# PERIODS AND GAUSS-MANIN CONNECTION FOR THE MUMFORD CURVE $y_2^{r_2}y_1^{r_1} - y_2^{r_2} - y_1^{r_1} + \lambda = 0$

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A method is introduced which allows to obtain explicit formulas for the periods $q(\lambda)$ of a family $(C_\lambda)$ of smooth curves. It gives expressions for $q(\lambda)$ in the vicinity of a point $\lambda_o$ for which the curve $C_{\lambda_o}$ is totally degenerate provided one knows Picard-Fuchs equations for differentials of the family $(C_\lambda)$.

Techniques from rigid analytic geometry are used, see [T]. We work with the notion of periods for p-adic Schottky groups as defined by Manin-Drinfeld, [MD]. The result can certainly be applied to the usual complex periods. In this approach it is basic that one has a canonical basis for the De Rham cohomology classes.

In this manuscript only one example is treated. The curves $C_\lambda^r$ given by the equation in the title are prestable and totally degenerate for $\lambda = 1$. The p-adic Schottky uniformization is constructed in section 2. In section 3 a crucial formula for the Gauss-Manin connection is explained. The main application is the expression for the periods in proposition 4 of section 4. For elliptic curves the result is classical, see [F]. It is planned to give a more complete account of this method in a joint paper with F. Herrlich. The relation to the work of B. Dwork, [D], shall be included.

## 1. The curve $C_\lambda^r$

Let K be a field of characteristic 0 and $r = (r_1, r_2)$ a pair of integers $\geq 2$. Assume that there is a primitive root of unity $\rho_i$ of order $r_i$ in K.

Let $(y_1,y_2)$ be a system of inhomogeneous coordinates for $\mathbb{P}\times\mathbb{P}$, where $\mathbb{P}$ is the projective line over K and let $\lambda$ be a parameter in K. The equation

$$y_2^{r_2}y_1^{r_1} - y_2^{r_2} - y_1^{r_1} + \lambda = 0$$

defines a projective curve $C_\lambda^r$ in $\mathbb{P}\times\mathbb{P}$.

If $u_i$, $v_i$ are homogeneous variables for $\mathbb{P}$ with $y_i = \frac{u_i}{v_i}$, then $C_\lambda^r$ is the set of zeroes of the bihomogeneous equation

$$u_2^{r_2}u_1^{r_1} - u_2^{r_2}v_1^{r_1} - v_2^{r_2}u_1^{r_1} + \lambda v_2^{r_2}v_1^{r_1} = 0$$

The curve $C_\lambda^r$ is non-singular if and only if $\lambda(\lambda-1)\neq 0$. The curve $C_\lambda^r$ is a union of $r_1\cdot r_2$ projective lines and prestable.

Let $\sigma_1$ (resp. $\sigma_2$) be the automorphism on $\mathbb{P}\times\mathbb{P}$ for which

$$y_1\circ\sigma_1 = \rho_1\cdot y_1,\quad y_2\circ\sigma_1 = y_2.$$

(resp. $y_1\circ\sigma_2 = y_1$, $y_2\circ\sigma_2 = \rho_2\cdot y_2$).

The restriction $\sigma_1|C_\lambda^r$ of $\sigma_i$ onto $C_\lambda^r$ is an automorphism of $C_\lambda^r$ and $\sigma_1\circ\sigma_2 = \sigma_2\circ\sigma_1$. Let G denote the group generated by $\sigma_1|C_\lambda^r$ and $\sigma_2|C_\lambda^r$. It is canonically isomorphic to $\mathbb{Z}/r_1\mathbb{Z}\oplus\mathbb{Z}/r_2\mathbb{Z}$.

The field of K-rational functions of $C_\lambda^r$ is generated by $y_1|C_\lambda^r$ and $y_2|C_\lambda^r$ if $\lambda\neq 0$, $\lambda\neq 1$. We will write in the sequel $y_i$ instead of $y_i|C_\lambda^r$ and define $x_i$ to be $y_i^{r_i}$. Then

$$\frac{dx_1}{x_1-1} = -\frac{dx_2}{x_2-1}$$

because the rational functions $x_1$, $x_2$ satisfy the relation

$$x_2x_1 - x_2 - x_1 + \lambda = 0$$

and thus

$$x_2dx_1 + x_1dx_2 - dx_2 - dx_1 = 0$$



$$(x_1-1)dx_2 + (x_2-1)dx_1 = 0.$$

Let $I := \{i = (i_1,i_2)\in\mathbb{Z}^2 : 1\leq i_1 < r_1,\ 1\leq i_2 < r_2\}$.

For $i = (i_1,i_2)\in I$ we define

$$\omega_i = \frac{dx_1}{y_1^{i_1}y_2^{i_2}(x_1-1)} = -\frac{dx_2}{y_1^{i_1}y_2^{i_2}(x_2-1)}$$

$$\omega_i' = \frac{dx_1}{y_1^{i_1}y_2^{i_2}}$$

Then the De Rham cohomology vectorspace $H^1_{DR}(C_\lambda^r)$ admits a direct decomposition

$$\bigoplus_{i\in I}\langle\omega_i,\omega_i'\rangle$$

where $\langle\omega_i,\omega_i'\rangle$ denotes the K-vectorspace of differentials generated by $\omega_i$ and $\omega_i'$. In fact $\langle\omega_i,\omega_i'\rangle$ is the eigenspace of the canonical action of G on $H^1_{DR}(C_\lambda^r)$ with respect to the character $\chi : G\to K^*$ for which $\chi(\sigma_1) = \rho_1^{-i_1}$, $\chi(\sigma_2) = \rho_2^{-i_2}$.

As dim $H^1_{DR}(C_\lambda^r) = 2(r_1-1)(r_2-1)$ the genus of $C_\lambda^r$ is $(r_1-1)(r_2-1)$.

## 2. p-adic uniformization

Let now K be complete with respect to non-archimedean valuation $|\,|$ and assume that $|\lambda-1| < 1$ and that $r_1\cdot r_2$ is prime to the characteristic of the residue field. I want to show that $C_\lambda^r$ is a Mumford curve. This will be achieved by constructing the non-archimedean or p-adic Schottky uniformization for $C_\lambda^r$.

Let z be a coordinate for $\mathbb{P}$, and $s\in K$, $|s-1| < 1$, $s\neq 1$ and let

$$\sigma_1(z) = \rho_1\cdot z$$

$$\sigma_2(z) = \frac{(s-\rho_2)z + (\rho_2-1)s}{(1-\rho_2)z + (\rho_2 s-1)}.$$

Then $\sigma_1$, $\sigma_2$ are elliptic fractional linear transformation of $\mathbb{P}$ and $\sigma_2$ has the multiplier $\rho_2$ and the fixed points 1 and s. One can show that the group $\langle\sigma_1,\sigma_2\rangle$ is discontinuous in the sense of [GP], Chap. I, §1, and that the commutator subgroup $\Gamma$ of $\langle\sigma_1,\sigma_2\rangle$ is a free group freely generated by $\{\gamma_i := \sigma_1^{i_1}\sigma_2^{i_2}\sigma_1^{-i_1}\sigma_2^{-i_2} : i \in I\}$, see [GH].

Let $z_1 := z$. $z_2 := \frac{z-s}{z-1}$. Then

$$z_1 = \frac{z_2-s}{z_1-1} \, .$$

Let $\Gamma_i$ the group generated by $\Gamma \cup \{\sigma_i\}$ $i = 1,2$.

Define

$$y_1 := \prod_{\gamma\in\Gamma_2} \frac{z_1 \circ \gamma}{(z_1\circ\gamma)(1)}$$

$$y_2 := \prod_{\gamma\in\Gamma_1} \frac{z_2 \circ \gamma}{(z_1\circ\gamma)(\infty)} \, .$$

Both products converge on the domain Z of ordinary points for $\Gamma$. They are both meromorphic on Z and are $\Gamma$-automorphic forms on Z with constant factors of automorphy, see [GP], Chap. II, §2.

A direct computation gives

$$y_1 \circ \sigma_1 = \rho_1 \cdot y_1$$

$$y_2 \circ \sigma_2 = \rho_2 \cdot y_2 .$$

One can conclude that $y_1^{r_1}$, $y_2^{r_2}$ are $\langle\sigma_1,\sigma_2\rangle$-automorphic and that $y_1,y_2$ are $\Gamma$-invariant, see [GP], Chap. III, §1, for the notions.

Let $\lambda := y_1^{r_1}(s)$

Proposition 1: The mapping $z \to (y_1(z), y_2(z))$ gives a bianalytic mapping between the Mumford curve $Z/\Gamma$ and the curve $C_\lambda^r$.

Proof: see [GH].

Remark: The mapping

$$s \to \lambda(s)$$

is a bianalytic mapping between $\{s \in K : |1-s| < 1\}$ and



$\{\lambda \in K : |1-\lambda| < 1\}$ with $\lambda(1) = 1$. Moreover $\lambda(s^{-1}) = \lambda(s)^{-1}$.

## 3. Gauss-Manin connection

There are canonical analytic $\Gamma$-automorphic forms with constant factors of automorphy such that $\alpha_i := \frac{du_i}{u_i}$ are analytic differentials on $C_\lambda^r$ and such that $\{\alpha_i : i \in I\}$ is a basis of the K-vectorspace of analytic differentials on $C_\lambda^r$, see [GP], Chap. II, §4.

Let $q_{ij} := \frac{u_i \circ \gamma_j}{u_i} \in K^*$. The matrix $q := (q_{ij})$ is the period matrix of $\Gamma$ with respect to the basis $\{\gamma_i : i \in I\}$, see [MD], §2. Also there are meromorphic functions $\zeta_i$ on Z such that $\zeta_i - \zeta_i \circ \gamma_j = \begin{cases} 1 : i = j \\ 0 : i \neq j \end{cases}$, see [G2], p. 387, and [G1], section 3.

The differentials $\beta_i := d\zeta_i$ are of the second kind and $\{\alpha_i : i \in I\} \cup \{\beta_i : i \in I\}$ is a basis of $H^1_{DR}(C_\lambda^r)$.

We consider now $C^r$ as a family of curves by letting $\lambda$ vary through $\{\lambda \in K : |\lambda - 1| < 1\}$. The Gauss-Manin connection $\nabla$ of $C^r$ is a connection

$$\nabla : H^1_{DR} \to H^1_{DR} \otimes \Omega$$

where $H^1_{DR}$ is the sheaf of De Rham cohomology classes of $C^r$ as family of curves over $S = \{s \in K : |s-1| < 1\}$ and $\Omega$ is the sheaf of analytic differentials on S.

The main result of [G1] is a proof of

Proposition 2: $\nabla(\alpha_i) = \sum_{i\in I} \beta_j \otimes \frac{dq_{ij}}{q_{ij}}$

$\nabla(\beta_i) = 0.$

We want to apply this formula to the differential of the first kind

$$\omega_i = \frac{dx_1}{y_1^{i_1}y_2^{i_2}(x_1-1)}.$$

Proposition 3:

$$\omega_i = F_i(\lambda) \cdot \sum_{j\in I} \rho_1^{i_1 j_1} \rho_2^{i_2 j_2} \alpha_j$$

with $F_i(\lambda) = \sum_{n=o}^{\infty} \frac{(\frac{i_1}{r_1})_n \cdot (\frac{i_2}{r_2})_n}{(n!)^2} (1-\lambda)^n$ and $(a)_n := \prod_{i=o}^{n-1} (a+i)$.

Proof: The method of proof consist in the following: It is well known that the cohomology class $\omega$ of $\frac{dx}{y}$, $y := x^a(x-1)^b(x-\lambda)^c$, satisfies the hypergeometric differential equation also known as Picard-Fuchs equation for $\omega$:

$$\lambda(1-\lambda)\nabla_\lambda^2(\omega) + [a+c-(a+b+2c)\lambda]\nabla_\lambda(\omega) - (a+b+c-1)\,\omega = 0$$

see for instance [M], p. 378 or [D], Chap. I, p. 8.

In our case $a = \frac{-i_1}{r_1}$, $b = -1 + \frac{i_2}{r_2}$, $c = \frac{i_2}{r_2}$.

A straightforward computation shows that the above $F_i$ is up to a constant the only power series solution of the above differential equation.

But $\omega_i$ being a differential of the first kind admits a representation

$$\omega_i = \sum_{j\in I} G_{ij}\alpha_j$$

with $G_{ij}$ analytic in S.

Thus $\nabla_\lambda(\omega_i) \equiv \sum_{j\in I} \dot{G}_{ij}\alpha_j$ mod H' when H' is the subspace generated by $\{\beta_i : i \in I\}$. Thus each $\dot{G}_{ij} = c_j \cdot F_i$ with $c_j \in K$, where the dot over $G_{ij}$ means the derivative with respect to $\lambda$. By considering the limit case for $s \to 1$ one obtains the above constants. For the details see [GH].

R. Coleman (Berkeley) has informed me that he has a completely different approach to this result.



## 4. Application to periods

The formulas for the Gauss-Manin connection and the Picard-Fuchs equation allow to derive an explicit expression for the logarithmic derivative of $q_{ij}$ with respect to the variable $\lambda$ in the domain

$$\{|\lambda-1| < 1\}.$$

Proposition 4:

$$\frac{\dot{q}_{ij}}{q_{ij}} = \sum_{k\in I} c_{ik} \cdot E_{kj}$$

with $c_{ik} = \frac{(\rho_1^{-i_1k_1} - 1)(\rho_2^{-i_2k_2} - 1)}{r_1 \cdot r_2}$

$$E_{kj} = \frac{A_{kj}}{(1-\lambda)\lambda^{\frac{k_1}{r_1}+\frac{k_2}{r_2}} \cdot F_k^2}$$

$$A_{kj} = 1 - \rho_1^{k_1j_1} - \rho_2^{k_2j_2} + \rho^{kj} = c_{-k,j} \cdot r_1 \cdot r_2$$

$$\lambda^a = (1-(1-\lambda))^a := \sum_{n=o}^{\infty} \binom{a}{n}(1-\lambda)^n \cdot (-1)^n$$

$$F_k = \sum_{n=o}^{\infty} \frac{(\frac{k_1}{r_1})_n \cdot (\frac{k_2}{r_2})_n}{(n!)^2} (1-\lambda)^n$$

which is the hypergeometric function ${}_2F_1(\frac{k_1}{r_1}, \frac{k_2}{r_2}; 1; 1-\lambda)$, see [MOS], Chap. II, (2.1).

Remark: In the special case $r_1 = r_2 = 2$ the index set I consists of (1,1) only and with $q := q_{11}$ one gets

$$\frac{\dot{q}}{q} = \frac{4}{(1-\lambda)\lambda \cdot {}_2F_1(\frac{1}{2}, \frac{1}{2}, 1; 1-\lambda)}$$

which is equivalent to a classical formula, see [F]. Be aware that $\lambda$ is not the Legendre parameter as our equation is $y_2^2y_1^2 - y_2^2 - y_1^2 + \lambda = 0$.

We sketch now a proof of proposition 4.

1) Let $\omega_i^* := \frac{\omega_i}{F_i} = \sum_{j\in I} \rho^{ij}\alpha_j$. Then

$$\nabla_\lambda(\omega_i^*) = \sum_{k\in I} E_{ik}\cdot\beta_k$$

with $E_{ik} := \sum_{j\in I} \rho^{ij}\frac{\dot{q}_{jk}}{q_{jk}}$

Let $L_i$ denote the operator

$$L_i = \lambda(1-\lambda)\nabla_\lambda^2 - [1-(\frac{i_1}{r_1}+\frac{i_2}{r_2}+1)(1-\lambda)]\nabla_\lambda - \frac{i_1 i_2}{r_1 r_2}.$$

It is known that

$$L_i(\omega_i) = 0.$$

Now

$$\nabla_\lambda(F_i\omega_i^*) = F_i\nabla_\lambda(\omega_i^*) + \dot{F}_i\omega_i^*$$
$$\nabla_\lambda^2(F_i\omega_i^*) = F_i\nabla_\lambda^2(\omega_i^*) + 2\dot{F}_i\nabla_\lambda(\omega_i^*) + \ddot{F}_i\omega_i^*$$

and $\nabla_\lambda^2(\omega_i^*) = \sum_{k\in I}\dot{E}_{ik}\beta_k$

Substituting into the equation $L_i(\omega_i) = 0$ and looking for the coefficient at $\beta_k$ which must be zero gives

$$\frac{\dot{E}_{ik}}{E_{ik}} = -2\frac{\dot{F}_i}{F_i} + \frac{1}{1-\lambda} - \frac{(\frac{i_1}{r_1}+\frac{i_2}{r_2})}{\lambda}.$$

Solving this differential equation gives

$$E_{ik} = \frac{A_{ik}}{(1-\lambda)\lambda^{(\frac{i_1}{r_1}+\frac{i_2}{r_2})}F_i^2}$$

with a constant $A_{ik}\in K$. $E_{ik}$ is considered as a Laurent series in $(1-\lambda)$; its residue at 1 is just $A_{ik}$.

In a joint work with F. Herrlich we determined the constants $A_{ik}$. A careful study of the action of $\Gamma$ on the Bruhat-Tits tree of $\mathbb{P}$ gives the result that the vanishing order ord $q_{ji}$ of $q_{ji}$ at the



point $s = \lambda = 1$ is as follows:

$$\text{ord } q_{ik} = \begin{cases} 4 : j = k \\ 2 : j \neq k \text{ and } j_1 = k_1 \text{ or } j_2 = k_2 \\ 1 : \text{otherwise}\end{cases}$$

Therefore the residue of $\frac{dq_{jk}}{q_{jk}}$ at $\lambda = 1$ is ord $q_{jk}$ and the residue of $E_{ik}d\lambda$ at $\lambda = 1$ is

$$\sum_{j\in I}\rho^{ij} + \sum_{\substack{j\in I\\ j_1=k_1}}\rho^{ij} + \sum_{\substack{j\in I\\ j_2=k_2}}\rho^{ij} + \rho^{ik}$$

As $\sum_{\substack{j\in I\\ j_1=k_1}}\rho^{ij} = \sum_{j_2=1}^{r_2-1}\rho^{i_1j_1}\rho^{i_2j_2} = -\rho_1^{i_1k_1}$ and

$\sum_{\substack{j\in I\\ j_2=k}}\rho^{ij} = -\rho_2^{i_2k_2}$ and $\sum_{j\in I}\rho^{ij} = 1$ this residue is

indeed $A_{ik}$.

2) Let $\bar{\Gamma} = \Gamma/[\Gamma,\Gamma]$ be the commutator factor group of $\Gamma$; if is a free $\mathbb{Z}$-module generated by the images $e_i$ of $\gamma_i$, $i\in I$.

Now G is canonically isomorphic to the factor group $\langle\sigma_1,\sigma_2\rangle/\Gamma$ and thus acts on $\bar{\Gamma}$ by inner automorphims; we consider $\bar{\Gamma}$ as G-module.

As $\sigma_1\gamma_i\sigma_1^{-1} = \sigma_1\cdot\sigma_1^{i_1}\sigma_2^{i_2}\sigma_1^{-i_1}\sigma_2^{-i_2}\cdot\sigma_1^{-1}$

$$= \sigma_1^{i_1+1}\sigma_2^{i_2}\sigma_1^{-i_1-1}\cdot\sigma_1\cdot\sigma_2^{-i_2}\cdot\sigma_1^{-1}$$
$$= \gamma_{i_1+1,i_2}\cdot\sigma_2^{i_2}\cdot\sigma_1\cdot\sigma_2^{-i_2}\cdot\sigma_1^{-1}$$
$$= \gamma_{i_1+1,i_2}\cdot\gamma_{1,i_2}^{-1}$$

and $\sigma_2\gamma_i\sigma_2^{-1} = \sigma_2\sigma_1^{i_1}\sigma_2^{i_2}\sigma_1^{-i_1}\sigma_2^{-i_2}\cdot\sigma_2^{-1}$

$$= \sigma_2\sigma_1^{+i_1}\sigma_2^{-1}\sigma_1^{-i_1}\cdot\sigma_1^{i_1}\cdot\sigma_2\cdot\sigma_2^{i_2}\sigma_1^{-i_1}\sigma_2^{-i_2-1}$$
$$= \gamma_{i_1,1}^{-1}\quad\gamma_{i_1,i_2+1}$$

the action of G is known.

Let M be the submodule of the group ring $\mathbb{Z}[G]$ generated by

$$a_i = (\sigma_1^{i_1} - 1) \cdot (\sigma_2^{i_2} - 1)$$

for all $(i_1, i_2) \in I$. It is easy to verify that the mapping

$$\kappa : \bar{\Gamma} \to M$$

which sends $e_i$ to $a_i$, $i \in I$, is indeed an isomorphism of G-modules. In order to be able to work with a simpler basis we consider $K \otimes M$ and let

$$w_i := \sum_{j \in I} \rho^{+ij} \cdot a_j \in K \otimes M$$

where $i \cdot j$ is the multiplication in I considered as multiplicative semi-group in the ring $J = \mathbb{Z}/r_1\mathbb{Z} \oplus \mathbb{Z}/r_2\mathbb{Z}$ and $\rho^i := \rho_1^{i_1} \cdot \rho_2^{i_2}$ for $i \in \mathbb{Z}/r_1\mathbb{Z} \oplus \mathbb{Z}/r_2\mathbb{Z}$.

Then $\quad w_i = \sum_{j \in J} \rho^{+ij} \sigma^j$

with $\quad \sigma^j = \sigma_1^{j_1} \cdot \sigma_2^{j_2}$ for $j \in J$

and $\quad \sigma_1 w_i = \sigma_1^{-i_1} \cdot w_i$

$$\sigma_2 w_i = \rho_2^{-i_2} \cdot w_i$$

This shows that $\{w_i : i \in I\}$ is a basis of $K \otimes M$ and thus

$$a_i = \sum_{j \in I} c_{ij} w_j$$

with a matrix $c = (c_{ij})$, $c_{ij} \in K$, of determinant $\neq 0$. In fact c is the inverse of the matrix

$$(\rho^{+ij})_{i,j \in I}$$

A straight forward computation gives: $c_{ij} = \dfrac{\rho_1^{i_1 j_1} - 1}{r_1} \cdot \dfrac{\rho_2^{i_2 j_2} - 1}{r_2}$

for any $i,j \in I$, $i = (i_1, i_2)$, $j = (j_1, j_2)$.



3) From 2) we get that

$$\alpha_i = \sum_{j \in I} c_{ij} \omega_j^*$$

Now $\quad \nabla(\alpha_i) = \sum_{k \in I} \beta_k \dfrac{\dot{q}_{ik}}{q_{ik}}$

$$= \sum_{j \in I} c_{ij} \Big( \sum_{k \in I} E_{jk} \beta_k \Big)$$

$$= \sum_{k \in I} \Big( \sum_{j \in I} c_{ij} E_{jk} \Big) \cdot \beta_k$$

and thus $\sum_{j \in I} c_{ij} E_{jk} = \dfrac{\dot{q}_{ik}}{q_{ik}}$ which completes the proof.

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