(x) = \frac{g'(x)}{x} - \frac{g(x)}{x^2}$. For $x = 0$: $f'(0) = \lim_{x \to 0} \frac{g(x)}{x^2} = \frac{3}{2}$. (Apply l'Hopital's rule twice.)
Further, $\lim_{x \to 0} f'(x) = \frac{3}{2} = f'(0)$, thus $f'(x)$ is a continuous function.
For $x \neq 0$: $f''(x) = \frac{g''(x)}{x} - \frac{2g'(x)}{x^2} + \frac{2g(x)}{x^3}$. For $x = 0$: $f''(0) = \lim_{x \to 0} \frac{\frac{g'(x)}{x} - \frac{g(x)}{x^2} - \frac{3}{2}}{x^3}$.
Suppose $g''(x)$ is continuous, then can use l'Hopital and get: $f''(0) = \lim_{x \to 0} \frac{g''(x)-3}{3x}$. Eventhough $\lim_{x \to 0} g''(x) = 3$, $\lim_{x \to 0} \frac{g''(x)-3}{3x}$ does not necessarily exist. (For example, for $g''(x) = 3 + \sqrt[3]{x}$.)
Thus, even if g is assumed to be twice differentiable, with $g''(x)$ continuous, f is in general not twice differentiable.
c. We start from the general statement, for any k: $\lim_{x \to 0} \frac{e^{-\frac{1}{x^2}}}{x^k} = 0$. (Present the limit in the form $\lim_{x \to 0} \frac{x^{-k}}{e^{\frac{1}{x^2}}}$ and apply l'Hopital several times.)
When computing $f^{(n)}(0)$ (for any n) one always gets the limit of the form: $\lim_{x \to 0} Pol(\frac{1}{x})e^{-\frac{1}{x^2}}$, where $Pol(\frac{1}{x})$ is a polynomial in the inverse powers of x. As observed above: this limit is zero, regardless of the degree of the polynomial.

7. All the statements are false.
A counterexample to v. is the function of 5.c. A counterexample to vi. is e.g. $\begin{cases} x^2 sin\frac{1}{x}, & x \neq 0 \\ 0, & x = 0 \end{cases}$.

Partial answers to some of the questions of HWK10

1. a. Suppose the minimum of $f(x)$ is achieved at $0 < x_{min} \leq \frac{1}{2}$. Write the Taylor expansion at that point:

$$0 = f(0) = f(x_{min}) + \underbrace{f'(x_{min})}_{=0}(0 - x_{min}) + f''(c)\frac{(x_{min} - 0)^2}{2} \qquad \text{for some } c \in (0, x_{min}) < \frac{1}{2}.$$

Thus $f''(c) = \frac{2}{(x_{min}-0)^2} \geq 8$. If $\frac{1}{2} \leq x_{min} < 1$ then consider $f(1) = f(x_{min}) + \underbrace{f'(x_{min})}_{=0}(1 - x_{min}) + f''(c)\frac{(x_{min}-1)^2}{2}$.

b. If the equation $f(x) = 0$ has n solutions, $x_1 < x_2 < \cdots < x_n$ then in each interval (x_i, x_{i+1}) there is a point of local min/max, i.e. $c_i \in (x_i, x_{i+1})$ such that $f'(c_i) = 0$. Thus the equation $f'(x) = 0$ has at least $(n-1)$ solutions. As $f'(x)$ is a polynomial of $degree = (n-1)$, the equation can have at most $(n-1)$ solutions. In total, one has: $f'(x)$ vanishes precisely $(n-1)$ times. Continue by induction.

d. If $f(\frac{1}{n}) = 0$ for $n \in \mathbb{N}$ then on each interval $(\frac{1}{n+1}, \frac{1}{n})$ the derivative vanishes. Thus $f'(x)$ has infinite number of zeros in any neighborhood of $x = 0$. By the assumption: f' is continuous, thus $f'(0) = \lim_{x \to 0} f'(x) = 0$. Now repeat the same procedure for $f'(x)$, then get: $f''(0) = \lim_{x \to 0} f''(x) = 0$. And so on. Thus: $f^{(n)}(0) = 0$ for any n. Thus the Taylor series of $f(x)$ at $x = 0$ is identically zero.

Finally, we claim that $f(x)$ is equal to its Taylor series. The sufficient condition for this equality is in terms of the remainder: $\lim_{n \to \infty} r_n(x) = 0$, where $r_n(x) = \frac{f^{(n)}(c)x^n}{n!}$. In our case: $|f^{(n)}(x)| \leq L$ thus $|r_n| \leq \frac{L|x^n|}{n!} \to 0$. So the function equals to its Taylor series.

Thus: $f(x) \equiv 0$.

2. a. Consider for example $f(x) = -x^2\chi(x)$, where $\chi(x)$ is the Dirichlet function. Alternatively: $f(x) = \begin{cases} -x^2 sin^2(\frac{1}{x}), & x \neq 0 \\ 0, & x = 0 \end{cases}$

b. Consider $f(x) = \begin{cases} sin\frac{1}{x}, & x \neq 0 \\ 10, & x = 0 \end{cases}$.

d. Suppose x_1, x_2 are the points of local maximum, then the continuous function $f(x)$ on $[x_1, x_2]$ must have a local minimum. As x_1, x_2 are loc.maxima, the local minimum is achieved in (x_1, x_2).

This does not hold for $f(x)$ non-continuous. e.g. $f(x) = \begin{cases} |x|, & 0 \neq |x| \leq 1 \\ 1, & x = 0 \end{cases}$.

e. Consider for example $f(x) = -x^4$.

f. Consider for example $f(x) = x^3$ in the interval $[-1, 1]$.

g. Consider for example $f(x) = \begin{cases} e^{sin\frac{1}{x} - \frac{1}{x}}, & x \in (0, 1] \\ 0, & x = 0 \end{cases}$. By the direct check: the function is differentiable on $[0, 1]$ and for $x > 0$: $f'(x) = \frac{1-cos\frac{1}{x}}{x^2}e^{sin\frac{1}{x} - \frac{1}{x}}$. Thus $f'(x) \geq 0$ on $[0, 1]$ and $f'(x) = 0$ at all the points $\frac{1}{x_n} = 2\pi n$.

3. i. Consider the function $f(x) = xln(x) - \frac{x^2+1}{2}$, we want to prove $f(x) \leq 0$. Note that $\lim_{x\to 0^+} f(x) < 0$ and $\lim_{x\to+\infty} f(x) < 0$. Thus it is enough to check the value of $f(x)$ at its maximal point.

For this we study $f'(x) = 1 + ln(x) - x$. We want to check when is $f'(x)$ positive/negative. Note that $\lim_{x\to 0^+} f'(x) < 0$ and $\lim_{x\to+\infty} f'(x) < 0$. Thus we check $f''(x) = \frac{1}{x} - 1$. As $f''(x) > 0$ for $x < 1$ and $f''(x) < 0$ for $x > 1$ we have: $f'(x)$ increases for $x < 1$ and decreases for $x > 1$. Therefore its maximum is at $x = 1$ and $f'(1) = 0$. Therefore $f'(x) \leq 0$ for $x > 0$. But then $f(x)$ is non-increasing on $x > 0$. And $\lim_{x\to 0^+} f(x) < 0$. Thus $f(x) < 0$ on $x > 0$.

(Note: using $ln(1+t) \leq t$ one could get immediately that $f'(x) \leq 0$ for $x > 0$.)

4. a. ii. $f'(x) = cos(cos(x))(-sin(x))$. Note that $|cos(x)| \leq 1 < \frac{\pi}{2}$. Thus the sign of $f'(x)$ is determined by $sin(x)$. $f' > 0$ for $x \in (-\pi + 2\pi n, 2\pi n)$ and $f' < 0$ for $x \in (2\pi n, \pi + 2\pi n)$. Thus $\{2\pi n\}$ are the local maxima, while $\{\pi + 2\pi n\}$ are the local minima.

iii. $f'(x) = 0$ means: $\frac{e^x+e^{-x}}{2} = cos(x)$. Note that $e^x + e^{-x} \geq 2$, with equality iff $x = 0$. (e.g. the inequality of averages) Thus there is only one point to check: $x = 0$. At this point: $f'(0) = 0 = f''(0)$, $f^{(3)}(0) = 2$. Thus this is a flex.

iv. The domain of definition of f is $(-1, 0) \cup (0, \infty)$, in this domain the function is infinitely differentiable. $f'(x) = \frac{ln(1+x)}{x^2}\Big(\frac{2x}{1+x} - ln(1+x)\Big)$. Thus $f'(x)$ vanishes only when the expression $g(x) = \frac{2x}{1+x} - ln(1+x)$ vanishes. So we study this new function. Its domain of definition is $(-1, \infty)$. And $g'(x) = \frac{2}{(1+x)^2} - \frac{1}{(1+x)}$. Thus $g(x)$ increases on $(-1, 1)$ and decreases on $(1, \infty)$. Thus $g(1) = 1 - ln(2) > 0$ is the global maximum. Thus $g(x)$ has one root on $(-1, 0)$ and one root on $(1, \infty)$. By the direct check: $g(0) = 0$, while the other root,$x_1 > 1$, is transcendental.

Altogether: $f'(x) > 0$ on $(-1, 0) \cup (0, x_1)$ and $f'(x) < 0$ on (x_1, ∞). Thus x_1 is a local maximum.

v. The function is continuous but not differentiable at $x = 0$, $x = 1$, these point should be checked separately. For $x > 1$ there are no critical points, for $x < 1$ there is an additional point: $x = -1$. $f''(-1) < 0$, thus $x = -1$ is a local maximum. Further, $f(x) \geq 0$, and $f(0) = 0$ thus $x = 0$ is the global minimum.

Finally, for $f \to \pm\infty$: $f(x) \to 0$. Thus it achieves the global maximum at some point. And $f(1) = 1$, while $f(-1) = e^{-2}$. Thus $x = 1$ is the global maximum.

vi. $f'(x) = 0$ for $x = 0$, $x = 1$ and $x = \frac{m}{m+n}$. If m is even then f' changes the sign at $x = 0$, thus this is an extremum. Similarly, if n is even then $x = 1$ is an extremum. Finally, the point $x = \frac{m}{m+n}$ is always an extremum as f' changes the sign.

vii. $f'(x) = cos(sin^6(x))6sin^5(x)cos(x)$. As $|sin(x)| \leq 1 < \frac{\pi}{2}$, the part $cos(sin^6(x))$ is always positive. Thus it is enough to study the expression $sin^5(x)cos(x)$, so the critical points of $f(x)$ are: $\{\frac{\pi n}{2}\}_{n\in\mathbb{Z}}$.

To classify these critical points one can check the sign of $sin^5(x)cos(x)$.

To classify via higher derivative(s) is cumbersome as $f''(x)$ and $f^{(3)}$ vanish at the points πn. Rather one can check the Taylor expansion. At the point $x = 0$: $f(x) = x^6 + 0(x^7)$, so this is a local minimum. By periodicity all the points $x = \pi n$ are local minima.

The points $\frac{\pi}{2} + \pi n$ are local maxima.

5. Let $(x, y = ax + b)$ be a point on the line, consider the square-of-distance: $f(x) = (x - x_0)^2 + (ax + b - y_0)^2$. We must find the minimum of this function. $f'(x) = 0$ gives $x = \frac{x_0+a(y_0-b)}{1+a^2}$. Substitute this into $f(x)$ to get the final answer.

6.b. We should prove: $f(g(\lambda_1 x_1 + \lambda_2 x_2)) \geq \lambda_1 f(g(x_1)) + \lambda_2 f(g(x_2))$.

Indeed, as $g(x)$ is convex up: $g(\lambda_1 x_1 + \lambda_2 x_2) \geq \lambda_1 g(x_1) + \lambda_2 g(x_2)$. As f decreases we have the needed inequality.

6.c.i. By the assumptions f is convex down, thus

$$f(\lambda a + (1-\lambda)b) \leq \lambda f(a) + (1-\lambda)f(b) \leq \lambda \cdot max(f(a), f(b)) + (1-\lambda) \cdot max(f(a), f(b)) = max(f(a), f(b)).$$

7. a. v. Present the function in the form $f(x) = x + ln\Big((1 + \frac{1}{ex})^x\Big)$. Thus, for $x \to \pm\infty$, $f(x) \sim x + \frac{1}{e}$.

Further, for $x \to 0$: $f(x) \to 0$, while for $x \to (-\frac{1}{e})^-$: $f(x) \to \infty$.

b. If $f(x)$ has a slant asymptote for $x \to \infty$ then $f(x) - ax - b \to 0$ (for some appropriate (a, b)). Thus the function $f(x) - ax - b$ is continuous on $[a, \infty)$ and with a finite limit at $x \to \infty$. So it is uniformly continuous. Thus $f(x)$ is uniformly continuous.

c. As $f(x)$ has a slant asymptote, we get that $p(x)$ is of degree at most 2. Further, $f(0) = 0$ gives $p(x) = xg(x)$, where the polynomial $g(x)$ is of degree at most one.

Finally, the condition "$x = -2$ is an extremum of $f(x)$" forces: $f(x) = \frac{ax^2}{x+1}$, while the condition $f(-2) = 4$ forces $f(x) = -\frac{x^2}{x+1}$.

d. Consider $f(x) = \frac{sin(e^x)}{1+x^2}$.

8. b. The function is defined on $[0, 1]$ is continuous, differentiable on $(0, 1)$ and decreases everywhere. For $a = 1$ this is a line, for $a > 1$ the function is convex up, for $0 < a < 1$ the function is convex down.

To check the limit of the curve $|x|^a + |y|^a = 1$ as $a \to \infty$, note that $\lim_{a\to\infty} \sqrt[a]{|x|^a + |y|^a} = max(|x|, |y|)$.

To check the limit of the curve $|x|^a + |y|^a = 1$ as $a \to 0^+$, note that $\lim\limits_{a\to 0^+} |x|^a = \begin{cases} 1, & x \neq 0 \\ 0, & x = 0 \end{cases}$.

Partial answers to some of the questions of HWK11

1. b. iv. Use $x = tan(\alpha)$. Alternatively, one can use $\int \frac{1}{(1+x^2)}dx = \frac{x}{(1+x^2)} + \int \frac{2x^2}{(1+x^2)^2}dx$.

v. $\int \frac{e^x(1-x)}{x^2}dx = -\frac{e^x(1-x)}{x} + \int \frac{-e^x x}{x}dx$

vi. $\int e^{ax} sin(bx)dx = \frac{e^{ax}sin(bx)}{a} - \int \frac{e^{ax}b\cdot cos(bx)}{a}dx = \frac{e^{ax}sin(bx)}{a} - \frac{e^{ax}b\cdot cos(bx)}{a^2} - \int \frac{e^{ax}b^2\cdot sin(bx)}{a^2}dx$.

Thus $\int e^{ax}sin(bx)dx = \frac{ae^{ax}sin(bx) - be^{ax}cos(bx)}{a^2+b^2}$.

2. $(2n-1)J_n + \frac{x}{(x^2+1)^n} = 2na^2 J_{n+1}$.

Partial answers to some of the questions of HWK12

1. a. i. The function is continuous for $x \neq 0$, thus it is integrable for any interval $[a, b]$ that does not contain 0. To check integrability near $x = 0$, split the Darboux sums according to $\int_0^a = \int_\epsilon^a + \int_0^\epsilon$. As $f(x)$ is bounded, $|\int_0^\epsilon f(x)dx| \leq \epsilon$. Thus the limits of the lower and upper Darboux sums coincide.

Alternatively, as has been proved in the class: a bounded function with at most a finite number of discontinuities is integrable.

ii. iii. The function is continuous in its domain of definition, thus it is integrable.

iv. Split Darboux sums according to $\int_0^a = \int_\epsilon^a + \int_0^\epsilon$. On $[\epsilon, a]$ the function has at most finite number of discontinuities (and is bounded), thus it is integrable. Further, $|\int_0^\epsilon f(x)dx| \leq \epsilon$. Thus the limits of the lower and upper Darboux sums coincide.

b. For any partition the lower Darboux sum is 0, the upper is 1.

c. Let S_f, s_f be the upper/lower Darboux sums for a given partition. By the triangle inequality: $|S_f - s_f| \geq |S_{|f|} - s_{|f|}|$. Thus from $|S_f - s_f| \to 0$ we get $|S_{|f|} - s_{|f|}| \to 0$. Thus $|f|$ is integrable. Further, $S_{|f|} \geq |S_f|$.

2. vi. $\int\limits_{\frac{1}{e}}^{e} |ln(x)|dx = \int\limits_{1}^{e} ln(x)dx - \int\limits_{\frac{1}{e}}^{1} ln(x)dx$

vii. Note that $\int\limits_0^{\frac{\pi}{2}} \frac{cos^5(x)dx}{cos^5(x)+sin^5(x)} \overset{t=\frac{\pi}{2}-x}{=} \int\limits_0^{\frac{\pi}{2}} \frac{sin^5(t)dt}{cos^5(t)+sin^5(t)}$. Thus $\int\limits_0^{\frac{\pi}{2}} \frac{cos^5(x)dx}{cos^5(x)+sin^5(x)} = \frac{1}{2}\int\limits_0^{\frac{\pi}{2}} \frac{cos^5(x)+sin^5(x)}{cos^5(x)+sin^5(x)}dx = \frac{\pi}{4}$.

viii. Integrate by parts twice.

x. Use the factorization $x^4 - 1 = (x-1)(x+1)(x^2+1)$, then split the fraction: $\frac{1}{x^4-1} = \frac{1}{2}(\frac{\frac{1}{x-1} - \frac{1}{x+1}}{2} - \frac{1}{x^2+1})$.

3.c. Present in the form $\int\limits_0^{\frac{\pi}{2}} sin(x)sin^{m-1}(x)dx$ and integrate by parts.

4. a. Let S_f, S_g, s_f, s_g be the upper/lower Darboux sums for some partition. Then $\lim(S_{f+g} - s_{f+g}) = lim(S_f - s_f) + lim(S_g - s_g) = 0$. Thus $f + g$ is integrable.

Integrability of $f^2(x)$. As f is bounded, can assume $|f(x)| \leq C$. Therefore:

$$S_{f^2(x)} - s_{f^2(x)} = \sum_i (x_{i+1} - x_i)\Big(sup_i(f^2(x)) - inf_i(f^2(x))\Big)$$

Note that $sup_i(f^2(x)) - inf_i(f^2(x)) \leq C(sup_i(f(x)) - inf_i(f(x)))$, thus for any partition $0 \leq S_{f^2(x)} - s_{f^2(x)} \leq C(S_{f(x)} - s_{f(x)})$. Thus $lim(S_{f^2(x)} - s_{f^2(x)}) = 0$.

Integrability of $f(x)g(x)$ follows now from the presentation $f(x)g(x) = \frac{(f+g)^2-(f-g)^2}{4}$.

b. If $f(x)$ is integrable then $|f(x)|$ is integrable as well. (By the triangle inequality: $S_{|f|} - s_{|f|} \leq S_f - s_f$.)

The converse does not hold, e.g. consider $f(x) = \begin{cases} 1, & x \in \mathbb{Q} \\ -1, & x \notin \mathbb{Q} \end{cases}$

c.d.e. Consider $f(x) = \begin{cases} 10^{100}, & x = \frac{1}{n},\ n \in \mathbb{N} \\ 0, & x \neq \frac{1}{n} \end{cases}$

f. Suppose $f(x_0) = c > 0$, then by continuity $f(x) > \frac{c}{2}$ for $x \in (x_0 - \epsilon, x_0 + \epsilon)$ (for some small enough ϵ). Thus $\int\limits_a^b f(x)dx > c\epsilon$.

g. It is enough to prove for the case: $f(x) = g(x)$ for $x \in [a, b] \setminus \{x_0\}$. In this case $g(x)$ is integrable and $\int\limits_a^b (f(x) - g(x))dx = \int\limits_a^{x_0} (f(x) - g(x))dx + \int\limits_{x_0}^b (f(x) - g(x))dx$. By the direct check of Darboux sums, both integrals vanish.

h. The proof in both cases is by the change of variable, $x \to -x$.

5. a. In all the cases the limits are the Darboux sums. Thus i. $lim(...) = \int_0^1 ln(1+x)dx = \cdots$.
ii. $lim(..) = \int_0^1 x^2 arctan(x)dx = \cdots$. iii. $liml(..) = \int_0^2 x^\alpha dx = \cdots$

b. In all the cases apply l'Hôpital's rule. e.g.
iii. $\lim(...) = \lim\limits_{x\to 0} \frac{e^{sin^2(x)}cos(x) - e^{sin^2(sin(x))}cos(sin(x))cos(x)}{2x} = \lim\limits_{x\to 0} \frac{e^{sin^2(x)}cos(x)}{2} \frac{1-e^{sin^2(sin(x))-sin^2(x)}cos(sin(x))}{x} = 0$.

iv. $lim(..) = \lim\limits_{x\to 0} \frac{tan(x)sin(\alpha tan(x))\frac{1}{cos^2(x)}}{1-cos(x)} = 2a.$

v. $lim(...) = \lim\limits_{x\to 0} \frac{f(x)g(x)}{f(x)} = g(0).$

Partial answers to some of the questions of HWK14

1. Here all the functions have constant signs, thus one can use the comparison criterion.

iiii. $\int_0^\infty \frac{xdx}{\sqrt{x^4+1}} \geq \int_0^\infty \frac{dx}{10x} = \infty$

iv. $\int_1^\infty e^{-x^2}dx \leq \int_1^\infty e^{-x}dx < \infty$

v. Use: $e^{-\sqrt{x}} < \frac{1}{x^2}$ for $x >> 1$.

vi. There are two problematic points: $x=0$ and $x=1$. At $x=0$:

$$\int_0^\epsilon \frac{x^{a-1}}{100}dx < \int_0^\epsilon x^{a-1}(1-x)^{b-1}dx < \int_0^\epsilon x^{a-1}dx$$

thus at $x=0$ the integral converges iff $a>0$. Similarly, at $x=1$ the integral converges iff $b>0$.

viii. The integral diverges at $x=1$, there $\frac{1}{x^2-1} \sim \frac{1}{x-1}$.

x. At $x=0$ the integral converges iff $\alpha > -1$. At ∞ the integral converges iff $\beta - \alpha > 1$.

xi. At $x=0$ the integral converges iff $\alpha > -1$. For $x=1$ use: $1 - cos(x-1) = \frac{sin^2(x-1)}{1+cos(x-1)}$. Thus at $x=1$ the integral converges iff $2\beta > -1$.

xii. Near $x=0$ present: $ln(sin(x)) = ln(x) + ln(\frac{sin(x)}{x})$. Note that $ln(\frac{sin(x)}{x}) \to 0$, thus the integral $\int_0^\epsilon \frac{ln(\frac{sin(x)}{x})}{\sqrt{x}}dx$ converges. (e.g. use $\int_0^\epsilon |\frac{ln(\frac{sin(x)}{x})}{\sqrt{x}}|dx < \int_0^\epsilon \frac{10}{\sqrt{x}}dx$).

It remains to check the convergence of $\int_0^\epsilon \frac{ln(x)}{\sqrt{x}}dx$. Integrate by parts: $\int_0^\epsilon \frac{ln(x)}{\sqrt{x}}dx = \lim\limits_{a\to 0} \frac{xln(x)}{\sqrt{x}}\Big|_a^\epsilon - \int_0^\epsilon \frac{dx}{\sqrt{x}}$. Now in the right hand side both terms converge to the finite limits.

xiv. Draw the graph of the function, it is a sequence of triangles, of height 1 (the length of basis is $\frac{2}{2^n}$). The area of such a triangle is $\frac{1}{2^n}$. Therefore $\int_2^\infty f(x)dx < \sum\limits_{n\geq 1} \frac{1}{2^n} = 1$.

2. a. i. At $x=0$: the integral converges iff $\alpha < 2$. (If $\alpha < 2$ then the convergence is absolute, if $\alpha \geq 2$ then the integral diverges.)

For $x\to\infty$, integrate by parts: $\int_1^\infty \frac{sin(x)}{x^\alpha}dx = \frac{-cos(x)}{x^\alpha}|_1^\infty - \alpha\int_1^\infty \frac{cos(x)}{x^{\alpha+1}}dx$. Thus, if $\alpha > 0$ then the integral converges.

As was shown in the class, the integral converges absolutely only for $\alpha > 1$. Thus, for $0 < \alpha \leq 1$ the convergence is conditional.

ii. The integral converges but not absolutely (conditionally), integrate by parts.

iii. Integrate by parts to get that the integral converges. The convergence is conditional (not absolute):

$$\int_1^\infty |cos(x^2)|dx > \int_1^\infty cos^2(x^2)dx > \int_1^\infty \frac{cos^2(x^2)+sin^2(x^2)}{100}dx = \infty.$$

iv. At $x=0$ the integral converges iff $\alpha > -1$, the convergence is absolute.

For $x\to\infty$, change the variable $e^x = t$, then integrate by parts. One gets that $\int_1^\infty x^\alpha cos(e^x)dx$ converges for any α. (Though for $\alpha \geq -1$ the convergence is condition.)

In total $\int_0^\infty x^\alpha cos(e^x)dx$ converges conditionally for $\alpha > -1$ the integral.

v. Compare to $\frac{1}{x}$.

4. a. Consider e.g. $\int_1^\infty x^\alpha cos(e^x)dx$ (from question 2).

b. Consider $f(x) = \begin{cases} n\cdot 2^n(x-n+\frac{1}{2^n}), & x\in[n-\frac{1}{2^n},n] \\ n\cdot 2^n(n+\frac{1}{2^n}-x), & x\in[n,n+\frac{1}{2^n}], \\ 0 \end{cases}$, cf. question 1.xiv.

c. Consider $f\geq 0$ which is "mostly" zero but have narrow bumps at the integer points. An example of such a function is given in question 7. If the width of the bumps goes to zero fast enough then $\int^{\infty} f(x)dx<\infty$. But $f(x)\not\to 0$ as $x\to\infty$.

d. Suppose $\lim\limits_{x\to\infty} f(x)\neq 0$. Then there exists a number $\epsilon>0$ and a sequence of points $x_n\to\infty$ such that $f(x_n)>\epsilon$. As $f(x)$ is uniformly continuous, there exists $\delta>0$ such that on each interval $(x_n-\delta,x_n+\delta)$: $f(x)>\frac{\epsilon}{2}$. Then, for each such interval: $\int\limits_{x_n-\delta}^{x_n+\delta} f(x)dx>2\delta\frac{\epsilon}{2}$. This contradicts the condition $\lim\limits_{\substack{N_1\to\infty\\N_2\to\infty}}\int\limits_{N_1}^{N_2} f(x)dx=0$.

e. Consider $f(x)=g(x)=\frac{sin(x)}{\sqrt{x}}$. As is shown in question 2: $\int^{\infty}\frac{sin(x)}{\sqrt{x}}dx<\infty$, but $\int^{\infty}\frac{sin^2(x)}{x}dx=\infty$.

5. As $\lim\limits_{t\to\infty} t^2f(t)=1$, one has for $t\gg 1$: $|f(t)|<\frac{100}{t^2}$. Thus $\int^{\infty} f(t)dt$ converges, even absolutely.

Suppose $a>0$, then $\lim\limits_{n\to\infty}\int\limits_a^{\infty} f(nx)dx=\lim\limits_{n\to\infty}\frac{\int\limits_{na}^{\infty} f(t)dt}{n}=0$.

Finally: $\lim\limits_{n\to\infty}\int\limits_a^{\infty} f(nx)dx=0$.

6. Present the equation in the form $y^2=\frac{8}{x}-4$. We need to compute the area of the domain: $0<x\leq 2$, $0\leq y\leq\sqrt{\frac{8}{x}-4}$. The area is: $\int\limits_0^2\sqrt{\frac{8}{x}-4}dx$. First we check the convergence of this integral at $x=0$. Note that $\int\limits_0^2\sqrt{\frac{8}{x}-4}dx<10\int\limits_0^2\frac{dx}{\sqrt{x}}<\infty$, so the integral converges.

To compute the integral use the substitution: $t^2=\frac{8}{x}-4$.

7. The function has a vertical asymptote at each $x\in\mathbb{N}$. It is enough to check the convergence of the integral on each interval $(n-\epsilon,n+\epsilon)$. Note that $\lim\limits_{x\to n}\frac{sin(\pi x)}{x-n}=1$. Thus $\int\limits_{n-\epsilon}^{n+\epsilon}\frac{dx}{\sqrt{|sin(\pi x)|}(1+e^x)}<\int\limits_{n-\epsilon}^{n+\epsilon}\frac{100dx}{\sqrt{|x-n|}}<\infty$.

In fact the whole integral $\int\limits_1^{\infty}\frac{dx}{\sqrt{|sin(\pi x)|}(1+e^x)}$ converges absolutely, by the slightly better comparison: $\int\limits_{n-\epsilon}^{n+\epsilon}\frac{dx}{\sqrt{|sin(\pi x)|}(1+e^x)}<\frac{10^{10}}{1+e^n}\int\limits_{n-\epsilon}^{n+\epsilon}\frac{dx}{\sqrt{|x-n|}}$. Thus

$$\int\limits_1^{\infty}\frac{dx}{\sqrt{|sin(\pi x)|}(1+e^x)}<10^{10}\int\limits_1^{\infty}\frac{dx}{1+e^x}<10^{10}\int\limits_1^{\infty}e^{-x}dx<\infty.$$

8. i. $f(x)$ is differentiable for $x\in\mathbb{R}$. In particular it is uniformly continuous on any finite interval. Note that $\lim\limits_{x\to-\infty} f(x)=\int\limits_0^0(..)dt=0$. Moreover, $\lim\limits_{x\to\infty} f(x)=\int\limits_0^{\infty}(..)dt<\infty$. Thus the function has two horizontal asymptotes.

In particular, as both limits are finite, $f(x)$ is uniformly continuous in $\mathbb{R}$.

$f'(x)=sin(e^x)$. Thus $f(x)$ increases for $e^x\in(2\pi n,2\pi n+\pi)$ and decreases for $e^x\in(2\pi n-\pi,2\pi n)$. The points $\{ln(2\pi n)\}_{n\in\mathbb{N}}$ are local minima, the points $\{ln(2\pi n-\pi)\}_{n\in\mathbb{N}}$ are local maxima.

$f''(x)=e^xcos(e^x)$. Thus $f(x)$ is convex down for $e^x\in(-\frac{\pi}{2}+2\pi n,\frac{\pi}{2}+2\pi n)$ and convex up for $e^x\in(\frac{\pi}{2}+2\pi n,\frac{3\pi}{2}+2\pi n)$. The points $\{ln(\frac{\pi}{2}+\pi n)\}_{n\in\mathbb{N}}$ are inflection points.

Similarly for ii. and iii.