Calculus2.ME, BGU, Spring 2015.
Some partial answers/hints to the homeworks

1. Hwk.1.5

(1) a. Diverges (both series diverge). b. Converges (bring to the common denominator).
c. Diverges. (Note that $\sum \frac{1}{ln^n(n)}$ converges. The convergence $\sum(\frac{1}{(n+10)ln(n)} - \frac{1}{ln^n(n)})$ would imply the convergence of $\sum(\frac{1}{(n+10)ln(n)}$. Thus the $\sum(\frac{1}{(n+10)ln(n)} - \frac{1}{ln^n(n)})$ diverges.)
d. Diverges for $s < 1$ (any t) and for $s = 1$, $t \leq 1$. Converges for $s > 1$ or $s = 1$, $t > 1$. (Integral comparison criterion).
e. Diverges. f. Converges (criterion of d'Alambert). g. Diverges.
h. Converges absolutely (comparison to $\sum \frac{1}{n^2}$).

(2) The domains of convergence are:
a. $|x| \leq 1$. b. $x \geq -\frac{3}{2}$. c. $x > -\frac{5}{4}$. d. $-\frac{1}{e} \leq x < \frac{1}{e}$. e. $-\frac{1}{2} \leq sin(x) < \frac{1}{2}$.
f. $|x+1| \leq 1$. g. $-1 \leq x < 1$. (The divergence at $x = 1$ occurs because $\sum\limits_{k=1}^{n} \frac{1}{k} < n$.)
h. $-1 \leq x < 1$. (Let $a_n = \frac{1 \cdot 4 \cdots (3n+1)}{2 \cdot 5 \cdots (3n+2)}$. To prove the divergence at $x = 1$ we present a_n in the form $a_n = \frac{4 \cdot 7 \cdots (3n+1)}{2 \cdot 5 \cdots (3n-1)} \frac{1}{(3n+2)} > \frac{1}{(3n+2)}$. Thus, for $x = 1$ the series diverges by comparison with $\sum \frac{1}{n}$. To check the convergence at $x = -1$ we use Leibnitz's criterion, for this we should prove that $a_n \to 0$. Alternatively, we should prove: $ln(a_n) \to -\infty$. But $ln(a_n) = -\sum\limits_{k=1}^{n} ln(1 + \frac{1}{3k+1})$. And $\sum\limits_{k=0}^{\infty} ln(1 + \frac{1}{3k+1}) = \infty$ by comparison to $\sum \frac{1}{n}$. Thus $a_n \to 0$ and we use Leibnitz criterion for $x = -1$.)
i. The series converges for any x.

(3) a. $f(x) = -ln(3-x)(2-x) = -ln(6) - ln(1 - \frac{x}{3}) - ln(1 - \frac{x}{2}) = -ln(6) + \sum\limits_{n \geq 1} \frac{(\frac{x}{3})^n + (\frac{x}{2})^n}{n}$. (The series converges for $-2 \leq x < 2$)
b. $f(x) = \frac{1 - cos(2x)}{2} = \cdots$.
c. $f(x) = \frac{1}{(1-x)^3} + \frac{x}{(1-x)^3}$. Now use the series for $\frac{1}{(1-x)^3}$, which can be obtained by the differentiation of the series for $\frac{1}{(1-x)}$.
d. Rewrite $f(x) = \frac{sin(2) - sin(2x)}{2}$. Now use the series for $sin(x)$.
e. First obtain the series for $arctan(x)$, using the series for $(arctan(x))' = \frac{1}{1+x^2}$. Then substitute x^3.
h. Rewrite $f(x) = x \cdot cos(x) - sin(x)$. Now use the series for $sin(x)$ $cos(x)$.

(4) a. A counterexample: $\sum \frac{(-1)^n}{n}$.
b. The proof goes via the main Cauchy theorem (as explained in the lectures).
c. A counterexample: $\sum \frac{(-1)^n}{ln(n)}$.

2. hwk5.3

5. iv. The limit does not exist, even the limit along $y = 0$ does not exist.

7.i. and ii. f is continuous at $(0, 0)$.

8. ii. f can be extended continuously to $(0, 0)$ by 0. iii. f can be extended continuously to $(0, 0)$ by 0. iv. f cannot be extended continuously to any point where $y = 0$. vi. f can be extended continuously to $\mathbb{R}^2$.

9. If one approaches the point $(1, 2)$ along the line $\{y = 2,\ x < 1\}$ then $f = 0$.

Note that when approaching any point of the boundary $\{x^2 + y^2 = 5,\ y > 2\}$ $f \to -\infty$. Therefore, if one goes to the point $(1, 2)$ along a path that approaches $\{x^2 + y^2 = 5,\ y > 2\}$ much faster than it approaches $(1, 2)$, then $f \not\to 0$. To construct an explicit example of such a path, consider: $y(x) = \sqrt{5 - x^2 - h(x)}$, where $\lim\limits_{x \to 1^-} h(x) = 0$. Then $f(x, y(x)) = ln(h(x))\sqrt{1 - x^2 - h(x)}$. We want $h(x)$ to vanish so fast that f will not tend to 0. For example, let $h(x) = e^{-\frac{1}{(1-x)^2}}$. Then $f(x, y(x)) = \frac{\sqrt{1 - x^2 - h(x)}}{(1-x)^2} \to \infty$. Therefore $\lim\limits_{(x,y) \to (1,2)} f(x, y)$ does not exist.

3. hwk6.0

1. a. i. f is not continuous at $(0, 0)$. ii. f is continuous at $(0, 0)$, $\partial_x f_{(0,0)} = 0 = \partial_y f_{(0,0)}$. f is not differentiable. iii. f is not continuous at $(0, 0)$. iv. f is continuous at $(0, 0)$. $\partial_y f_{(0,0)} = 0$, $\partial_x f_{(0,0)} = 1$. f is not differentiable v. $\partial_x f_{(0,0)} = 0 = \partial_y f_{(0,0)}$ and f is differentiable at $(0, 0)$. The derivatives are not continuous. vi. f is continuous at $(0, 0)$. $\partial_y f_{(0,0)} = \frac{1}{ln(3)}$, $\partial_x f_{(0,0)} = \frac{1}{\sqrt{2}}$. f is not differentiable.

vii. $\partial_x f$ does not exists at the points $(0, y)$ for $y \neq 0$. $\partial_y f$ does not exists at the points $(x, 0)$ for $x \neq 0$. $\partial_x f|_{(0,0)} = 0 = \partial_y f|_{(0,0)}$. f is not differentiable.

b. f is continuous at $(0, 0)$ iff $\alpha > 0$. f is differentiable at $(0, 0)$ iff $\alpha > \frac{1}{2}$.

c. By checking $\lim_{x \to y} f(x,y)$ one gets: $g(x) = \frac{1}{x}$, to ensure the continuity of f. Then, for $x \neq 0$: $\partial_x f|_{x,x} = -\frac{1}{2x^2} = \partial_y f|_{x,x}$.

d. The continuity of f forces: $g(x,y) = 2cos(x+y)$. Then $\partial_x f|_{z=0} = -2sin(x+y) = \partial_y f|_{z=0}$, $\partial_z f|_{z=0} = 0$. (To see this one can present f near $z=0$ in the form: $2cos(x+y)(1 - \frac{z^2}{3!} + \frac{z^4}{5!} - \cdots))$. The function is differentiable in $\mathbb{R}^3$.

e. i. f is continuous at $(0,0)$. $\partial_x f|_{(0,0)} = 0 = \partial_x y|_{(0,0)}$. f is not differentiable.

ii. $\partial_x f|_{(0,0)} = 0 = \partial_x y|_{(0,0)}$ f is differentiable at $(0,0)$.