Sketchy solutions of Midterm, Hedva1.EE, 201.1.981

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(1) By the direct check: $y_n \geq x_n$. Then, from the definition of y_n one has: $y_n \geq y_{n+1} \geq x_n$. From the definition of x_n one has: $x_n \leq x_{n+1} \leq y_n$. Therefore:
• the sequence x_n is non-decreasing and bounded from above (e.g. by y_1),
• the sequence y_n is non-increasing and bounded from below (e.g. by $x_1 > 0$).
Thus there exist the finite limits: $x_n \to x$ and $y_n \to y$.
These limits satisfy: $x = \frac{2}{\frac{1}{x}+\frac{1}{y}}$ and $y = \sqrt{xy}$. From this one gets: $x = y$.

(2) Fix any $\epsilon > 0$. We claim that there exist N such that for any $n > N$: at least one of the bounds $|a_n - b_1| < \frac{\epsilon}{4^{k+1}}$, ..., $|a_n - b_k| < \frac{\epsilon}{4^{k+1}}$ is satisfied.
(proof: Otherwise one can construct a subsequence a_{n_j} that does not approach any of $b_1, \ldots, b_k$. As a_{n_j} is bounded, it has a finite partial limit. Which is not one of $b_1, \ldots, b_k$. Contradiction.)

Therefore for $n > N$ holds: all the factors $|a_n|$, $|a_n - b_1|, \ldots$, $|a_n - b_k|$ are bounded (e.g. by 4) and at least one of them is smaller than $\frac{\epsilon}{4^{k+1}}$. Therefore $|a_n \prod_{i=1}^{k}(a_n - b_i)| \leq \epsilon$. Thus $\lim_{n\to\infty} a_n \prod_{i=1}^{k}(a_n - b_i) = 0$.

(3) Draw the graph of $1 - \frac{1}{x}$. Draw the graph of $sin(x)$. We are looking for the intersection points of the two graphs. Note that for $x > 1$: $0 < 1 - \frac{1}{x} < 1$. Note that $sin(\pi n) = 0$, while $sin(\frac{\pi}{2} + 2\pi n) = 1$. Look at the graph.

(a) By the mean value theorem there is a solution in each interval $(2\pi n, \frac{\pi}{2} + 2\pi n)$ and in each interval $(\frac{\pi}{2} + 2\pi n, \pi + 2\pi n)$, $n \geq 1$. Note that there are no solutions in the intervals $(2\pi n + \pi, 2\pi n + 2\pi)$, as $sin(x) < 0$ there.

(b) By part (a) there is an infinite sequence of solutions, $x_n \to \infty$. Thus $sin(x_n) = 1 - \frac{1}{x_n} \to 1$. By the continuity of $sin(x)$ and $arcsin(x)$: these solutions approach the points $\{\frac{\pi}{2} + 2\pi k\}_{k\in\mathbb{N}}$. Therefore the differences $x_{n+1} - x_n$ tend either to 0 or to 2π (depending on whether n is odd or even). Thus $sin(\frac{x_{n+1}-x_n}{2}) \to 0$.

(4) The initial domain of definition of f is: $x \neq 0$ and $x \neq \frac{\pi}{2} + \pi n$. We would like to extend $f(x)$ in a continuous way to the biggest possible domain.

We check $\lim_{x\to 0} f(x)$. Note that $\lim_{x\to 0} x \cdot ln(1 + \frac{1}{x^4}) = 0$. (Indeed: $\lim_{x\to 0}(\frac{1}{|x|^4})^x = (\lim_{x\to 0}\frac{1}{(|x|)^x})^4 = 1$, as follows e.g. from $\sqrt[n]{n} \to 1$.) Therefore $\lim_{x\to 0} f(x) = 1$.

Thus one can extend the definition of f to $x = 0$ and get a continuous function.

We check $\lim_{x\to\frac{\pi}{2}+\pi n} f(x)$. It is enough to check the part $|sin(x)|^{\frac{1}{|cos(x)|}}$. By periodicity it is enough to check $\lim_{x\to\frac{\pi}{2}} |sin(x)|^{\frac{1}{|cos(x)|}}$. Present:

$$|sin(x)|^{\frac{1}{|cos(x)|}} = \Big|1 - (1 - sin(x))\Big|^{\frac{1}{|cos(x)|}} = \Big|1 - \frac{cos^2(x)}{1 + sin(x)}\Big|^{\frac{1}{|cos(x)|}} = \Big(\Big|1 - \frac{cos^2(x)}{1 + sin(x)}\Big|^{\frac{1+sin(x)}{cos^2(x)}}\Big)^{\frac{|cos(x)|}{1+sin(x)}}.$$

Thus $\lim_{x\to\frac{\pi}{2}} |sin(x)|^{\frac{1}{|cos(x)|}} = 1$.

Thus one can extend the definition of f to the points $\frac{\pi}{2} + \pi n$, by $f(\frac{\pi}{2} + \pi n) = 1 + (\frac{\pi}{2} + \pi n) ln(1 + \frac{1}{(\frac{\pi}{2} + \pi n)^4})$, to get a continuous function.

In total: one can extend f to a continuous function on the whole $\mathbb{R}$, therefore f is uniformly continuous on any finite interval (by Cantor's theorem).

It remans to check the behavior of f at infinity. Note that $|sin(x)|^{\frac{1}{|cos(x)|}}$ is periodic and uniformly continuous on any finite interval. Thus it is uniformly continuous on the whole $\mathbb{R}$.

Note that $\lim_{x\to\pm\infty} x \cdot ln(1 + \frac{1}{x^4}) = 0$. (For example, $\lim_{x\to\pm\infty}(1 + \frac{1}{x^4})^x = 1$.) As has been proved in the class: if a function is continuous on $\mathbb{R}$ and its limits as $x \to \pm\infty$ are finite, then f is uniformly continuous on the whole $\mathbb{R}$. Therefore $x \cdot ln(1 + \frac{1}{x^4})$ is uniformly continuous on the whole $\mathbb{R}$.

In total: f is uniformly continuous on the whole $\mathbb{R}$.