SOME BIRATIONAL INVARIANTS OF ALGEBRAIC VARIETIES

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The notion of differential forms plays an important role in the birational geometry of algebraic varieties. In particular, the plurigenera and the HODGE-numbers are fundamental birational invariants. This paper deals with the more general differential tensor forms, "which hopefully will lead us to a new birational geometry of algebraic varieties" [8]. In fact for any smooth projective variety Y and any YOUNG-diagram T the finite-dimensional vector space $H^0(Y,\mathcal{T}^T)$ of the global T-symmetrical tensor forms on Y is a birational invariant of Y. This follows from the fact, that a rational map gives a morphism outside a subvariety of codimension ≥ 2.

Let T be a YOUNG-diagram with r squares and f rows. In case f=1 one has $\mathcal{T}^T = \bigwedge^r \Omega^1$, i.e. $\dim_k H^0(Y,\mathcal{T}^T)$ is a HODGE-number. In case depth T = dim Y and r = depth T · length T (i.e. T is a rectangle) holds $\mathcal{T}^T = \mathcal{L}(f\cdot K_Y)$, where by K_Y is denoted the canonical divisor of Y. Therefore in this case $\dim_k H^0(Y,\mathcal{T}^T) = P_f$ is a plurigenus of Y.

First in this paper a generalization of the LEFSCHETZ-theorems is shown concerning the restriction of differential tensor forms onto the section with a hypersurface (Theorems 1, 3). Then we consider smooth complete intersections $Y \subseteq P^n_k$.

Some of the results are the following:

1. $H^q(Y,\mathcal{T}^T(p)) = 0$ if $0 \leq q < \dim Y - \min\{r, \text{depth } T \cdot \text{codim } Y\}$
   and (depth $T < q$ or $p < r-q$).

2. In particular (p=q=0):
   $H^0(Y,\mathcal{T}^T) = 0$ if $r < \dim Y$ or depth T · codim Y < dim Y.
   In case depth T = 1 (i.e. $\mathcal{T}^T = S^r\Omega^1$) follows:
   $H^0(Y, S^r\Omega^1) = 0$ if $r < \dim Y$ or codim Y < dim Y (see [1]).
   In case codim Y = 1 (i.e. $Y \subset P^n$ is a hypersurface) follows:
   $H^0(Y,\mathcal{T}^T) = 0$ if depth T $\neq$ dim Y (see [2]).

3. For any integer $a \in N$ there exist a complexe 2-dimensional complete intersection $Y \subset P^4$ and an integer $b > a$ with the properties
   $H^0(Y, S^r\Omega^1) = 0$ for all $0 < r \leq a$ and $H^0(Y, S^r\Omega^1) = 0$ for all $r \geq b$.

§ 1. YOUNG - diagrams

As well known by representation theory tensor space splits in a direct sum of subspaces [9]. Each of these subspaces consists of all tensors with a well-defined property of symmetry, which corresponds to a YOUNG-diagram. Analogous the sheaf $\mathcal{T}^r = \overset{r}{\otimes}\Omega^1$ of germs of differential tensor forms splits in the direct sum of subsheaves. We remind of some facts:

For any integer $r>0$ let $D(r)$ denote the set of all YOUNG-diagrams by r cells. Let be $T \in D(r)$, M the set of cells of T, S the permutation group of M, $P \subseteq S$ be the subgroup of those permutations of M, which leave each cell in its row, $Q \subseteq S$ be the subgroup of all permutations, which leave each cell in its column. Moreover let R be any order relation in M, $\alpha_R : M \longrightarrow \{1, 2, \ldots, r\}$ denotes the 1 to 1 mapping, which preserves the order, and $\beta_R : S \longrightarrow S_r$ the group isomorphism defined by $\beta_R(p) = \alpha_R \circ p \circ \alpha_R^{-1}$. Finally let k be a field with $\mathrm{Char}\, k = 0$ or $\mathrm{Char}\, k > r$ and A be the group algebra over S_r. If one sets $c(T, R) := \sum_{q\in Q} \sum_{p\in P} \chi(q)\cdot\beta_R(q\cdot p)$, then there exists $m(T) \in k^*$ with $c(T,R)\cdot c(T,R) = m(T)\cdot c(T,R)$ in A.

Therefore $e(T,R) := \frac{1}{m(T)}\cdot c(T,R)$ is an idempotent of A.

For any distinct YOUNG-diagrams $T, T' \in D(r)$ and any order relations R, R' of their cells the idempotents $e(T,R)$ and $e(T',R')$ of A are perpendicular to each other. Moreover for any YOUNG-diagram $T \in D(r)$ there exist an integer $n(T) > 0$ and $n(T)$ pairwise distinct orders of the cells of T R_i^T ($i = 1, 2, \ldots, n(T)$) with the following properties:

$e(T,R_i^T)\cdot e(T,R_j^T) = 0$ for $i \neq j$ and $\sum_{T\in D(r)} \sum_{i\in\{1,\ldots,n(T)\}} e(T,R_i^T) = 1.$

For any $T \in D(r)$ let $R(T)$ be the lexicographic order of the cells of T corresponding row index and column index. The idempotent $e(T, R(T)) \in A$ is denoted by $e(T)$. Now let V be a d-dimensional k-vector space, V^* the dual vector space and $W^r = \overset{r}{\otimes} V^*$ the space of the contravariant tensors on V of rank r.

For any product $w = a_1 \otimes a_2 \otimes \ldots \otimes a_r$ ($a_i \in V^*$) and any permutation $p \in S_r$ we set $p(w) = a_{i_1} \otimes a_{i_2} \otimes \ldots \otimes a_{i_r}$ where $p(i_j) = j$. Then W^r becomes an A-left-modul.

Therefore the tensor space W^r splits in the direct sum $W^r = \sum_{T\in D(r)} \sum_{i\in\{1,2,\ldots,n(T)\}} W(T, R_i^T)$. Moreover the subspaces $W(T, R_i^T)$ are both $GL(k,d)$ - invariant and $GL(k,d)$ - irreducible. For any orders R_1, R_2 of the cells of the YOUNG-diagram T the subspaces $W(T, R_1)$ and $W(T, R_2)$ are isomorphic to each other. From depth $T > \dim V$ follows $W(T,R(T))=0$.

§ 2. The sheaf of germs of T-symmetrical tensor forms

Let X be an algebraic variety over the algebraically closed field k, A the group algebra over k and $\mathcal{T}^r = \overset{r}{\otimes}\Omega^1$ the sheaf of germs of differential tensor forms of rank r on X. In case Char k = 0 or Char k > r the $\Gamma(U,\mathcal{O})$-modul $\Gamma(U,\mathcal{T}^r)$ becomes an A-left-modul and therefore the sheaf $\mathcal{T}^r$ splits in the direct sum $\mathcal{T}^r = \bigoplus_{T\in D(r)} \bigoplus_{i\in\{1,2,\dots,n(T)\}} \mathcal{T}(T, R_i^T)$ of coherent subsheaves. For arbitrary orders R_1, R_2 of cells of the YOUNG-diagram T the sheaves $\mathcal{T}(T, R_1)$ and $\mathcal{T}(T, R_2)$ on X are isomorphic to each other. The sheaf $\mathcal{T}(T, R(T))$ is denoted by $\mathcal{T}^T$ and is called the sheaf of germs of T-symmetrical tensor forms. If X is smooth then each of these subsheaves $\mathcal{T}(T,R)$ is locally free. Moreover in this case from depth T > dim X follows $\mathcal{T}^T = 0$. If $X \subseteq P^n$ is a projective variety, then $\dim_k H^0(X,\mathcal{T}^T)$ becomes a birational invariant of X. Simple examples show the independence of these invariants. In case length T = 1 the sheaf $\mathcal{T}^T$ is equal to $\Omega^r = \overset{r}{\wedge}\Omega^1$. In case depth T = 1 holds $\mathcal{T}^T = S^r\Omega^1$.

§ 3. Restriction of tensor forms onto the intersection with a hypersurface

Let k be an algebraically closed field, $H \subset P_k^n$ a hypersurface of degree m, X, $Y = X \cap H \subseteq P^n$ smooth projective varieties (dim Y = dim X - 1). Then one has the following short exact sequences of sheaves:

$$0 \longrightarrow \mathcal{O}_X(-m) \longrightarrow \mathcal{O}_X \longrightarrow \mathcal{O}_Y \longrightarrow 0$$
$$0 \longrightarrow \mathcal{T}_X^r(-m) \longrightarrow \mathcal{T}_X^r \longrightarrow \mathcal{A}^r \longrightarrow 0 \qquad (1)$$
$$0 \longrightarrow \mathcal{O}_Y(-m) \longrightarrow \mathcal{A}^1 \longrightarrow \Omega_Y^1 \longrightarrow 0 \quad \text{with } \mathcal{A}^r = \mathcal{O}_Y \otimes \mathcal{T}_X^r .$$

By considering the tensor product of the complex $0 \longrightarrow \mathcal{O}_Y(-m) \longrightarrow \mathcal{A}^1$ one gets the exact sequence of sheaves on Y:

$$0 \longrightarrow \mathcal{L}^0 \longrightarrow \mathcal{L}^1 \longrightarrow \dots \longrightarrow \mathcal{L}^{r-1} \longrightarrow \mathcal{A}^r \longrightarrow \mathcal{T}_Y^r \longrightarrow 0 , \qquad (2)$$

where $\mathcal{L}^s$ denotes the direct sum $\bigoplus_{\binom{r}{s}} \mathcal{A}^s((s-r)\cdot m)$ of $\binom{r}{s}$ specimens of the sheaf $\mathcal{A}^s((s-r)\cdot m)$.

<u>Theorem 1.</u>

For any integer a holds: If deg H is high enough the restriction map $\varphi^*: H^q(X,\mathcal{T}_X^r(p)) \longrightarrow H^q(Y,\mathcal{T}_Y^r(p))$ is a monomorphism for all $p \leq a$ and all $q < \dim X - r$ and an isomorphism for all $p \leq a$ and all $q < \dim Y - r$.

Corollary.

If deg H is high enough the restriction map of the global (i.e. q=0) tensor forms of rank r is a monomorphism for all $r<\dim X$ and an isomorphism for all $r<\dim Y$.

Theorem 2.

Assuming $\deg H \geq 2$ and $H^q(X,\mathcal{T}^r_X(p))=0$ for all $q,r\in N$, $p\in Z$ with the properties $q+r<\dim X$ and ($p<r-q$ or $q>r$). Then follows $H^q(Y,\mathcal{T}^r_Y(p))=0$ for all $q,r\in N$, $p\in Z$ with the properties $q+r<\dim Y$ and ($p<r-q$ or $q>r$).

Proof.

The theorems can be proved by researching into the exact homology sequences we obtain from (1) and (2).

In case of T-symmetrical tensor forms more precise results hold true. In this case the number of rows depth T takes over the part of rank r. Assume Char k = 0 or Char k > r. Then the sheaf $\mathcal{T}^r_X$ splits in the direct sum of locally free subsheaves $\mathcal{T}_X(T, R^T_i) \cong \mathcal{T}^T_X$ on X. Therefore the following short sequence is exact:

$$0 \longrightarrow \mathcal{T}^T_X(-m) \longrightarrow \mathcal{T}^T_X \longrightarrow \mathcal{A}^T \longrightarrow 0 \quad \text{with } \mathcal{A}^T = \mathcal{O}_Y \otimes \mathcal{T}^T_X .$$

Moreover for each $s\in\{0,1,\dots,r-1\}$ the $\Gamma(U,\mathcal{O}_Y)$ - modul $\Gamma(U,\mathcal{B}^s)$ becomes an A-left-modul and the sheaf $\mathcal{B}^s$ splits in the direct sum $\mathcal{B}^s = \bigoplus_{T\in D(r)} \bigoplus_{i\in\{1,\dots,n(T)\}} \mathcal{B}^s(T, R^T_i)$ of subsheaves. If we denote the sheaf $\mathcal{B}^s(T, R(T))$ by $\mathcal{B}^s(T)$ we have $\mathcal{B}^s(T,R)\cong\mathcal{B}^s(T)$ for any order R of the cells of T. One can show $\mathcal{B}^s(T)=0$ for $s\in\{0,1,\dots,r-t-1\}$ (t = depth T)[6]. Therefore one has the following exact sequence of sheaves on Y:

$$0 \longrightarrow \mathcal{B}^{r-t}(T) \longrightarrow \mathcal{B}^{r-t+1}(T) \longrightarrow \dots \longrightarrow \mathcal{B}^{r-1}(T) \longrightarrow \mathcal{A}^T \longrightarrow \mathcal{T}^T_Y \longrightarrow 0,$$

where $\mathcal{B}^s(T)$ is isomorphic to a direct summand of the sheaf $\mathcal{B}^s$ on Y. Using the corresponding homology sequences one obtains the following:

Theorem 3.

Assume $T\in D(r)$ and Char k = 0 or Char k > r. Then for any integer a holds: If deg H is high enough the restriction map $\varphi^*: H^q(X,\mathcal{T}^T_X(p)) \longrightarrow H^q(Y,\mathcal{T}^T_Y(p))$ is a monomorphism for all $p\leq a$ and all $q<\dim X$ - depth T and an isomorphism for all $p\leq a$ and all $q<\dim Y$ - depth T.

Corollary.

If deg H is high enough the restriction map of the global T-symmetrical tensor forms (i.e. $p=q=0$) is a monomorphism in case depth $T<\dim X$ and is an isomorphism in case depth $T<\dim Y$.

Application.

For any integer a there exist a complex 2-dimensional complete intersection $Y\subset P^4$ and an integer $b>a$ with the properties:

$H^0(Y, S^r\Omega^1)=0$ for all $0<r\leq a$ and $H^0(Y, S^r\Omega^1)\neq 0$ for all $r\geq b$.

This shows the independence of the birational invariants $\dim_k H^0(Y, S^r\Omega^1)$.

Proof.

Let $X\subset P^4$ be a smooth hypersurface of degree $m_1\geq 6$. Then $H^0(X, S^r\Omega^1)=0$ holds for each integer $r>0$ [2]. Let $H\subset P^4$ be a hypersurface of degree m_2 and assume that X and H are in general position, i.e. $Y := X\cap H$ is a smooth 2-dimensional complete intersection. If m_2 is high enough then holds:

1. For any integer $0<r\leq a$ the restriction map $\varphi^*: H^0(X,S^r\Omega^1_X)\longrightarrow$ $\longrightarrow H^0(Y,S^r\Omega^1_Y)$ is an isomorphism, i.e. $H^0(Y,S^r\Omega^1_Y)=0$ for $0<r\leq a$.
2. Y is a surface of general type and $c_1^2-c_2=(m_1m_2-5m_1-5m_2+15)m_1m_2>0$. Therefore by a result of BOGOMOLOV [1] an integer $b>a$ exists with $H^0(Y, S^r\Omega^1)\neq 0$ for all $r\geq b$.

We previously said, that the sheaf $\mathcal{B}^s(T)$ is isomorphic to a direct summand of $\mathcal{B}^s=\bigoplus_{\binom{r}{s}}\mathcal{A}^s((s-r)m)$. On the other hand the sheaf $\mathcal{A}^s$ splits in a direct sum

$\mathcal{A}^s=\bigoplus_{T'\in D(s)}\ \bigoplus_{i\in\{1,\dots,n(T')\}}\mathcal{A}(T', R_i^{T'})$. More precisely one can show that the sheaf $\mathcal{B}^s(T)$ is isomorphic to a direct summand of the subsheaf

$\bigoplus_{\binom{r}{s}}\ \bigoplus_{T'\in D(s,t)}\ \bigoplus_{i\in\{1,\dots,n(T')\}}(\mathcal{A}(T', R_i^{T'}))((s-r)\cdot m)$ of $\mathcal{B}^s$,

where $t=$ depth T and $D(s,t):=\{T'\in D(s)$: depth $T'\leq t\}$.

Therefore the homology sequences give the following:

Theorem 4.

Assume $\deg H\geq 2$, $T\in D(r)$, $t=$ depth T, Char $k=0$ or Char $k>r$ and $H^q(X,\mathcal{T}^{T'}_X(p))=0$ for all $s\in\{1,2,\dots,r\}$ and all $T'\in D(s,t)$, $q\in N$, $p\in Z$ with the properties $q<\dim X-\min\{s$, depth $T'\cdot\operatorname{codim} X\}$ and (depth $T'<q$ or $p<s-q$). Then follows $H^q(Y,\mathcal{T}^T_Y(p))=0$ for all $q\in N$, $p\in Z$ with the properties $q<\dim Y-\min\{r$, depth $T\cdot\operatorname{codim} Y\}$ and (depth $T<q$ or $p<r-q$).

§ 4. On cohomology of the projective space

We consider the sheaves $\mathcal{O}, \Omega^1, \mathcal{T}^r, \mathcal{T}(T,R), \mathcal{T}^T$ on P^n and restrict to such problems resolution of which is useful for research into complete intersections. From the well known short exact sequence $0 \longrightarrow \Omega^1 \longrightarrow \bigoplus_{n+1} \mathcal{O}(-1) \longrightarrow \mathcal{O} \longrightarrow 0$ we get the following exact sequence of sheaves on P^n:

$$0 \longrightarrow \mathcal{T}^r \longrightarrow \bigoplus_{c_r} \mathcal{O}(-r) \longrightarrow \bigoplus_{c_{r-1}} \mathcal{O}(1-r) \longrightarrow \dots \longrightarrow \bigoplus_{c_1} \mathcal{O}(-1) \longrightarrow \mathcal{O} \longrightarrow 0$$

with $c_s = \binom{r}{s} \cdot (n+1)^s$. Therefore we have $H^q(P^n, \mathcal{T}^r(p)) = 0$ for $q \neq n$ and ($q > r$ or $p < r-q$).

Theorem 5.

Assume $T \in D(r)$ and Char $k = 0$ or Char $k > r$. Then holds
$H^q(P^n, \mathcal{T}^T(p)) = 0$ for $q \neq n$ and ($q >$ depth T or $p < r-q$) and
$H^q(P^n, \mathcal{T}^T) = 0$ for $q \neq n$ and length $T > 1$.

Proof.

The groups $\Gamma(U, \bigoplus_{c_s} \mathcal{O}(-s))$ are A-left-moduls and we obtain an exact sequence of sheaves on P^n:

$$0 \longrightarrow \mathcal{T}^T \longrightarrow \bigoplus_{d_r} \mathcal{O}(-r) \longrightarrow \bigoplus_{d_{r-1}} \mathcal{O}(1-r) \longrightarrow \dots \longrightarrow \bigoplus_{d_{r-t-1}} \mathcal{O}(t+1-r) \longrightarrow \bigoplus_{d_{r-t}} \mathcal{O}(t-r) \longrightarrow 0$$

(t = depth T) with some integers $d_s \in N$. By considering the homology sequences the theorem follows.

Remark.

Generally speaking the dimension of the cohomology groups $H^q(P^n_k, S^r\Omega^1)$ depends on the ground field k. In case Char $k = r$ ($\neq 0$) we have for instance

$$H^1(P^2_k, S^r\Omega^1) \neq 0 \text{ because the cycle } \tau \text{ with } \tau_{ij} = \left(\frac{x_i}{x_j} \cdot d\, \frac{x_j}{x_i}\right)^r \quad (i,j \in \{0,1,2\})$$

is not homologous to zero. On the other hand in case $r > 1$ and Char $k = 0$ or Char $k > r$ from theorem 5. one obtains $H^1(P^2_k, S^r\Omega^1) = 0$.

§ 5. T-symmetrical tensor differential forms on complete intersections

Let k be an algebraically closed field and $H_1, H_2, \dots, H_{n-d} \subset P^n_k$ hypersurfaces with the property that for each $j \in \{1,2,\dots,n-d\}$ the variety $H_1 \cap H_2 \cap \dots \cap H_j$ is the smooth complete intersection of these hypersurfaces.
Let $X = H_1 \cap H_2 \cap \dots \cap H_{n-d-1}$ and $Y = H_1 \cap H_2 \cap \dots \cap H_{n-d}$. Then from theorem 2. follows:

Theorem 6.

For any $q, r \in N$, $p \in Z$ with $q+r < \dim Y$ and ($p < r-q$ or $q > r$) the cohomology group $H^q(Y, \mathcal{T}^r(p))$ is trivial.

Corollary.

The complete intersection Y has no global tensor form the rank of which is less than $\dim Y$. In particular in case $\mathrm{Char}\, k = 0$ or $\mathrm{Char}\, k > r$ the complete intersection Y has no global T-symmetrical tensor form the rank of which is less than $\dim Y$.

Moreover in case of T-symmetrical tensor forms holds:

Theorem 7.

Let be $T \in D(r)$ and $\mathrm{Char}\, k = 0$ or $\mathrm{Char}\, k > r$. Then the complete intersection Y has no global T-symmetrical tensor form if $\mathrm{depth}\, T \cdot \mathrm{codim}\, Y < \dim Y$.
In particular Y has no global symmetrical differential form if $\mathrm{codim}\, Y < \dim Y$.
Any smooth hypersurface $H \subset P_k^n$ has no global T-symmetrical tensor form if $\mathrm{depth}\, T < \dim H$ (see [1] and [2]).

Theorem 8.

Assume $T \in D(r)$ and $\mathrm{Char}\, k = 0$ or $\mathrm{Char}\, k > r$. Then holds:

$$H^q(Y, \mathcal{T}^T(p)) = 0 \quad \text{if} \quad q < \dim Y - \min\{ r, \mathrm{depth}\, T \cdot \mathrm{codim}\, Y \} \quad \text{and} \quad (q > \mathrm{depth}\, T \ \text{ or } \ p < r-q).$$

In particular in case of an irreducible smooth projective hypersurface $H \subset P_k^n$:

$$H^q(H, \mathcal{T}^T(p)) = 0 \quad \text{if} \quad q < \dim H - \mathrm{depth}\, T \quad \text{and} \quad (q > \mathrm{depth}\, T \ \text{ or } \ p < r-q).$$

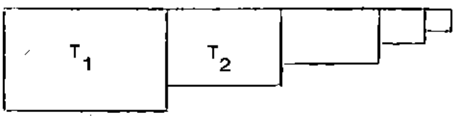
Proof.

Theorem 8. follows from the theorems 4. and 5. In The special case $q = 0$ one obtains the theorem 7.

By considering sections by hyperplanes the following can be proved:

Theorem 9.

Assume $\mathrm{Char}\, k = 0$ or $\mathrm{Char}\, k > r$ and the YOUNG-diagram $T \in D(r)$ consists of the rectangles $T_1, T_2, \ldots, T_a$.

T =


Setting $L_i = \mathrm{length}\, T_i$, $t_i = \mathrm{depth}\, T_i$, $r_i = L_i \cdot t_i$ ($i = 1, 2, \ldots, a$) one has $\mathrm{length}\, T = \sum L_i$, $r = \sum r_i$ and $\mathrm{depth}\, T = \mathrm{depth}\, T_1 = t_1 > t_2 > \ldots > t_a > 0$.

Moreover for any smooth complete intersection $Y \subseteq P_k^n$ of the multidegree $(m_1, m_2, \dots, m_{n-d})$ we set $p(Y) := m_1 + m_2 + \dots + m_{n-d} - n - 1$. Then holds:

1. If there exists an integer $s \in \{1, 2, \dots, a-1\}$ with the properties
$(p(Y) + \dim Y)\cdot(L_1 + \dots + L_a) < r + \min\{(L_{s+1} + \dots + L_a), m_1 - 2, \dots, m_{n-d} - 2\}$
and $(r_{s+1} + r_{s+2} + \dots + r_a < t_s$ or $t_{s+1}\cdot \operatorname{codim} Y < t_s)$
then the cohomology group $H^0(Y, \mathcal{T}^T)$ is trivial.
2. If T is a rectangle (i.e. $a = 1$) then from $p(Y) + \dim Y <$ depth T follows $H^0(Y, \mathcal{T}^T) = 0$.

§ 6. Symmetrical differential forms on complete intersections

In case of symmetrical differential forms (i.e. depth $T = 1$, Char $k = 0$ or Char $k > r$) from theorem 8. for any $q, r \in N$ ($r > 0$), $p \in Z$ follows:

$H^0(Y, S^r\Omega^1) = 0$ if $r < \dim Y$ or $\operatorname{codim} Y < \dim Y$

$H^0(Y, (S^r\Omega^1)(p)) = 0$ if $p < r$ and ($r < \dim Y$ or $\operatorname{codim} Y < \dim Y$)

$H^1(Y, (S^r\Omega^1)(p)) = 0$ if $p < r-1$ and ($r < \dim Y - 1$ or $\operatorname{codim} Y < \dim Y - 1$)

$H^q(Y, (S^r\Omega^1)(p)) = 0$ if $1 < q < \dim Y - r$ or $1 < q < \dim Y - \operatorname{codim} Y$.

T h e o r e m 10.

There exists a 2-dimensional smooth complex complete intersection $Y \subset P^4$ with global symmetrical differential forms of all degrees $r \geq 2$ and with the property $c_1^2 - c_2 < 0$.

P r o o f.

Let be $X, H \subset P^4$ the hypersurfaces $x_o^m + x_1^m + x_2^m + x_3^m + x_4^m = 0$ and $a_o\cdot x_o^m + a_1\cdot x_1^m + a_2\cdot x_2^m + a_3\cdot x_3^m + a_4\cdot x_4^m = 0$ resp. where the complex numbers a_o, a_1, a_2, a_3, a_4 are pairwise distinct. Let $Y := X \cap H$ and $\mathcal{A}^1 := \mathcal{O}_Y \otimes \Omega_X^1$. Then each of the differentials

$$H^0(Y, S^2\Omega_Y^1) \xrightarrow{d_o} H^1(Y, \mathcal{A}^1(-m)) \xrightarrow{d_1} H^2(X, \Omega_X^1(-2m)) \xrightarrow{d_2} H^3(X, \mathcal{O}_X(-3m)) \xrightarrow{d_3} H^4(P^4, \mathcal{O}_{P^4}(-4m))$$

is monomorphic.

In case $m \geq 5$ there are integers b_i $(i=0,1,2,3,4)$ with $0 < b_i < m$ and $b_o + b_1 + b_2 + b_3 + b_4 = 4m$. The inverse image of the cocycle $f \in C^4(\mathfrak{U}, \mathcal{O}_{P^4}(-4m))$ defined by the equation $f_{o1234} = x_o^{-b_o}\cdot x_1^{-b_1}\cdot x_2^{-b_2}\cdot x_3^{-b_3}\cdot x_4^{-b_4}$ is the global symmetrical differential form $\omega \in H^0(Y, S^2\Omega^1_Y)$ with

$$\omega = x_0^{-b_0} \cdot x_1^{m-1-b_1} \cdot x_2^{m-1-b_2} \cdot x_3^{m-1-b_3} \cdot x_4^{m+2-b_4} \cdot \sigma \quad \text{and}$$

$$\sigma = (a_2-a_1)a_3 \cdot x_3 \cdot d\frac{x_1}{x_4} \circ d\frac{x_2}{x_4} + (a_1-a_3)a_2 \cdot x_2 \cdot d\frac{x_1}{x_4} \circ d\frac{x_3}{x_4} +$$

$$+ (a_3-a_2)a_1 \cdot x_1 \cdot d\frac{x_2}{x_4} \circ d\frac{x_3}{x_4}.$$ Therefore in case $m \geq 5$ holds $H^2(Y, S^2\Omega^1) \neq 0$.

In case $m \geq 7$ there are those integers b_i with $b_2, b_3 < m-1$. Then $\frac{1}{x_2 \cdot x_3} \cdot \omega$ is a global section of the sheaf $(S^2\Omega^1_Y)(-2)$ and

$\frac{x_0^2}{x_2 x_3} \cdot \omega \circ d\frac{x_1}{x_0}$ is a global section of the sheaf $S^3\Omega^1_Y$. Therefore in case $m \geq 7$ one has both $H^0(Y, S^2\Omega^1) \neq 0$ and $H^0(Y, S^3\Omega^1) \neq 0$, that means $H^0(Y, S^r\Omega^1) \neq 0$ for all $r \geq 2$. On the other hand for all m with $2 \leq m \leq 8$ we have $c_1^2 - c_2 = (m^2 - 10m + 15) \cdot m^2 < 0$.

References.

1. Bogomolov, F.A.: Holomorphic symmetrical tensors on projective surfaces (Russian). Usp. mat. nauk 33, 5, 171-172 (1978).
2. Brückmann, P.: Differential tensor forms on algebraic varieties (Russian). Isv. akad. nauk. SSSR, ser. mat. 35, 1008-1036 (1971).
3. Brückmann, P.: Zur Kohomologie von projektiven Hyperflächen. Beitr. Alg. Geom. 2, 87-101 (1974).
4. Brückmann, P.: Zur Kohomologie von vollständigen Durchschnitten mit Koeffizienten in der Garbe der Keime der Differentialformen I, II. Math. Nachr. 71, 203-210 (1976) and Math. Nachr. 77, 307-318 (1977).
5. Brückmann, P.: Über die Beschränkung von Tensor-Differentialformen auf den Schnitt mit einer algebraischen Hyperfläche. Wiss. Hefte PH Köthen 2, 57-71 (1979).
6. Brückmann, P.: T-symmetrische Tensor-Differentialformen und vollständige Durchschnitte von Hyperflächen. Dissertation B, MLU Halle-Wittenberg 1984.
7. Sakai, F.: Symmetric powers of the cotangent bundle and classification of algebraic varieties. Lecture Notes in Math. 732, Springer 1979.
8. Sakai, F.: Differential Tensor Forms in Algebraic Geometry. Preprint Dep. of Math., Fac. of Sci., Saitama Univ., Urawa, Japan (1979).
9. Serre, J.-P.: Faisceaux algebriques coherents. Ann. of Math. 61, 197-278 (1955).
10. Weyl, H.: The classical groups. 2nd. ed., Princeton 1946.

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