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SINGULAR ELEMENTS OF SEMI-SIMPLE ALGEBRAIC GROUPS

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Four years ago Steinberg lectured in Moscow on classes of elements of semisimple algebraic groups [12]. In a very modest sense my talk may be viewed as a continuation of Steinbergs lecture. However, I shall not try to give a report on all the problems posed by Steinberg. Instead of that I shall concentrate on some recent results concerning one of Steinbergs problems. The problem is : Study the variety of unipotent elements thoroughly.

Let G be a semisimple algebraic group over an algebraically closed field $K = \overline{K}$. We want to study the conjugacy classes of elements $x \in G$. This problem can be decomposed into two parts, corresponding to the Jordan decomposition $x = x_s \cdot x_u$ of x into its semisimple and unipotent parts. The conjugacy classes of semisimple elements are obviously classified by T/W, where T is a maximal torus and W is the Weyl group. Thus, associating to x the conjugacy class $\overline{x}_s$ of x_s, one obtains a morphism $G \to T/W$, the fibres of which are unions of conjugacy classes of G.

Steinberg, Springer and Kostant studied the classes of elements of the most general type, that is, the regular elements ([11]. [9], [7]).

DEFINITION. — x is regular if and only if the dimension of its centralizer $Z_G(x)$ is minimal, i.e. equals the rank r of G.

Steinberg and Kostant have obtained various characterizations of regular elements, for instance the following ones.

THEOREM. —

(i) *x is regular if and only if x is contained only in finitely many Borel groups.*

(ii) *x is regular if and only if $G \to T/W$ is regular,* i.e. *smooth, at x.*

A good deal is known about the regular elements. For instance, they form an open dense subset in G whose complement is an algebraic set of codimension 3, and each fibre of $G \to T/W$ contains exactly one regular class. For a singular x one has $\dim Z_G(x) \geqslant r + 2$.

DEFINITION. — x is subregular if and only if $\dim Z_G(x) = r + 2$.

It was Grothendieck who recommended to study the subregular elements. In fact, he conjectured most of what is going to follow after reading a paper of mine on a mysterious connection between Weyl groups and rational singularities [2].

I am going to explain two characterizations of subregular elements, which correspond to the two characterizations of regular elements given by Steinberg. First I shall talk about the one using Borel groups.

Let D be the projective variety of all Borel groups of G, and let Y be the subvariety of $G \times D$ consisting of all pairs (x, B) such that $x \in B$. One has natural morphisms $Y \to G$ and $Y \to T$ such that the diagram formed by them, $G \to T/W$ and $T \to T/W$ commutes.

The following theorem was proved by Grothendieck and, as far as the unipotent fibre is concerned, already by Springer [10].

THEOREM. – *The following diagram is a simultaneous resolution of the singularities of the fibres of $G \to T/W$:*

$$\begin{array}{ccc} Y & \to & G \\ \downarrow & & \downarrow \\ T & \to & T/W \end{array}$$

The term "resolution" is explained by the following definition.

DEFINITION. – A resolution of $X \to S$ is a commutative diagram

$$\begin{array}{ccc} Y & \to & X \\ \downarrow & & \downarrow \\ T & \to & S \end{array}$$

where $Y \to X$ is proper and surjective,
$T \to S$ is finite and surjective,
$Y \to T$ is regular,
$Y_t \to X_{s(t)}$ is a resolution of singularities for all fibres Y_t, $t \in T$.

In general, a morphism does not admit a resolution. It is a very particular property of $G \to T/W$ and the singularities of its fibres that it has a resolution.

In order to study the singularity of a fibre X_s at a point $x \in X_s$, we consider the reduced exceptional fibre F_x over x in the resolution $Y_t \to X_s$. For unipotent x by construction $F_x = \{B \in D \mid x \in B\}$. The regular x are those with F_x a point, by Steinbergs theorem. We shall now describe the F_x for $x \in G$ subregular unipotent and G simple. – It is easy to reduce the consideration of the general situation to this case.

Choose a Borel group B_0, let Δ be the corresponding system of simple positive roots, and P_a for $a \in \Delta$ the parabolic group generated by B_0 and U_{-a}. The fibres of $G/B_0 \to G/P_a$ are projective lines in D called lines of type a. Let (n_{ab}) be the Cartan matrix, and $n'_{ab} = -n_{ab}$ if $-n_{ab} \neq \text{char } K$, and $n'_{ab} = 1$ otherwise.

DEFINITION. – A Dynkin curve is a connected curve in D, the components of which are lines of type a, $a \in \Delta$, such that any component of type a intersects n'_{ab} components of type b.

Tits and Steinberg proved the following.

THEOREM. – *Let G be simple.*

(i) *There is exactly one conjugacy class of subregular unipotent elements.*

(ii) *A unipotent $x \in G$ is subregular if and only if its exceptional fibre F_x is a Dynkin curve. All Dynkin curves occur as exceptional fibres.*

The first statement follows of course from Dynkins classification of all unipotent classes [4], if $K = \mathbf{C}$, and this proof carries over to the more general case of good characteristic. The second statement is exactly the analogue of Steinbergs first characterization of regular elements.

Exceptional curves of the type mentioned above are well known in algebraic geometry. They occur in the theory of rational singularities. This notion was introduced by M. Artin [1].

DEFINITION. – Let V be an algebraic surface, $v \in V$ a normal point and $f : V' \to V$ the minimal resolution of singularities. (V, v) is a rational singularity if for the higher direct images of the structure sheaf $(R^i f_* \mathcal{O}_{V'})_v = 0$, $i > 0$.

THEOREM. – *(V, v) is a rational singularity with* emb. $\dim_v V \leqslant 3$ *if and only if the reduced exceptional curve over v is isomorphic to a Dynkin curve of type A_r, D_r or E_r with self-intersection* -2 *for all components.*

Hence the theorem of Tits and Steinberg means that the unipotent variety has a rational singularity "along" its subregular orbit.

For $K = \mathbf{C}$, in the category of complex analytic spaces – to which I shall switch from now on – the rational singularities with embedding – dimension 3 admit the following beautiful description.

PROPOSITION. – The rational singularities with emb. dim $\leqslant 3$ are exactly the singularities of $\mathbf{C}^2/\Gamma$, Γ a finite subgroup of $SL(2, \mathbf{C})$.

The finite subgroups of $SL(2, \mathbf{C})$ are well known, they are the cyclic groups, and the binary dihedral, tetrahedral, octahedral and icosahedral groups. For example, if Γ is the binary icosahedral group, the corresponding Dynkin curve is that of E_8, and $\mathbf{C}^2/\Gamma \subset \mathbf{C}^3$ is the set of zeros of the equation

$$x^2 + y^3 + z^5 = 0 .$$

This equation is now almost one hundred years old – it first occurs in a paper of H.A. Schwarz [8] in 1872. Note that the equation is weighted homogeneous, this notion being defined as follows :

DEFINITION. – $\Sigma a_{i_1 \dots i_n} x_1^{i_1} \dots x_n^{i_n}$ with $w_1 i_1 + \dots + w_n i_n = v$ is weighted homogeneous of weight $(w_1, \dots, w_n)$ and degree v.

The equations of all $\mathbf{C}^2/\Gamma$ were determined by F. Klein in 1874 (see e.g. [6]).

PROPOSITION. – The equations of $\mathbf{C}^2/\Gamma$ are weighted homogeneous. Their weights and degrees are given in the following table.

type	weight	degree
A_r	$\left(1, \frac{r+1}{2}, \frac{r+1}{2}\right)$	$r+1$
D_r	$(2, r-2, r-1)$	$2r-2$
E_6	$(3, 4, 6)$	12
E_7	$(4, 6, 9)$	18
E_8	$(6, 10, 15)$	30

Up to analytic isomorphism the equations, which have to describe isolated singularities, are uniquely determined by these weights and degrees.

In order to give our second description of subregular unipotent elements, we need one more notion, that of "universal deformation" ("universal unfolding" in Thoms theory of singularities).

DEFINITION. – Let X_0 be a complex space, $x \in X_0$. A deformation of the germ (X_0, x) is a flat morphism $(X, x) \to (T, t)$ together with an isomorphism of (X_0, x) with the germ (X_t, x) of the fibre over t.

$(X, x) \to (T, t)$ is semi-universal if for all

$(X', x) \to (T', t')$ there exists a $g : (T', t') \to (T, t)$

with uniquely determined $dg|_{t'}$, such that X' is isomorphic to $X \times_T T'$.

Tjurina [13] and Schlesinger-Kas proved independently :

THEOREM. – *For isolated complete intersections semi-universal deformations exist and are unique.*

It is an easy consequence of this theorem that one can give a very explicit description of the universal deformation. For the sake of simplicity, I shall explain this only for the case of hypersurfaces.

COROLLARY. – Let X_0 be the hypersurface in $\mathbf{C}^n$ given by $f(z) = 0$, and $0 \in X_0$. The universal deformation of $(X_0, 0)$ is the germ at the origin of $\mathbf{C}^n \times \mathbf{C}^k \to \mathbf{C}^1 \times \mathbf{C}^k$, where (z, t) maps to $(F(z, t), t)$ and $F(z, t) = f(z) + \sum_{i=1}^{k} g_i(z)\, t_i$, where the polynomials $1, g_1, \ldots, g_k$ represent a basis of $\mathbf{C}\{z_1, \ldots, z_n\}/(f, \partial f/\partial z_i)$.

COROLLARY. – The universal deformation of $(\mathbf{C}^2/\Gamma, o)$ is defined by a weighted homogeneous polynomial F of degree ν_r and weight $(w_1, w_2, w_3, \nu_1, \ldots, \nu_{r-1})$, where $\nu_1 \leqslant \cdots \leqslant \nu_r$ are the degrees of a minimal set of generating W-invariant polynomials.

The following result was conjectured by Grothendieck.

THEOREM. – *Let G be a simple complex Lie group of type A_r, D_r or E_r. Then a unipotent $x \in G$ is subregular if and only if there exists a factorization of map-germs*

$$\begin{tikzcd} (G, x) \arrow[dr, "\pi"] \arrow[dd] & \\ & (X, x) \arrow[dl, "\varphi"] \\ (T/W, e) & \end{tikzcd}$$

where π is regular and φ is the universal deformation of the corresponding Kleinian singularity $\mathbf{C}^2/\Gamma$.

The idea of the proof is very simple. It suffices to prove the corresponding statement for x a subregular nilpotent element in the Lie algebra. $\mathfrak{g} \to \mathfrak{t}/W$ given by a set $\varphi_1, \ldots, \varphi_r$ of G-invariant polynominals. X is a transversal subspace of dimension $r + 2$, intersecting the orbit of x in x. As pointed out by Varadarajan [15], it follows from the Jacobson-Morosov-Lemma, that $\varphi_i | X$ is weighted homogeneous of degree ν_i and weight $(w_1, w_2, w_3, \nu_1, \ldots, \nu_{r-1})$. From this one deduces $\mathrm{rank}_x(d\varphi_1, \ldots, d\varphi_{r-1}) = r - 1$ and subsequently $\varphi_r = F$.

COROLLARY. — The set of subregular $x \in G$ forms a nonsingular submanifold of codimension 3.

COROLLARY. — Any deformation of a Kleinian singularity admits a resolution.

This has independently been proved by Tjurina [14] and for A_r by Kas [5], using methods developed in [2].

The universal deformation is a nonsingular fibration over the set of regular semisimple classes. In order to analyze this fibration, one needs the fundamental group of its base space. The following result was conjectured by Tits and is proved in [3].

PROPOSITION. — Let H_{reg} be the space of regular elements in a complex Cartan algebra. $\pi_1(H_{\mathrm{reg}}/W)$ has a presentation with Δ as set of generators and with relations

$$\underbrace{\alpha\beta\alpha\ldots}_{m_{\alpha\beta}\text{ factors}} = \underbrace{\beta\alpha\beta\ldots}_{m_{\alpha\beta}\text{ factors}},$$

where $(m_{\alpha\beta})$ is the Coxetermatrix.

Applying this proposition for E_8 and Picard-Lefschetz-theory, one obtains :

COROLLARY. — $\{z \in \mathbf{C}^5 \mid \|z\| = 1,\ z_1^2 + z_2^2 + z_3^2 + z_4^3 + z_5^5 = 0\}$ is an exotic 7-sphere representing Milnors standard generator of Θ_7.

Thus we see that there is a relation between exotic spheres, the icosahedron and E_8. But I still do not understand why the regular polyhedra come in. It is perhaps interesting to note that Klein in his lectures on the icosahedron emphasizes his indebtedness to Lie dating back to the years 1869-70, when Lie and Klein studied together at Berlin and Paris.

Klein writes : "At that time we jointly conceived the scheme of investigating geometric or analytic forms susceptible of transformation by means of groups of changes. This purpose has been of directing influence in our subsequent labours, though these may have appeared to lie far asunder. Whilst I primarily directed my attention to groups of discrete operations, and was thus led to the investigation of regular solids and their relations to the theory of equations, Professor Lie attacked the more recondite theory of continued groups of transformations, and therewith of differential equations".

Maybe the two theories do not lie so far asunder.

REFERENCES

[1] ARTIN M. — On isolated rational singularities of surfaces. *Amer. J. Math.*, 88, 1966, p. 129-136.

[2] BRIESKORN E. — Die Auflösung der rationalen Singularitäten holomorpher Abbildungen, *Math. Ann.*, 178, 1968, p. 255-270.

[3] Brieskorn E. — Die Fundamentalgruppe des Raumes der regulären Orbits einer endlichen komplexen Spiegelungsgruppe. Inventiones math., 12, 1971, p. 57-61.
[4] Dynkin E.B. — Semisimple subalgebras of semisimple Lie algebras, *Mat. Sbornik N.S.*, 30, 72, 1952, p. 349-462.
[5] Kas A. — On the resolution of certain holomorphic mappings, *Amer. J. Math.*, 90, 1968, p. 789-804.
[6] Klein F. — *Vorlesungen über das Ikosaeder und die Auflösung der Gleichung vom fünften Grade*, Leipzig 1884.
[7] Kostant B. — The principal three-dimensional subgroup and the Betti numbers of a complex simple group. *Amer. J. Math.*, 81, 1959, p. 973-1032.
[8] Schwarz H.A. — Uber diejenigen Fälle, in welchen die Gaussische hypergeometrische Reihe eine algebraische Function ihres vierten Elements darstellt. *J. Reine Angew. Math.*, 75, 1872, p. 292-335.
[9] Springer T.A. — Some arithmetical results on semisimple Lie algebras. *Publ. I.H.E.S.*, No. 30, 1966.
[10] Springer T.A. — The unipotent variety of a semisimple group. *Proc. Bombay Colloq. on Algebraic Geometry*, 1968, p. 373-391.
[11] Steinberg R. — Regular elements of semisimple algebraic groups. *Publ. I.H.E.S.*, No. 25, 1965.
[12] Steinberg R. — Classes of elements of semisimple algebraic groups. *Proceedings of the International Congress of Mathematicians*, Moscow, 1966, p. 277-289.
[13] Tjurina G.N. — Flat locally semi-universal deformations of isolated singularities of complex spaces, *Izv. Akad. Nauk S.S.S.R. Ser. Mat.*, 33, 1969, p. 1026-1058.
[14] Tjurina G.N. — Resolution of singularities for flat deformations of rational double points, *Funkcional. Anal. i. Priložen.*, 4, 1970, p. 77-83.
[15] Varaldarajan V.S. — On the ring of invariant polynomicals on a semisimple Lie algebra, *Amer. J. Math.*, 90, 1968, p. 308-317.

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