F. CANO et R. PIEDRA

Characteristic polygon of surface singularities

0. INTRODUCTION

Lez Z be a regular noetherian scheme, let X be a closed two-dimensional subscheme of Z and let P be a closed point of X. The aim of this paper is to associate a polygon $\Delta(P)$ to P which turns to be an intrinsic invariant of the singularity of X at P.

Actually $\Delta(P)$ is the polygon defined by Hironaka in |4|, but for a selected choice of the "tangential parameters", the selection being made for reaching certain maximum in the space of polygons.

If D is a "tranversal regular hypersurface" we shall define an intermediate invariant $\Delta_D(P)$ at §3 which may be used for the control of the resolution algorithms in the same way as in |3|.

The invariat $\Delta(P)$ is expected to be useful for formulating a "fine" version of the resolution game |8| as well as an intrinsic invariant for the analysis of the singularities.

1. PRELIMINARIES

Here we shall recall some results and notations needed in the sequel. Most of them are contained in |4| or |6|.

(1.1) Let R be the completion of the local ring of Z at P, let I be the ideal of X in R and let M be the maximal ideal of R. The residual field k = R/M is supossed to be arbitrary unless otherwise would be specifield.

(1.2) The graded ring $Gr_M(R)$ with respect to the M-adic filtration is a polynomial ring over the field k. Let us denote by E(I) the minimum k-submodule of the homogeneus part of degree one $Gr_M(R)_1$ of $Gr_M(R)$ such that

$$\mathrm{In}_M(I) = (\mathrm{In}_M(I) \cap K[E(I)])\, \mathrm{Gr}_M(R) \tag{1.2.1}$$

where $\mathrm{In}_M(I)$ denotes the initial ideal of I in $\mathrm{Gr}_M(R)$. E(I) defines the strict tangent space of X at P, i.e. the maximum vector subspace of the tangent space of Z at P which leaves invariat the tangent cone of X acting by translations.

We shall suppose that the codimension of E(I) in $\mathrm{Gr}_M(R)_1$ is two. In this case the strict tangent space and the tangent cone of X at P agree as reduced subschemes of the tangent space of Z at P. Actually, this is the "general case" for resolution purposes (|3|).

(1.3) A regular sequence $x=(x_1,x_2)$ in R is called a "system of tangential parameters" iff there exists a regular sequence $y = (y_1,\dots,y_r)$ such that $t = (x,y)$ is a regular system of parameters and the following condition is verified

$$E(I) = Y_1\mathrm{Gr}_M(R)_1 + \dots + Y_r\mathrm{Gr}_M(R)_1 \tag{1.3.1}$$

where $Y_i = \mathrm{In}_M(y_i)$, $i=1,\dots,r$.

A subset of I, $f = (f_1,\dots,f_m)$ is called a "tangential base" of I iff $\mathrm{In}_M(f_i)$, $i=1,\dots,m$, generates the initial ideal $\mathrm{In}_M(I)$. Systems of tangential parameters and tangential basis always exist (|6|).

(1.4) Let us fix a regular system of parameters $t=(x,y)$ as in (1.3). Let $e : \mathbb{R}^2 \longrightarrow \mathbb{R}$ be a positively linear function. We shall consider the following filtration in R

$$R_{e,\nu} = (x^\alpha y^\beta ;\ |\beta| + e(\alpha) \leq \nu)R \tag{1.4.1}$$

The associated graded ring (resp. initial ideal) will be denoted by $\mathrm{Gr}_e(R)$ (resp. $\mathrm{In}_e(I)$). there is an isomorphism of k-algebras

$$\lambda_e : \mathrm{Gr}_e(R) \longrightarrow \mathrm{Gr}_M(R) \tag{1.4.2}$$

such that $\lambda_e(\mathrm{In}_e(x_i)) = X_i$ and $\lambda_e(\mathrm{In}_e(y_j)) = Y_j$, $i=1,2$, $j=1,\dots,r$. The isomorphism λ_e does not preserve the natural graded structure of $\mathrm{Gr}_e(R)$.

(1.5) <u>Definition</u>.- (Hironaka |6|). "Let $t=(x,y)$ be a regular system of parameters, x being a system of tangential parameters and y verifying (1.3.1). The polygon $\Delta_{x,y}(I)$ is defined by

(1.5.1) $$\Delta_{x,y}(I) = \bigcap_e \{\alpha \in \mathbb{R}^2;\ e(\alpha) \geq 1\}$$

where e ranges over all positively linear functiones that

(1.5.2) $$\lambda_e(In_e(I)) = In_M(I)$$

The "characteristic polygon $\Delta_x(I)$" is defined by

$$\Delta_x(I) = \bigcap_y \Delta_{x,y}(I)$$

where y ranges over all regular sequences such that t=(x,y) is a regular system of parameters and (1.3.1) is verified ".

(1.6) Let $f=(f_1,\dots,f_m)$ be a set of elements of R, let $d_i=ord_M(f_i)$, $i=1,\dots,m$. For each f_i, the polygon $\Delta_{x,y}(f_i)$ is defined to be

(1.6.1) $$\Delta_{x,y}(f_i) = \bigcap_e \{\alpha \in \mathbb{R}^2;\ e(\alpha) \geq 1\}$$

where e ranges over all positively linear functions e such that

(1.6.2) $$f_i \in R_{e,d_i}.$$

The polygon $\Delta_{x,y}(f)$ is defined to be the convex hull of

(1.6.3) $$\bigcup_{i=1,\dots,m} \Delta_{x,y}(f_i)$$

(1.7) Definition.- (Hironaka |6|). "Let t=(x,y) be as in (1.5). Let $f=(f_1,\dots,f_m)$ be a tangential base of I such that

(1.7.1) $$In_M(f_i) \in k[Y_1,\dots,Y_r] \subset Gr_M(R)$$

$i=1,\dots,m$. A vertex v of $\Delta_{x,y}(f)$ is said to be "well prepared" iff the following condition is verified.

(1.7.2) Let e be a positively linear function on $\mathbb{R}^2$ such that e(v)=1 and e >1 on $\Delta_{x,y}(f) - \{v\}$. Let J be the ideal of $Gr_M(R)$ generated by $\lambda_e(In_e(f_i))$, $i=1,\dots,m$. Then, there exists no $k[X_1,X_2]$ - automorphism σ of $Gr_M(R)$ such that

a) $\sigma(Y_i) = Y_i + c_iX^v$ $c_i \in k$, $i=1,\dots,m$.

b) σ (J) is generated by $\sigma(J) \cap k[Y]$

(in (1.7.2) we denote $X=(X_1,X_2)$).

(1.8) Remark that if (1.7.2) fails to be true and $v \notin \mathbb{Z}^2$, then σ = identity.

(1.9) Theorem.- (|6|). With notations as above, one has that

(1.9.1) $$\Delta_{x,y}(f) = \Delta_x(I)$$

iff every vertex of $\Delta_{x,y}(f)$ is well prepared.

(1.10) From the proof of the abore theorem,the following useful result may be deduced:

(1.10.1) " If v is a well prepared vertex of $\Delta_{x,y}(f)$, then v is a vertex of $\Delta_x(I)$ also.

2. VERY WELL PREPARATION

We shall study the relevant coordinate changes in order to obtain new polygons.

(2.1) A set $\Delta \subset \mathbb{R}^2_0$ is said to be a "discrete F-set" iff it is positively convex and it has its vertices on $(\mathbb{Z}_o/d)^2$ for some $d \in \mathbb{Z}_o$. For a discrete F-set Δ we shall denote by

(2.1.1) $$v_i(\Delta) = (\alpha_i(\Delta), \beta_i(\Delta))$$

$i=1,\ldots,t$ its vertices, arranged by increasing abscissas. We shall denote by $l_i(\Delta)$ the length of the segment joining $v_i(\Delta)$ and $v_{i+1}(\Delta)$ and we shall denote $-1/\varepsilon_i(\Delta)$ the slope of this segment.

(2.2). For a discrete F-set ,the "characteristic sequence" $s(\Delta)$ of Δ will be defined by.

(2.2.1) $$s(\Delta) = (\alpha_1(\Delta), \beta_1(\Delta), \varepsilon_1(\Delta), -l_1(\Delta), \ldots, {}_t(\Delta), -l_t(\Delta),)$$

Given two discrete F-set, we shall write $\Delta \leq \Delta'$ iff $s(\Delta')$ is bigger than $s(\Delta)$ for the lexicographic order. This gives a total ordering in the discrete F-sets strictly finer than the inclusion ordering.

(2.3) Let us identify $\mathbb{R}^2 \leq P^2(\mathbb{R})$, the added infinitum line corresponding to the third homogeneus coordinate equal zero. Let us consider the subset $\mathbb{H}$ of $P^2(\mathbb{R})$ defined by.

(2.3.1) $$\mathbb{H} = \{[(-a,1,b)];\ a,b \in \mathbb{Z}_0,\ (a,b) \neq (0,0)\}.$$

Given a discrete F-set Δ, we shall define the set $T(\Delta)$ as the set of all straigth lines $L \in P^2(\mathbb{R})$ such that L meets Δ only at the border and $L \cap \mathbb{H} \neq \emptyset$.

(2.4) Lemma.- Let $t=(x_1,x_2,y)$ be a regular system of parameters as in (1.3) and let $f=(f_1,\ldots,f_m)$ be a tangential base of I such that $In_M(f_i) \in K[Y]$. Let us fix $L \in T(\Delta)$, where $\Delta = \Delta_{x,y}(f)$. Let σ be a $K|X_1,Y|$-automorphism of $Gr_M(R)$ such that

(2.4.1)
$$\sigma(X_2) = X_2 + \sum_B \lambda_B X_1^a Y^B$$

where $[(-a,1, |B|)] \in L \cap \mathbb{H}$ and $\lambda_B \in k$. Let us consider

(2.4.2)
$$x'_2 = x_2 + \sum_B g_B x_1^a y^B$$

where the residual class of g_B is λ_B. Then

i) If s is the first index such that $v_s(\Delta) \in L \cap \Delta$, then Δ and $\Delta' = \Delta_{x',y}(f)$, where $x'=(x_1,x'_2)$, have exactly the same vertices until $v_s(\Delta) = v_s(\Delta')$.

ii) A vertex $v_i(\Delta) = v_i(\Delta')$ is well prepared with respect to (x,y) iff it is well prepared with respect to (x',y), for each $i \leq s$.

iii) If $e : \mathbb{R}^2 \longrightarrow \mathbb{R}$ is a positively linear function such that e=1 defines L, then

(2.4.3)
$$\Delta' \subset \{ \alpha \in \mathbb{R}^2_o ;\ e(\alpha) \geq 1 \}.$$

Proof.- Let $t=(x,y)$, $t'=(x',y)$ and let us denote the ideals of (1.4.1) by $R_{e,t;\nu}$ to indicate their dependence on the parameters. Let $d_i = ord_M(f_i)$, $i=1,\ldots,m$ and let us denote by.

(2.4.4)
$$E(x,y;\ f_i;\ L) \qquad i = 1,\ldots,m$$

the set of all positively linear functions $e : \mathbb{R}^2 \longrightarrow \mathbb{R}$ such that the slope of e=1 is strictly smaller than the slope of L and

(2.4.5)
$$f_i \in R_{e,t;d_i}$$

Now, in viem of (1.6), to prove i) and iii) it sufices to prove that

(2.4.6)
$$E(x,y;\ f_i;\ L) = E(x',y;\ f_i;\ L)$$

for all i=1,...,. But this is a consequence of the fact that

(2.4.7) $$R_{e,t;\nu} = R_{e,t';\nu} \qquad \nu \in \mathbb{R}$$

for all e such that the slope of e=1 is strictly smaller than the slope of L. Moreover, for such an e, one has an isomorphism of graded k-algebras

(2.4.8) $$\psi : Gr_{e,t}(R) \longrightarrow Gr_{e,t'}(R)$$

given by

$$\psi(In_{e,t}(x_1)) = In_{e,t'}(x_1)$$

(2.4.9) $$\psi(In_{e,t}(x_1)) = In_{e,t'}(x'_1)$$

$$\psi(In_{e,t}(y_i) = In_{e,t'}(y_i) \qquad i=1,\dots,r$$

which is compatible with the k-isomorphism of (1.4.2). Thus the condition (1.7.2) may be enounced equivalently for t or t' and ii) is proven.

(2.5) Corollary.- With notations as above, if $\Delta = \Delta_x(I)$ and $\Delta' = \Delta_{x'}(I)$, then Δ and Δ' have exactly the same vertices until $v_s(\Delta) = v_s(\Delta')$ and

(2.5.1) $$\Delta' \subset \{\alpha \in \mathbb{R}^2_0;\ e(\alpha) \geq 1\}.$$

Proof.- We can choose f, t=(x,y) in such a way that

(2.5.2) $$\Delta_x(I) = \Delta_{x,y}(f)$$

(this is always possible |4|,|6|). Now, the first assertion follows from (1.10.1) and the second from the fac that

(2.5.3) $$\Delta_{x'}(I) \subseteq \Delta_{x,y}(f)$$

(see |6|).

(2.6) Let us fix a coefficient set $C \subset R$. The remaining of this paragraph is devoted to standarize, relatively to C, the modifications on f, x_2, y which one must to make for reaching $\Delta_{x'}(I)$ from $\Delta_{x,y}(f)$, where $x=(x_1,x_2)$, $x'=(x_1,x'_2)$ are both systems of tangential parameters.

(2.7) Let t=(x,y) be a regular system of parameters as in (1.3) and let $f=(f_1,\dots,f_m)$ be a tangential base of I such that $In_M(f_i) \in k|Y|$ for all i. For a not well-prepared vertex v of $\Delta_{x,y}(f)$ there are two posibilities (which exclude ono to another).

(2.7.1) "Condition (1.7.2) fails with σ = identity".

(2.7.2) "Condition (1.7.2) fails with $\sigma \neq$ identity".

If (2.7.1) holds we shall say that " y is x-prepared with respect to f and v".

(2.8.) If (2.7.2) holds, then there is a unique sequence $g = (g_1, \ldots g_r)$ with $g_i \in C$, $g \neq 0$, such that if we set

(2.8.1) $$y'_i = y_i + g_i x^v \qquad i = 1, \ldots, r$$

then there are two posibilities:

(2.8.2) $$\Delta_{x,y'}(f) \subset \Delta_{x,y}(f) - \{v\}.$$

(2.8.3) $\Delta_{x,y'}(f) = \Delta_{x,y}(f)$ and y' is x-prepared with respect to f and v.

We shall call the change $y \longmapsto y'$ a "well preparation change relatively to C, t, f and v".

Notice that since $|v| < 1$ we have also $\mathrm{In}_M(f_i) \in k|Y'|$.

(2.9) If (2.7.1) holds, let us consider the set

(2.9.1) $$\Lambda = \{j \in [1,m] ; \quad v \notin \Delta_{x,y}(f_j)\}.$$

Let j be the smallest index among those $j \in [1,m] - \Lambda$ such that $d_j = \mathrm{ord}_M(f_j)$ is the minimum possible. Now, for each couple (i,α) with $i \in \Lambda$, $\alpha \in \mathbb{Z}^2$, such that there exists $d \in \mathbb{Z}_0$ with

(2.9.2) $$(d_j - d_i - d)\, v = \alpha$$

one can find element $h_{j\alpha i} \in R$ with

(2.9.3) $$h_{j\alpha i} = \sum_{|\beta|=d} h_{j\alpha i} y^\beta$$

where $h_{j\alpha i} \in C$, in such a way that if we make

(2.9.4) $$f'_j = f_j - \textstyle\sum_\alpha x^\alpha \sum_{i \in \Lambda} h_{j\alpha i}\, f_i$$

then one has that $v \notin \Delta_{x,y}(f'_j)$ (see |6|). Thus, by making changes in f as (2.9.4) one may reach $\Lambda = |1,m|$ and so $v \notin \Delta_{x,y}(f')$, f' being the new tangential base obtained.

A change $f \longmapsto f'$ as above will be called a "well preparation change of tangential base, relatively to C, t, f and v".

(2.10) <u>Remark</u>.- If v and v' are two vertices as iu (2.9), then there is a conmutativity in the following sense. If $f \longmapsto f'$ is a well preparation relatively to v and $f' \longmapsto f''$ is a well preparation relatively to v' (the status of v' does not change if we made $f \longmapsto f'$), the coefficients $h_{j\alpha i\beta}$ of $f' \longmapsto f''$ are the same as the coefficients of the well preparation change $f \longmapsto f'''$ relatively to v'.

(2.11) The changes $y \longmapsto y'$ and $f \longmapsto f'$ of (2.8) and (2.9) defines both convergent situations by making them successively in all possible vertices and the limits $(x,y^{\sim})$ and $f^{\sim}$ verify that

$$\Delta_x(I) = \Delta_{x,y^{\sim}}(f^{\sim})$$

(see |6|).

(2.12) Definition.- Let $t = (x_1,x_2,y)$ be as in (1.3) and let $f = (f_1,\dots,f_m)$ be a tangential base of I such that $In_M(f_i) \in k|Y|$ for all i. Let us suppose that $\Delta_{x,y}(f) = \Delta_x(I)$. A sequence

(2.12.1) $$S = \{((x(j),\ y(j),\ f(j))\}_{j\geq 0}$$

with $x(j) = (x_1, x_2(j))$, will be calles a "very well preparation sequence for I beginning at t,f and relatively to C" iff one has that

a) $x(0) = x$, $y(0) = y$, $f(0) = f$

b) Let us denote $\Delta(j) = \Delta_{x(j),y(j)}(f(j))$. Let L_1 be the element of $T(\Delta(0))$ of the smallest slope and let $L_j \in T(\Delta(j-1))$ be the element of the smallest slope strictly bigger than the slope of L_{j-1}, for $j \geq 2$. Then, there exixt $g_B \in C$ such that

(2.12.2) $$x_2(j) = x_2(j-1) + \textstyle\sum_B g_B\, x_1 \ |y(j-1)|^B, \quad j \geq 1$$

where $[(-a,1,|B|)] \in L_j \cap \mathbb{H}(\Delta(j-1))$.

c) The changes $y(j-1) \mapsto y(j)$ and $f(j-i) \mapsto f(j)$ are obtained from the following algorithm: take the first vertex v of $\Delta_{x(j),y(j-1)}(f(j-1))$ wich is not well prepared and make a well preparation of $y(j-1)$ followed from a well preparation of $f(j-1)$ and repeat. The algorithm stops when $\Delta(j)$ has all its vertices well prepared until the vertex of biggest abscissa in $L(j+1) \cap \Delta(j)$.

(2.13.) Remark.- The algorithm in c) is always finite. Actually, let Γ be the triangle defined by the x-axis, $L(j)$ and the line passing throught the vertex of smallest abscissa in $L(j) \cap \Delta(j-1)$ and having the smallest slope $-1/m$, $m \in \mathbb{Z}_0$, strictly bigger than the slope of $L(j)$. Then, the vertex v in the algorithm may alway be taken in Γ.

(2.14) The limit of a very well preparation sequence is defined in an obvions way. If $(t^{\sim}, f^{\sim})$, with $t^{\sim} = (x^{\sim}, y^{\sim})$, is the limit of S, then one has that

(2.14.1) $$\Delta_{x^{\sim}}(I) = \Delta_{x^{\sim}, y^{\sim}}(f^{\sim}).$$

This is a consequence of lemma (2.4) since a vertex v remains unchanged from a certain step of the sequence and this is compatible with the limit change, because we deal with initial forms (with respect to the filtrations of (1.4)).

(2.15) Theorem.- "Let $x = (x_1, x'_2)$ and $x' = (x_1, x'_2)$ be two systems of tangential parameters. Let $t = (x, y)$ and f be such that

(2.15.1) $$\Delta_{x,y}(f) = \Delta_x(I).$$

Then, there exists a very well preparation sequence S beginning at t,f and relatively to C such that if $((x^{\sim}, y^{\sim}), f^{\sim})$ is its limit one has that

(2.15.2) $$\Delta_{x^{\sim}, y^{\sim}}(f^{\sim}) = \Delta_{x'}(I).$$

Proof.- Let $t' = (x', y')$ be as in (1.3). We have that

(2.15.3) $$\sum_{i=1,\dots,r} Y_i \, Gr_M(R) = \sum_{i=1,\dots,r} Y'_i \, Gr_M(R)$$

so, there exists a unit $u \in R$ such that

(2.15.4) $$u\, x'_2 = x_2 + \sum_{a,B} g_{a,B}\, x_1^a\, y^B$$

where $g_{a,B} \in C$. We can suppose u=1. Let $\Delta = \Delta_x(I)$. First, if for each positively linear function e such that $e \geq 1$ on Δ and e=1 intersects Δ one has that $In_e(x_2) = In_e(x'_2)$ (the filtration relatively to $t = (x,y)$), then, by applying the lemma (2.4), all the vertices of $\Delta_{x',y}(f)$ are well prepared and the trivial sequence solves the problem. If this is not the case, let us select e such that e=1 have the smallest slope and $In_e(x'_2) \neq In_e(x_2)$. Then (2.15.4) takes the form

(2.15.5) $$u\, x'_2 = x_2 + \sum_{e(-a,1) = |B|} g_{a,B}\, x_1^a\, y^B + \sum_{e(-a,1) > |B|} g_{a,B}\, x_1^a\, y^B$$

Now, set

(2.15.6) $$x_2(1) = x_2 + \sum_{e(-a,1) = |B|} g_{a,B}\, x_1^a\, y^B$$

Then, after making well preparations succesively to obtain $y \longmapsto y(1)$ and $f \longmapsto f(1)$ as in (2.12.), in view of (2.4) and (2.13) one has that

(2.15.7) $$u\, x_2' = x_2(1) + \sum_{e(-a,1) > |B|} g_{a,B}\, (1) x_1^a\, |y(1)|^B$$

Now, by repeating this procedure and taking limits one reach the first situations.

(2.16) Corollary.- Let t and f be as above. Then the supremum of the set

(2.16.1) $\{\Delta = \Delta_{x',y'}(f');\ x' = (x_1, x_2'),\ t = (x',y')$ is as in (1.3) and f' is a tangential base$\}$.

is the same as the supremum of the set

(2.16.2) $\{\ \Delta = \Delta_{x^{\sim},y^{\sim}}(f^{\sim});\ ((x^{\sim},y^{\sim}),f^{\sim})$ is the limit of a sequence of very well preparation beginning at $t,f,\}$.

3. THE CHARACTERISTIC POLYGON $\Delta_{x_1}(I)$.

(3.1) Definition.- Let x_2 be an element of $M \subset R$ such that there exists x_2 with $x = (x_1, x_2)$ being a system of transversal parameters. Then the "characteristic polygon $\Delta_{x_1}(I)$" is defined to be the supremum of the set (2.16.1).

(3.2) Remark.- $\Delta_{x_1}(I)$ always exist. Indeed $v_1(\Delta_{x,y}(f)) + \mathbb{R}_o^2$ is an upper bound for the elements in (2.16.1).

(3.3) We shall use the following elementary fact

Lemma.- Let k be no numerable and algebraically closed. Let $\{C_i\}$ be a sequence of constructible sets of $\mathbb{A}^n(k)$ such that $C_i \neq 0$ and $C_i \geq C_{i+1}$ for all i. Then the intersection of the whole family is not empty.

(3.4) Theorem.- Let us suppose that R has a coefficient field k which is algebraically closed and no numerable. Let x_1 be as in (3.1). Then, for each

$t = (x_1,x_2,y)$, f as in (2.12.), there exists a sequence of well preparation beginning at t,f and relatively to k such that

(3.4.1) $$\Delta_{x_1}(I) = \Delta_{x^\sim,y^\sim}(f^\sim)$$

where $(x^\sim = (x_1,x_2^\sim),\ y^\sim, f^\sim)$ is the limit of the sequence.

<u>Proof</u>.- Let $\Delta^* = \Delta_{x_1}$ and let H_i, $i \geq 1$, be the elements of $T(\Delta^*)$ arranged by increasing slope. For each $i \geq 1$, let us consider the regions

(3.4.2) $$U_i = ((\alpha_{j(i)},0) + \mathbb{R}_0^2) \cap \{e_i(\alpha) \geq 1\} - \Delta^*$$

(3.4.3) $$R_i = U_i - U_{i+1}$$

where $\alpha_{j(i)}$ is the abscissa of the first vertex in $\Delta^* \cap H_i$ and $e_i = 1$ defines H_i.

In view of (2.16), for each $n \geq 1$, there exists a very well preparation sequence S(n) such that

(3.4.4) $$L_i(n) = H_i \qquad i = 1,\dots,n.$$

where $L_i(n)$ denotes the lines which appear in (2.12). Thus, the vertices of Δ^* and the vertices of the polygon $\Delta(n)$ given by the limit of S(n) agree until $v_{j(n)}(\Delta^*)$.

For a vertex v, let us denote by $g_{i,v}$ and $h_{j\alpha i\beta,v}$ the coefficients in (2.8.1) and (2.9.3) and for a line L, let us denote by $g_{B,L}$ the coefficients in (2.12.2). Now, the conditions on the coefficients which participate on the changes of S(n) in order to have no vertices in R_l, $l = 1,\dots,n$, are polynomial relations

(3.4.5) $$\rho_l(\{g_{i,v}\}, \{h_{j\ i\beta,v}\}, \{g_{B,H_t}\}) = 0$$

where $t = 1,\dots,l$ and v ranges over $R_1 \cup \dots \cup R_l$. The relations of (3.4.5) depend only on the initial data and ρ_l for a fixed l do not depend on the $n \geq 1$.

Because $\Delta^* = \Delta_{x_1}$, $\rho_l = 0$ has non empty solution for each l. Thus, by applying lemma (3.3), the projection of all solutions over the space of

$$\{g_{i,v},\ h_{j\alpha i\beta,v},\ g_{B,H_1}\}$$

where v ranges over R_1, is not empty. A similar argument shows that there is a common solution of (3.4.5) for all . This solution gives us the construction of the desired very well preparation sequence

4. THE CHARACTERISTIC POLYGON $\Delta(I)$

(4.1.) We shall construct a polygon $\Delta(I)$ which depends only on I. It is the maximum of the $\Delta_x(I)$ for an ordering introduced below and it is strongly related with polygon $\Delta_{x_1}(I)$ of the preceding paragraph.

(4.2) Let Δ be a discrete F-set and let $e(x,y) = x+y$. We shall define

(4.2.1) $\delta(\Delta) = \min\{e(\alpha);\ \alpha \in \Delta\}$.

$p(\Delta) = (\gamma(\Delta), \psi(\Delta))$ = vertex of lowest abcissa such that $e(p(\Delta)) = \delta(\Delta)$.

(4.2.3) $lp(\Delta)$ = length of the segment of slope -1 in Δ.

4.2.4) $sim(\Delta)$ = simetric of Δ with respect to the diagonal $x-y = 0$.

Let us consider the set

(4.2.5) $A = \mathbb{R}^3 \times \{\text{discrete F-sets}\}^2$

with the lexicographic order. We shall define $t(\Delta) \in A$ as follows

(4.2.6)
$$t(\Delta) = (\delta(\Delta),\ -lp(\Delta),\ \psi(\Delta),\ \Delta \cap ((\gamma(\Delta),0) + \mathbb{R}^2_0),\ sim(\Delta) \cap ((\gamma(sim(\Delta)),0) + R^2_0))).$$

Then t defines a monic mapping from the discrete F-sets to A. We shall denote the induced order by $\prec$, i.e.

(4.2.7) $\Delta \prec \Delta' \iff t(\Delta) \leq t(\Delta')$.

(4.3) Remark.- If $x=(x_1,x_2)$ and $x'=(x_1,x_2')$ are system of tangential parameters, then one has that

(4.3.1) $\Delta_x(I) \leq \Delta_{x'}(I) \iff \Delta_x(I) \prec \Delta_{x'}(I)$.

This is a consequence of the fact that the changes involved in a very well preparation sequence do not affect the vertices of the polygon until $p(\Delta)$. Another consequence of this is that if $x=(x_1,x_2)$, $x'=(x_1',x_2')$ are systems of tangential parameters, then one has that

(4.3.2) $\delta(\Delta_x(I)) = \delta(\Delta_{x'}(I))$

and thus δ is an intrinsic character of the singularity which may be calculated directly from any $\Delta_x(I)$. (For proving (4.3.2) it suffices to divide $(x_1,x_2) \longmapsto (x'_1,x'_2)$ in $(x_1,x_2) \longmapsto (x_1,x'_2)$ and $(x_1,x'_2) \longmapsto (x'_1,x'_2)$ which is always possible).

(4.4) Definition.- We shall say that $\Delta(I)$ is the "characteristic polygon of I" iff $\Delta(I)$ is the supremum of

(4.4.1) $$\{\Delta_x(I);\ x \text{ is a system of tangential parameters}\}.$$

(4.5) Remark.- As a consequence of (4.3.2), $\Delta(I)$ always exists, since elements in (4.4.1) are bounded by

(4.5.1) $$(0,\delta(\Delta_x(I)) + \mathbb{R}^2_0 .$$

(4.6) Definition.- We shall say that a system of tangential parameters x is "adecuate" iff the couple

(4.6.1) $$(-\,\mathrm{lp}\,(\Delta_x(I)),\ \psi(\Delta_x(I)))$$

is maximum over all polygons ine (4.4.1)) for the lexicographic order.

(4.7) Given $x = (x_1,x_2)$, one may obtain an adecuate system $x' = (x'_1,x'_2)$ by making a linear change

(4.7.1) $$\begin{pmatrix} x'_1 \\ x'_2 \end{pmatrix} = \begin{pmatrix} g_1 & g_2 \\ g_3 & g_4 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}$$

where $g_i \in C$ (coefficient set), $i=1,\ldots,4$. This follows from lemma (2.4) and a calculation with initial forms.

(4.8) Theorem.- Let $t=(x,y)$ and f be such that x is adecuate and $\Delta_{x,y}(f) = \Delta_x(I)$ (with assumptions as usual for f and y). Let $x'=(x'_1,x'_2)$ be another adecuate system of tangential parameters. Then, after making if necessary an order change in (x'_1,x'_2), there exists a very well preparation sequence S beginning at t,f and relatively to C such that if $((x_1,x_2^{\sim},y^{\sim}),f^{\sim})$ is the limit of S, there exists a very well preparation sequence S' beginning at $(x_2^{\sim},x_1,y^{\sim})$, $f^{\sim}$ and relatively to C such that if $((x_2^{\sim},x_1^{\sim},y^{\sim\sim}),f^{\sim\sim})$ is the limit of S' one has that

(4.8.1) $$\Delta_{x^{\sim},y^{\sim\sim}}(f^{\sim\sim}) = \Delta_{x'}(I)$$

where $x^{\sim} = (x_1^{\sim}, x_2^{\sim})$.

Proof.- We may suppose that (x_1, x_2') is a tangential system of parameters. By theorem (2.15), there exists S such that

(4.8.2) $$\Delta_{x''}(I) = \Delta_{x''',y^{\sim}}(f^{\sim})$$

where $x'' = (x_1, x_2')$ and $x''' = (x_1, x_2^{\sim})$. Now it is enougth to prove that $(x_1', x_2^{\sim})$ is a tangential system of parameters such that

(4.8.3) $$\Delta_{x''''}(I) = \Delta_{x'}(I)$$

where $x'''' = (x_1', x_2^{\sim})$ and $x' = (x_1', x_2')$. In this case, a similar argument gives us the thesis. Now, the proof of (4.8.3) follows from a systhematic use of lemma (2.4) and an analysis of the proof of theorem (2.15).

(4.9) Corollary.- a) Let us suppose that R has a coefficient field k which is algebraically closed and no numerable. Then, there exists a system of tangential parameters $x=(x_1,x_2)$ such that

(4.9.1) $$\Delta_x(I) = \Delta(I).$$

b) Moreover $\Delta(I)$ may be reached in the following way: Let $t=(x,y)$, f be such that x is adecuate and such that $\Delta_x(I) = \Delta_{x,y}(f)$. Let $((x_1, x_2^{\sim}, y^{\sim}), f^{\sim})$ be the limit of a very well preparation sequence beginning at t, f and relatively to k such that

(4.9.2) $$\Delta_{x_1}(I) = \Delta_{x',y^{\sim}}(f^{\sim})$$

where $x' = (x_1, x_2^{\sim})$. Let $((x_2^{\sim}, x_1^{\sim}, y^{\sim}), f^{\sim\sim})$ be the limit of a very well preparation sequence beginning at $(x_2^{\sim}, x_1, y^{\sim})$, f and relatively to k such that

(4.9.3) $$\Delta_{x_2^{\sim}}(I) = \Delta_{x'^{\sim},y^{\sim}}(f^{\sim\sim})$$

where $x'^{\sim} = (x_2^{\sim}, x_1^{\sim})$. Let $x^{\sim} = (x_1^{\sim}, x_2^{\sim})$. Then, one has thas

(4.9.4) $$\Delta(I) = \Delta_{x^{\sim},f^{\sim}}(f^{\sim\sim})$$

or

(4.9.5) $$\Delta(I) = \Delta_{x',y^{\sim}}(f^{\sim\sim}).$$

Proof.- Let us suppose that

(4.9.6) $$\Delta_{x^{\sim}}(I) \succ \Delta_{x'^{\sim}}(I)$$

If $\Delta_{x^{\sim}}(I)$ is not $\Delta(I)$, then, ther exist $x'' = (x_1', x_2')$ such that

(4.9.7) $$\Delta_{x^{\sim}}(I) \not\succeq \Delta_{x''}(I)$$

By theorem (4.8), we may suppose that

A. $((x_1, x_2', y''), f'')$ is the limit of a very well preparation sequence beginning at t,f and relatively to k and $((x_2', x_1', y'), f')$ is the limit of a very well preparation sequence beginning at $(x_2', x_1, y''), f''$ and relatively to k.

B. There exists $((x_2^{\sim}, x_1'', y'''), f''')$ which is the limit of a very well preparation sequence begining at $(x_2^{\sim}, x_1^{\sim}, y^{\sim\sim}), f^{\sim\sim}$ and relatively to k. And there exists $((x_1'', x_2'', y''''), f'''')$ which is the limit of a very well preparation sequence beginning at $(x_1'', x_2^{\sim}, y'''), f'''$ and relatively to k, in such a way that

(4.9.8) $$\Delta_{(x_1'', x_2^{\sim})}(I) = \Delta_{(x_1', x_2^{\sim})}(I)$$

(4.9.9) $$\Delta_{(x_1'', x_2'')}(I) = \Delta_{(x_1', x_2')}(I)$$

Now, from the statement A one has that

(4.9.10) $$\Delta_{(x_1, x_2^{\sim})}(I) \geq \Delta_{(x_1, x_2')}(I)$$

But, since we have that

(4.9.11) $$\Delta_1(\Delta_{(x_1, x_2^{\sim})}(I)) = \Delta_1(\Delta_{(x_1, x_2')}(I))$$

where $\Delta_1(\Delta) = \Delta \cap (\alpha \geq \gamma(\Delta))$, then in view of (4.9.7) necessarily

(4.9.12) $$\Delta_{(x_1, x_2^{\sim})}(I) = \Delta_{(x_1, x_2')}(I)$$

On the other haud, from the statement B we have that

(4.9.13) $$\Delta_{x_2^{\sim}}(I) = \Delta_{(x_2^{\sim}, x_1^{\sim})}(I) \geq \Delta_{(x_2^{\sim}, x_1')}(I)$$

Let us denote $\Delta_2(\Delta) = \text{sim}\ [(\text{sim}\Delta) \cap (\alpha \geq \gamma(\text{sim}\Delta))]$. Then, from (4.9.13) one has that

(4.9.14) $$\Delta_2(\Delta_{(x_1^{\sim}, x_2^{\sim})}(I) \geq \Delta_2(\Delta_{(x_1', x_2^{\sim})}(I)) = \Delta_2(\Delta_{(x_1', x_2')}(I)).$$

Since Δ_2 is not changed by $x_2^{\sim} \mapsto x_2''$. Now, we have that

(4.9.15) $\Delta_1(\Delta_{(x_1^{\sim},x_2^{\sim})}(I)) = \Delta_1(\Delta_{(x_i,x_2^{\sim})}(I)) = \Delta_1(\Delta_{(x_1,x_2'')}(I)) = \Delta_1(\Delta_{(x_1',x_2')}(I))$

Thus, from (4.9.14) and (4.9.15) and since $(x_1^{\sim},x_2^{\sim})$ and (x_1',x_2') are adecuate

(4.9.16) $$\Delta_{(x_1',x_2')}(I) \prec \Delta_{(x_1^{\sim},x_2^{\sim})}(I)$$

which is the desired contradiction.

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